Completion

Theorem 1 (Completion) *If $\big(\mathcal{V}, \langle\,\cdot\,,\,\cdot\,\rangle_{\mathcal{V}}\big)$ is any inner product space, then there exists a Hilbert space $\big(\mathcal{H}, \langle\,\cdot, \,,\,\cdot\,\rangle_{\mathcal{H}}\big)$ and a map $U : \mathcal{V} \to \mathcal{H}$ such that*

(i) U is 1–1

(ii) U is linear

(iii) $\langle U\mathbf{x}, U\mathbf{y}\rangle_{\mathcal{H}} = \langle \mathbf{x}, \mathbf{y}\rangle_{\mathcal{V}}$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{V}$

(iv) $U(\mathcal{V}) = \big\{\ U\mathbf{x}\ \big|\ \mathbf{x} \in \mathcal{V}\ \big\}$ is dense in $\mathcal{H}$. If $\mathcal{V}$ is complete, then $U(\mathcal{V}) = \mathcal{H}$.

Remark 2

(a) $\mathcal{H}$ is called the completion of $\mathcal{V}$.

(b) U gives a one-to–one correspondance between elements of $\mathcal{V}$ and elements of $U(\mathcal{V})$. So we can think of U as giving the new name $U\mathbf{x}$ to each $\mathbf{x} \in \mathcal{V}$ and we can think of $U(\mathcal{V})$ as being the same as $\mathcal{V}$ but with the names of the elements changed. Thus, we can think of $\mathcal{V}$ as being $U(\mathcal{V}) \subset \mathcal{H}$. Using this point of view, the above theorem says that any inner product space can be extended to a complete inner product space. I.e. can have its holes filled in.

Motivation.

The hard part of the proof is to make a guess as to what $\mathcal{H}$ should be. That's what we'll do now. A good strategy is to work backwards. Suppose that, somehow, we have found a suitable $\mathcal{H}$ with $\mathcal{V} \subset \mathcal{H}$. If we can find a way to describe each element of $\mathcal{H}$ purely in terms of elements of $\mathcal{V}$, then we can turn around and take that as the definition of $\mathcal{H}$.

Because $\mathcal{V}$ is dense in $\mathcal{H}$, each element of $\mathcal{H}$ may be written as the limit of a sequence in $\mathcal{V}$, and each such sequence is Cauchy. Thus specifying an element of $\mathcal{H}$ is equivalent to specifying a Cauchy sequence in $\mathcal{V}$. But there is not a one–to–one correspondance between elements of $\mathcal{H}$ and Cauchy sequences in $\mathcal{V}$, because many different Cauchy sequences in $\mathcal{V}$ converge to the same element of $\mathcal{H}$. For example, if $\big\{\mathbf{x}_n\big\}_{n\in\mathbb{N}} \subset \mathcal{V}$ converges to $\mathbf{x} \in \mathcal{H}$, then $\big\{\mathbf{x}_n + \frac{1}{2^n}\mathbf{x}_1\big\}_{n\in\mathbb{N}} \subset \mathcal{V}$ and $\big\{e^{1/n}\mathbf{x}_n + \frac{(-1)^n}{n}\mathbf{x}_2\big\}_{n\in\mathbb{N}} \subset \mathcal{V}$ also converge to $\mathbf{x}$. To get a one–to–one correspondance, we can identify each $\mathbf{x} \in \mathcal{H}$ with the *set of all* Cauchy sequences in $\mathcal{V}$ that converge to $\mathbf{x}$.

Outline of Proof. First define

$$\mathcal{V}' = \Big\{\ \{\mathbf{x}_n\}_{n\in\mathbb{N}}\ \Big|\ \{\mathbf{x}_n\}_{n\in\mathbb{N}} \text{ is a Cauchy sequence in } \mathcal{V}\Big\}$$

That is, $\mathcal{V}'$ is the set of all Cauchy sequences in $\mathcal{V}$. Next we define two Cauchy sequences $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ and $\{\mathbf{y}_n\}_{n\in\mathbb{N}}$ in $\mathcal{V}$ to be "equivalent", written $\{\mathbf{x}_n\}_{n\in\mathbb{N}} \sim \{\mathbf{y}_n\}_{n\in\mathbb{N}}$, if and only if

$$\lim_{n\to\infty} \|\mathbf{x}_n - \mathbf{y}_n\|_{\mathcal{V}} = 0$$

This definition is rigged so that any two convergent sequences have the same limit if and only if they are equivalent. Next, if $\{\mathbf{x}_n\}_{n\in\mathbb{N}} \in \mathcal{V}'$, we define the "equivalence class of $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$" to be the set

$$\big[\{\mathbf{x}_n\}_{n\in\mathbb{N}}\big] = \Big\{\ \{\mathbf{y}_n\}_{n\in\mathbb{N}} \in \mathcal{V}'\ \Big|\ \{\mathbf{y}_n\}_{n\in\mathbb{N}} \sim \{\mathbf{x}_n\}_{n\in\mathbb{N}}\ \Big\}$$

of all Cauchy sequences that are equivalent to $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$. We shall shortly prove

Lemma 3 *$\sim$ is an equivalence relation*[1]*. In particular, if $\{\mathbf{x}_n\}_{n\in\mathbb{N}},\ \{\mathbf{y}_n\}_{n\in\mathbb{N}} \in \mathcal{V}'$ then*

$$\text{either}\ \big[\{\mathbf{x}_n\}_{n\in\mathbb{N}}\big] = \big[\{\mathbf{y}_n\}_{n\in\mathbb{N}}\big] \qquad \text{or} \qquad \big[\{\mathbf{x}_n\}_{n\in\mathbb{N}}\big] \cap \big[\{\mathbf{y}_n\}_{n\in\mathbb{N}}\big] = \emptyset$$

If you think of a Cauchy sequence as one person and an equivalence class of Cauchy sequences as a "family" of related people, then the above Lemma says, that the whole world is divided into a collection of nonoverlapping families. Next, we define

$$\mathcal{H} = \Big\{ \ \big[\{\mathbf{x}_n\}_{n\in\mathbb{N}}\big] \ \Big|\ \{\mathbf{x}_n\}_{n\in\mathbb{N}} \in \mathcal{V}' \ \Big\}$$

as the set of all "families" and prove

Lemma 4 *If $\{\mathbf{x}_n\}_{n\in\mathbb{N}},\ \{\mathbf{y}_n\}_{n\in\mathbb{N}} \in \mathcal{V}'$ then $\lim_{n\to\infty} \langle \mathbf{x}_n, \mathbf{y}_n\rangle_{\mathcal{V}}$ exists.*

Lemma 5 *Define, for each $\big[\{\mathbf{x}_n\}_{n\in\mathbb{N}}\big] \in \mathcal{H}$, $\big[\{\mathbf{y}_n\}_{n\in\mathbb{N}}\big] \in \mathcal{H}$ and $\alpha \in \mathbb{C}$,*

$$\begin{aligned} \big[\{\mathbf{x}_n\}_{n\in\mathbb{N}}\big] + \big[\{\mathbf{y}_n\}_{n\in\mathbb{N}}\big] &= \big[\{\mathbf{x}_n + \mathbf{y}_n\}_{n\in\mathbb{N}}\big] \\ \alpha\big[\{\mathbf{x}_n\}_{n\in\mathbb{N}}\big] &= \big[\{\alpha\mathbf{x}_n\}_{n\in\mathbb{N}}\big] \\ \big\langle \big[\{\mathbf{x}_n\}_{n\in\mathbb{N}}\big]\,,\ \big[\{\mathbf{y}_n\}_{n\in\mathbb{N}}\big] \big\rangle_{\mathcal{H}} &= \lim_{n\to\infty} \langle \mathbf{x}_n, \mathbf{y}_n\rangle_{\mathcal{V}} \end{aligned}$$

Each of these operations is well–defined. For example, if $\{\mathbf{x}_n\}_{n\in\mathbb{N}} \sim \{\mathbf{x}'_n\}_{n\in\mathbb{N}}$ (so that $\big[\{\mathbf{x}_n\}_{n\in\mathbb{N}}\big] = \big[\{\mathbf{x}'_n\}_{n\in\mathbb{N}}\big]$) and $\{\mathbf{y}_n\}_{n\in\mathbb{N}} \sim \{\mathbf{y}'_n\}_{n\in\mathbb{N}}$, then

$$\lim_{n\to\infty} \langle \mathbf{x}_n, \mathbf{y}_n\rangle_{\mathcal{V}} = \lim_{n\to\infty} \langle \mathbf{x}'_n, \mathbf{y}'_n\rangle_{\mathcal{V}}$$

Finally, we define $U : \mathcal{V} \to \mathcal{H}$ by

$$U\mathbf{x} = \big[\{\mathbf{x}, \mathbf{x}, \mathbf{x}, \cdots\}\big]$$

The conclusions of Theorem 1 are now proven as a series of Lemmata.

Lemma 6 *$\mathcal{H}$ with the operations of Lemma 5 is an inner product space.*

Lemma 7 *$\mathcal{H}$ is complete*

Lemma 8 *U is linear and obeys $\langle U\mathbf{x}, U\mathbf{y}\rangle_{\mathcal{H}} = \langle \mathbf{x}, \mathbf{y}\rangle_{\mathcal{V}}$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{V}$.*

Lemma 9 *U is one–to–one.*

Lemma 10 *$U(\mathcal{V})$ is dense in $\mathcal{H}$.*

Lemma 11 *If $\mathcal{V}$ is complete, then $U(\mathcal{V}) = \mathcal{H}$.*

■

(1) A binary relation $\sim$ on a set S is an equivalence relation if and only if, for all $s, t, u \in S$, (1) $s \sim s$ (reflexivity) (2) if $s \sim t$, then $t \sim s$ (symmetry) and (3) if $s \sim t$ and $t \sim u$, then $s \sim u$ (transitivity).

So now we just have to prove all of the Lemmata.

Lemma 3 $\sim$ *is an equivalence relation. In particular, if* $\{\mathbf{x}_n\}_{n\in\mathbb{N}},\ \{\mathbf{y}_n\}_{n\in\mathbb{N}} \in \mathcal{V}'$ *then*

$$\textit{either } \big[\{\mathbf{x}_n\}_{n\in\mathbb{N}}\big] = \big[\{\mathbf{y}_n\}_{n\in\mathbb{N}}\big] \quad \textit{or} \quad \big[\{\mathbf{x}_n\}_{n\in\mathbb{N}}\big] \cap \big[\{\mathbf{y}_n\}_{n\in\mathbb{N}}\big] = \emptyset$$

Proof: The three equivalence relation axioms are trivially verified, so we only prove the last claim. Suppose that $\big[\{\mathbf{x}_n\}_{n\in\mathbb{N}}\big] \cap \big[\{\mathbf{y}_n\}_{n\in\mathbb{N}}\big] \neq \emptyset$ but $\big[\{\mathbf{x}_n\}_{n\in\mathbb{N}}\big] \neq \big[\{\mathbf{y}_n\}_{n\in\mathbb{N}}\big]$. As $\big[\{\mathbf{x}_n\}_{n\in\mathbb{N}}\big]$ and $\big[\{\mathbf{y}_n\}_{n\in\mathbb{N}}\big]$ intersect, there is a Cauchy sequence $\{\mathbf{u}_n\}_{n\in\mathbb{N}}$ that is in both $\big[\{\mathbf{x}_n\}_{n\in\mathbb{N}}\big]$ and $\big[\{\mathbf{y}_n\}_{n\in\mathbb{N}}\big]$. Consequently $\{\mathbf{u}_n\}_{n\in\mathbb{N}}$ is equivalent to both $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ and $\{\mathbf{y}_n\}_{n\in\mathbb{N}}$ so that

$$\lim_{n\to\infty} \|\mathbf{u}_n - \mathbf{x}_n\|_{\mathcal{V}} = 0 \text{ and } \lim_{n\to\infty} \|\mathbf{u}_n - \mathbf{y}_n\|_{\mathcal{V}} = 0$$

On the other hand, as $\big[\{\mathbf{x}_n\}_{n\in\mathbb{N}}\big]$ and $\big[\{\mathbf{y}_n\}_{n\in\mathbb{N}}\big]$ are different, there is a Cauchy sequence $\{\mathbf{v}_n\}_{n\in\mathbb{N}}$ that is in one of them (say $\big[\{\mathbf{x}_n\}_{n\in\mathbb{N}}\big]$) and not in the other (say $\big[\{\mathbf{y}_n\}_{n\in\mathbb{N}}\big]$). Since $\{\mathbf{v}_n\}_{n\in\mathbb{N}}$ is in $\big[\{\mathbf{x}_n\}_{n\in\mathbb{N}}\big]$,

$$\lim_{n\to\infty} \|\mathbf{v}_n - \mathbf{x}_n\|_{\mathcal{V}} = 0$$

By the triangle inequality

$$\|\mathbf{v}_n - \mathbf{y}_n\|_{\mathcal{V}} \le \|\mathbf{v}_n - \mathbf{x}_n\|_{\mathcal{V}} + \|\mathbf{x}_n - \mathbf{u}_n\|_{\mathcal{V}} + \|\mathbf{u}_n - \mathbf{y}_n\|_{\mathcal{V}}$$

But this forces $\lim\limits_{n\to\infty} \|\mathbf{v}_n - \mathbf{y}_n\|_{\mathcal{V}} = 0$, which contradicts the assumption that $\{\mathbf{v}_n\}_{n\in\mathbb{N}}$ is not in $\big[\{\mathbf{y}_n\}_{n\in\mathbb{N}}\big]$.

■

Lemma 4 *If* $\{\mathbf{x}_n\}_{n\in\mathbb{N}},\ \{\mathbf{y}_n\}_{n\in\mathbb{N}} \in \mathcal{V}'$ *then* $\lim\limits_{n\to\infty} \langle \mathbf{x}_n, \mathbf{y}_n\rangle_{\mathcal{V}}$ *exists.*

Proof: We first observe that

$$\begin{aligned} \big|\, \langle \mathbf{x}_n, \mathbf{y}_n\rangle_{\mathcal{V}} - \langle \mathbf{x}_m, \mathbf{y}_m\rangle_{\mathcal{V}} \,\big| &\le \big|\, \langle \mathbf{x}_n - \mathbf{x}_m, \mathbf{y}_n\rangle_{\mathcal{V}} \,\big| + \big|\, \langle \mathbf{x}_m, \mathbf{y}_n - \mathbf{y}_m\rangle_{\mathcal{V}} \,\big| \\ &\le \|\mathbf{x}_n - \mathbf{x}_m\|_{\mathcal{V}}\, \|\mathbf{y}_n\|_{\mathcal{V}} + \|\mathbf{x}_m\|_{\mathcal{V}}\, \|\mathbf{y}_n - \mathbf{y}_m\|_{\mathcal{V}} \end{aligned} \tag{1}$$

Since $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ and $\{\mathbf{y}_n\}_{n\in\mathbb{N}}$ are both Cauchy, both $\{\|\mathbf{x}_m\|_{\mathcal{V}}\}_{m\in\mathbb{N}}$ and $\{\|\mathbf{y}_n\|_{\mathcal{V}}\}_{n\in\mathbb{N}}$ are bounded and both $\|\mathbf{x}_n - \mathbf{x}_m\|_{\mathcal{V}}$ and $\|\mathbf{y}_n - \mathbf{y}_m\|_{\mathcal{V}}$ converge to zero as $m, n \to \infty$. So the same is true for $\big|\, \langle \mathbf{x}_n, \mathbf{y}_n\rangle_{\mathcal{V}} - \langle \mathbf{x}_m, \mathbf{y}_m\rangle_{\mathcal{V}} \,\big|$. So $\{\langle \mathbf{x}_n, \mathbf{y}_n\rangle_{\mathcal{V}}\}_{n\in\mathbb{N}}$ is a Cauchy sequence of complex numbers. Since $\mathbb{C}$ is complete, this Cauchy sequence converges. ■

Lemma 5 *Define, for each* $\big[\{\mathbf{x}_n\}_{n\in\mathbb{N}}\big] \in \mathcal{H}$, $\big[\{\mathbf{y}_n\}_{n\in\mathbb{N}}\big] \in \mathcal{H}$ *and* $\alpha \in \mathbb{C}$,

$$\begin{aligned} \big[\{\mathbf{x}_n\}_{n\in\mathbb{N}}\big] + \big[\{\mathbf{y}_n\}_{n\in\mathbb{N}}\big] &= \big[\{\mathbf{x}_n + \mathbf{y}_n\}_{n\in\mathbb{N}}\big] \\ \alpha\big[\{\mathbf{x}_n\}_{n\in\mathbb{N}}\big] &= \big[\{\alpha\mathbf{x}_n\}_{n\in\mathbb{N}}\big] \\ \big\langle \big[\{\mathbf{x}_n\}_{n\in\mathbb{N}}\big]\,,\ \big[\{\mathbf{y}_n\}_{n\in\mathbb{N}}\big] \big\rangle_{\mathcal{H}} &= \lim_{n\to\infty} \langle \mathbf{x}_n, \mathbf{y}_n\rangle_{\mathcal{V}} \end{aligned}$$

Each of these operations is well–defined.

Proof: Let $\{\mathbf{x}_n\}_{n\in\mathbb{N}} \sim \{\mathbf{x}'_n\}_{n\in\mathbb{N}}$ and $\{\mathbf{y}_n\}_{n\in\mathbb{N}} \sim \{\mathbf{y}'_n\}_{n\in\mathbb{N}}$. Replace, in (1), $\mathbf{x}_m$ and $\mathbf{y}_m$ by $\mathbf{x}'_n$ and $\mathbf{y}'_n$ respectively. This gives

$$\big|\,\langle \mathbf{x}_n, \mathbf{y}_n\rangle_{\mathcal{V}} - \langle \mathbf{x}'_n, \mathbf{y}'_n\rangle_{\mathcal{V}}\,\big| \le \|\mathbf{x}_n - \mathbf{x}'_n\|_{\mathcal{V}}\,\|\mathbf{y}_n\|_{\mathcal{V}} + \|\mathbf{x}'_n\|_{\mathcal{V}}\,\|\mathbf{y}_n - \mathbf{y}'_n\|_{\mathcal{V}}$$

Since $\{\mathbf{x}_n\}_{n\in\mathbb{N}} \sim \{\mathbf{x}'_n\}_{n\in\mathbb{N}}$, we have that $\lim\limits_{n\to\infty} \|\mathbf{x}_n - \mathbf{x}'_n\|_{\mathcal{V}} = 0$. Since $\{\mathbf{y}_n\}_{n\in\mathbb{N}} \sim \{\mathbf{y}'_n\}_{n\in\mathbb{N}}$, we have that $\lim\limits_{n\to\infty} \|\mathbf{y}_n - \mathbf{y}'_n\|_{\mathcal{V}} = 0$. Since $\big\{\|\mathbf{y}_n\|_{\mathcal{V}}\big\}_{n\in\mathbb{N}}$ and $\big\{\|\mathbf{x}'_n\|_{\mathcal{V}}\big\}_{n\in\mathbb{N}}$ are bounded, $\lim\limits_{n\to\infty} \big|\,\langle \mathbf{x}_n, \mathbf{y}_n\rangle_{\mathcal{V}} - \langle \mathbf{x}'_n, \mathbf{y}'_n\rangle_{\mathcal{V}}\,\big| = 0$. So the inner product is well–defined. Similarly

$$\begin{aligned} \|(\mathbf{x}_n + \mathbf{y}_n) - (\mathbf{x}'_n + \mathbf{y}'_n)\|_{\mathcal{V}} &\le \|\mathbf{x}_n - \mathbf{x}'_n\|_{\mathcal{V}} + \|\mathbf{y}_n - \mathbf{y}'_n\|_{\mathcal{V}} \overset{n\to\infty}{\longrightarrow} 0 \\ \|\alpha\mathbf{x}_n - \alpha\mathbf{x}'_n\|_{\mathcal{V}} &\le |\alpha|\,\|\mathbf{x}_n - \mathbf{x}'_n\|_{\mathcal{V}} \overset{n\to\infty}{\longrightarrow} 0 \end{aligned}$$

so that addition and scalar multiplication are well–defind. ■

Lemma 6 *$\mathcal{H}$ with the operations of Lemma 5 is an inner product space.*

Proof: There are nine vector space axioms and three inner product axioms to be checked. The proofs are essentially trivial and very similar. I'll just verify the first vector space axiom and first half of the first inner product axiom. Let $\big[\{\mathbf{x}_n\}_{n\in\mathbb{N}}\big], \big[\{\mathbf{y}_n\}_{n\in\mathbb{N}}\big] \in \mathcal{H}$ and $\alpha \in \mathbb{C}$.

$$\begin{aligned} \big[\{\mathbf{x}_n\}_{n\in\mathbb{N}}\big] + \big[\{\mathbf{y}_n\}_{n\in\mathbb{N}}\big] &= \big[\{\mathbf{x}_n + \mathbf{y}_n\}_{n\in\mathbb{N}}\big] = \big[\{\mathbf{y}_n + \mathbf{x}_n\}_{n\in\mathbb{N}}\big] \\ &= \big[\{\mathbf{y}_n\}_{n\in\mathbb{N}}\big] + \big[\{\mathbf{x}_n\}_{n\in\mathbb{N}}\big] \\ \big\langle \big[\{\mathbf{x}_n\}_{n\in\mathbb{N}}\big], \alpha\big[\{\mathbf{y}_n\}_{n\in\mathbb{N}}\big]\big\rangle_{\mathcal{H}} &= \big\langle \big[\{\mathbf{x}_n\}_{n\in\mathbb{N}}\big], \big[\{\alpha\mathbf{y}_n\}_{n\in\mathbb{N}}\big]\big\rangle_{\mathcal{H}} = \lim_{n\to\infty}\langle \mathbf{x}_n, \alpha\mathbf{y}_n\rangle_{\mathcal{V}} = \alpha\lim_{n\to\infty}\langle \mathbf{x}_n, \mathbf{y}_n\rangle_{\mathcal{V}} \\ &= \alpha\big\langle \big[\{\mathbf{x}_n\}_{n\in\mathbb{N}}\big]\,,\,\big[\{\mathbf{y}_n\}_{n\in\mathbb{N}}\big]\big\rangle_{\mathcal{H}} \end{aligned}$$

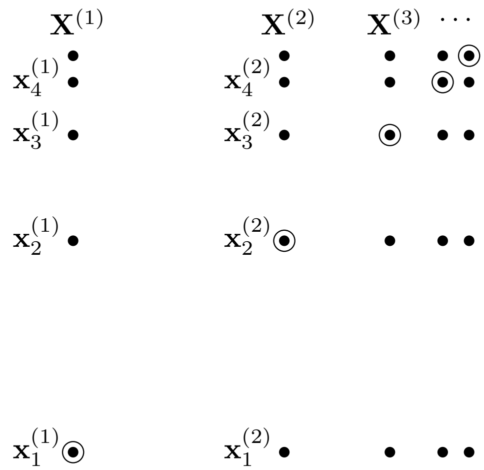
■

Lemma 7 *$\mathcal{H}$ is complete*

Proof: Let $\{\mathbf{X}^{(n)} \in \mathcal{H}\}_{n\in\mathbb{N}}$ be a Cauchy sequence. We must prove that it has a limit, $\mathbf{X} \in \mathcal{H}$, as $n \to \infty$. Each $\mathbf{X}^{(n)}$ is an equivalence class of Cauchy sequences in $\mathcal{V}$. Say $\mathbf{X}^{(n)} = \big[\big\{\mathbf{x}^{(n)}_\ell\big\}_{\ell\in\mathbb{N}}\big]$. We shall guess $\mathbf{X} = \big[\big\{\mathrm{x}_n\big\}_{n\in\mathbb{N}}\big]$ by choosing each x_n to be an $\mathbf{x}^{(n)}_{\ell_n}$ with ℓ_n chosen larger and larger as n increases. Here we go:

$$\begin{gathered} \big\{\mathbf{x}^{(1)}_\ell\big\}_{\ell\in\mathbb{N}} \text{ Cauchy} \Rightarrow \exists\,\ell_1 \in \mathbb{N} \text{ s.t. } \ell \ge \ell_1 \Rightarrow \big\|\mathbf{x}^{(1)}_\ell - \mathbf{x}^{(1)}_{\ell_1}\big\|_{\mathcal{V}} < 1. \text{ Pick } \mathrm{x}_1 = \mathbf{x}^{(1)}_{\ell_1}. \\ \big\{\mathbf{x}^{(2)}_\ell\big\}_{\ell\in\mathbb{N}} \text{ Cauchy} \Rightarrow \exists\,\ell_2 > \ell_1 \text{ s.t. } \ell \ge \ell_2 \Rightarrow \big\|\mathbf{x}^{(2)}_\ell - \mathbf{x}^{(2)}_{\ell_2}\big\|_{\mathcal{V}} < \tfrac{1}{2}. \text{ Pick } \mathrm{x}_2 = \mathbf{x}^{(2)}_{\ell_2}. \\ \big\{\mathbf{x}^{(3)}_\ell\big\}_{\ell\in\mathbb{N}} \text{ Cauchy} \Rightarrow \exists\,\ell_3 > \ell_2 \text{ s.t. } \ell \ge \ell_3 \Rightarrow \big\|\mathbf{x}^{(3)}_\ell - \mathbf{x}^{(3)}_{\ell_3}\big\|_{\mathcal{V}} < \tfrac{1}{3}. \text{ Pick } \mathrm{x}_3 = \mathbf{x}^{(3)}_{\ell_3}. \\ \vdots \\ \big\{\mathbf{x}^{(n)}_\ell\big\}_{\ell\in\mathbb{N}} \text{ Cauchy} \Rightarrow \exists\,\ell_n > \ell_{n-1} \text{ s.t. } \ell \ge \ell_n \Rightarrow \big\|\mathbf{x}^{(n)}_\ell - \mathbf{x}^{(n)}_{\ell_n}\big\|_{\mathcal{V}} < \tfrac{1}{n}. \text{ Pick } \mathrm{x}_n = \mathbf{x}^{(n)}_{\ell_n}. \\ \vdots \end{gathered}$$

In the example sketched below, the $\mathbf{x}^{(n)}_{\ell_n}$'s are circled (in the simplest case in which ℓ_n happens to be n).



Proof that $\{\mathrm{x}_n\}_{n\in\mathbb{N}}$ *is Cauchy:* Let $\varepsilon > 0$. By the triangle inequality

$$\begin{aligned} \big\|\mathrm{x}_n - \mathrm{x}_m\big\|_{\mathcal{V}} &\le \big\|\mathbf{x}_{\ell_n}^{(n)} - \mathbf{x}_\ell^{(n)}\big\|_{\mathcal{V}} + \big\|\mathbf{x}_\ell^{(n)} - \mathbf{x}_\ell^{(m)}\big\|_{\mathcal{V}} + \big\|\mathbf{x}_\ell^{(m)} - \mathbf{x}_{\ell_m}^{(m)}\big\|_{\mathcal{V}} \\ &= \big\|\mathbf{x}_{\ell_n}^{(n)} - \mathbf{x}_\ell^{(n)}\big\|_{\mathcal{V}} + \big[\big\|\mathbf{x}_\ell^{(n)} - \mathbf{x}_\ell^{(m)}\big\|_{\mathcal{V}} - \|\mathbf{X}^{(n)} - \mathbf{X}^{(m)}\|_{\mathcal{H}}\big] \\ &\qquad + \big\|\mathbf{X}^{(n)} - \mathbf{X}^{(m)}\|_{\mathcal{H}} + \big\|\mathbf{x}_\ell^{(m)} - \mathbf{x}_{\ell_m}^{(m)}\big\|_{\mathcal{V}} \end{aligned} \tag{2}$$

for any $\ell \in \mathbb{N}$.

- If $\ell \ge \ell_n$, the first term is smaller than $\frac{1}{n}$.
- By definition, $\big\|\mathbf{X}^{(n)} - \mathbf{X}^{(m)}\big\|_{\mathcal{H}} = \lim_{\ell\to\infty} \big\|\mathbf{x}_\ell^{(n)} - \mathbf{x}_\ell^{(m)}\big\|_{\mathcal{V}}$. So there is a natural number $N_{n,m}$ such that the second term is smaller than $\frac{\varepsilon}{4}$ whenever $\ell \ge N_{n,m}$.
- By hypothesis, the sequence $\{\mathbf{X}^{(n)}\}_{n\in\mathbb{N}}$ is Cauchy. So there is a natural number $\tilde{N}$ such that the third term is smaller than $\frac{\varepsilon}{4}$ whenever $n, m \ge \tilde{N}$.
- Finally, if $\ell \ge \ell_m$, the last term is smaller than $\frac{1}{m}$.

Choose any natural number $N \ge \max\big\{\tilde{N}, \frac{4}{\varepsilon}\big\}$. I claim that $\|\mathrm{x}_n - \mathrm{x}_m\|_{\mathcal{V}} < \varepsilon$ whenever $n, m \ge N$. To see this, let $n, m \ge N$. Now choose ℓ to be any natural number bigger than $\max\big\{N_{n,m}, \ell_n, \ell_m\big\}$. Then the four terms in (2) are each smaller than $\frac{\varepsilon}{4}$.

Proof that $\mathbf{X} = \lim_{n\to\infty} \mathbf{X}^{(n)}$*:* Let $\varepsilon > 0$. By definition

$$\|\mathbf{X} - \mathbf{X}^{(n)}\|_{\mathcal{H}} = \lim_{m\to\infty} \big\|\mathrm{x}_m - \mathbf{x}_m^{(n)}\|_{\mathcal{V}} = \lim_{m\to\infty} \big\|\mathbf{x}_{\ell_m}^{(m)} - \mathbf{x}_m^{(n)}\|_{\mathcal{V}}$$

By the triangle inequality

$$\big\|\mathbf{x}_{\ell_m}^{(m)} - \mathbf{x}_m^{(n)}\big\|_{\mathcal{V}} \le \big\|\mathbf{x}_{\ell_m}^{(m)} - \mathbf{x}_{\ell_n}^{(n)}\big\|_{\mathcal{V}} + \big\|\mathbf{x}_{\ell_n}^{(n)} - \mathbf{x}_m^{(n)}\big\|_{\mathcal{V}} \tag{3}$$

- Since the sequence $\big\{\mathrm{x}_n = \mathbf{x}_{\ell_n}^{(n)}\big\}_{n\in\mathbb{N}}$ is Cauchy, there is a natural number N' such that the first term is smaller than $\frac{\varepsilon}{2}$ whenever $n, m \ge N'$.
- By the construction of ℓ_n, the second term is smaller than $\frac{1}{n}$ whenever $m \ge \ell_n$.

Choose any natural number $N \ge \max\big\{N', \frac{2}{\varepsilon}\big\}$. I claim that $\|\mathbf{X} - \mathbf{X}^{(n)}\|_{\mathcal{H}} < \varepsilon$ whenever $n \ge N$. To see this, let $n \ge N$. Now for all m bigger than $\max\big\{N', \ell_n\big\}$, the two terms in (3) are each smaller than $\frac{\varepsilon}{2}$.

■

Lemma 8 *U is linear and obeys $\langle U\mathbf{x}, U\mathbf{y}\rangle_{\mathcal{H}} = \langle \mathbf{x}, \mathbf{y}\rangle_{\mathcal{V}}$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{V}$.*

Proof: $\langle U\mathbf{x}, U\mathbf{y}\rangle_{\mathcal{H}} = \big\langle \big[\{\mathbf{x}, \mathbf{x}, \mathbf{x}, \cdots\}\big]\,,\, \big[\{\mathbf{y}, \mathbf{y}, \mathbf{y}, \cdots\}\big]\big\rangle_{\mathcal{H}} = \lim_{n\to\infty} \langle \mathbf{x}, \mathbf{y}\rangle_{\mathcal{V}} = \langle \mathbf{x}, \mathbf{y}\rangle_{\mathcal{V}}$. The proof of linearity is similar. ■

Lemma 9 *U is one–to–one.*

Proof: If $\mathbf{x}, \mathbf{y} \in \mathcal{V}$, then

$$U\mathbf{x} = U\mathbf{y} \iff \big[\{\mathbf{x}, \mathbf{x}, \mathbf{x}, \cdots\}\big] = \big[\{\mathbf{y}, \mathbf{y}, \mathbf{y}, \cdots\}\big] \iff \lim_{n\to\infty} \|\mathbf{x} - \mathbf{y}\|_{\mathcal{V}} = 0 \iff \mathbf{x} = \mathbf{y}$$

■

Lemma 10 *$U(\mathcal{V})$ is dense in $\mathcal{H}$.*

Proof: Let $\mathbf{X} = \big[\{\mathbf{x}_n\}_{n\in\mathbb{N}}\big] \in \mathcal{H}$. Then I claim that the sequence $\big\{U\mathbf{x}_m\big\}_{m\in\mathbb{N}}$ converges in $\mathcal{H}$ to $\mathbf{X}$. To check this, it suffices to observe that

$$\big\|\mathbf{X} - U\mathbf{x}_m\big\|_{\mathcal{H}} = \lim_{n\to\infty} \|\mathbf{x}_n - U(\mathbf{x}_m)_n\|_{\mathcal{V}} = \lim_{n\to\infty} \|\mathbf{x}_n - \mathbf{x}_m\|_{\mathcal{V}}$$

Since $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$ is Cauchy, this converges to zero as $m \to \infty$. ■

Lemma 11 *If $\mathcal{V}$ is complete, then $U(\mathcal{V}) = \mathcal{H}$.*

Proof: Let $\mathbf{X} \in \mathcal{H}$. We must find $\mathbf{x} \in \mathcal{V}$ with $U\mathbf{x} = \mathbf{X}$. By Lemma 10,

$$\begin{aligned}
&\exists\ \big\{\mathbf{x}_n \in \mathcal{V}\big\}_{n\in\mathbb{N}} \ \text{ s.t. } \ \mathbf{X} = \lim_{n\to\infty} U\mathbf{x}_n \\
&\implies \big\{U\mathbf{x}_n\big\}_{n\in\mathbb{N}} \text{ is Cauchy in } \mathcal{H} \\
&\implies \big\{\mathbf{x}_n\big\}_{n\in\mathbb{N}} \text{ is Cauchy in } \mathcal{V}\text{, by Lemma 8} \\
&\implies \mathbf{x} = \lim_{n\to\infty} \mathbf{x}_n \text{ exists, since } \mathcal{V} \text{ is complete} \\
&\implies U\mathbf{x} = \lim_{n\to\infty} U\mathbf{x}_n \text{ by Lemma 8} \\
&\implies U\mathbf{x} = \mathbf{X}
\end{aligned}$$

■