

## Commutative Algebra Fall 2020 (Part II)

### Problem Set 2

(1) (Term Orders)

Show that there is only one term order for monomials in a single variable.

(2) (Term Orders)

Show that deg lex and deg rev lex orders coincide in  $\mathbb{K}[x_1, x_2]$ .

(3) (Division Algorithm)

Let  $f = x_1^3 x_2^3 + 2x_2^3$ ,  $f_1 = 2x_1 x_2^2 + 3x_1 + 4x_2^2$ ,  $f_2 = x_2^2 - 2x_2 - 2 \in \mathbb{Q}[x_1, x_2]$  and  $F = \{f_1, f_2\}$ .

Let  $\preceq$  be the lex order.

Find a reduced  $r$  such that  $f \xrightarrow{F}_+ r$  and find  $u_1, u_2$  such that

$$f = u_1 f_1 + u_2 f_2 + r.$$

(4) (Gröbner Bases)

Let  $A = (a_{ij}) \in \mathbb{K}^{m \times n}$  be an  $m \times n$ -matrix with entries in  $\mathbb{K}$ . Let  $f_i = a_{i1}x_1 + \cdots + a_{in}x_n \in \mathbb{K}[x_1, \dots, x_n]$  and  $I = (f_1, \dots, f_m)$ .

(i) Show:  $f_1, \dots, f_m$  is a Gröbner basis for the ideal  $I$  with respect to lex if  $A$  can be brought in row echelon form by permuting its rows.

(ii) Determine  $\text{in}_{\preceq}(I)$  for  $\preceq = \text{lex}$  (in terms of the row-echelon form of  $A$ ).