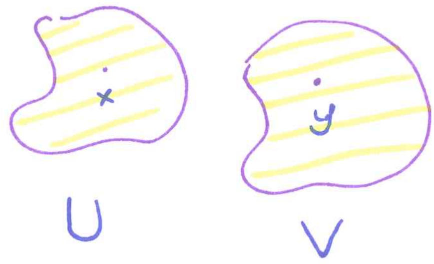
☆ Spaces of Continuous Functions

→ Let X be a compact Hausdorff space.

→ For any pair $x, y \in X$ $\exists$ a neighborhood U of x and V of y such that $U \cap V = \emptyset$ "disjoint".



→ The Hausdorff condition implies convergence of a sequence or net to a unique limit.

→ X is compact if every open cover of X has a finite subcover.

→ Let $\mathcal{C}(X)$ denote the space of continuous $\mathbb{C}$-valued functions on X.

☆ $\mathcal{C}(X)$ is a commutative algebra over the field $\mathbb{C}$: For $f_1, f_2 \in \mathcal{C}(X)$ and $\lambda \in \mathbb{C}$, we have

(a) $(f_1 + f_2)(x) = f_1(x) + f_2(x)$

(b) $(\lambda f_1)(x) = \lambda f_1(x)$

(c) $(f_1 f_2)(x) = f_1(x) f_2(x)$

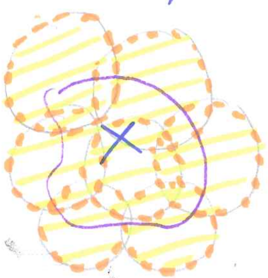
☆ Each $f \in \mathcal{C}(X)$ is bounded on X :

$$X \xrightarrow{\quad f \quad} f(X) \subset \mathbb{C}$$

compact — must be compact

Argument Sketch:

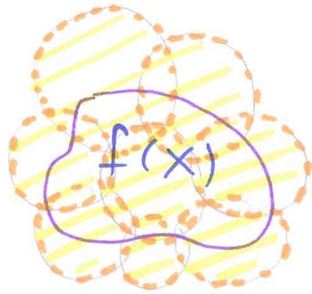
$V = f^{-1}(U)$



$$\xleftarrow{\quad f^{-1} \quad}$$

pullback / inverse image

U



Open cover $\{V_i\}_{i \in I}$ for X

Arbitrary open cover $\{U_i\}_{i \in I}$ for $f(X)$

$\rightarrow$ X compact $\Rightarrow$ $\exists$ finite subcover $\{V_i\}_{i=1}^{n}$ for X

$\therefore$ $\{U_i\}_{i=1}^{n}$ must be finite subcover for $f(X)$

$\therefore f(X)$ is compact $\blacksquare$

☆ Therefore the "norm" of any $f \in \mathcal{C}(X)$

$$\|f\|_\infty := \sup\{|f(x)| : x \in X\}$$

is finite. Some properties of norm $\|\cdot\|_\infty$:

ⓐ $\|f\|_\infty = 0 \iff f = 0$
iff

ⓑ $\|\lambda f\|_\infty = |\lambda| \, \|f\|_\infty \quad \forall \lambda \in \mathbb{C}$

ⓒ $\|f+g\|_\infty \le \|f\|_\infty + \|g\|_\infty$

ⓓ $\|fg\|_\infty \le \|f\|_\infty \|g\|_\infty$ Example $f=1/x$ $g=x$ $C[1,2]$

☆ "Metric" on $\mathcal{C}(X)$: For $f, g \in \mathcal{C}(X)$

$$\rho(f,g) := \|f-g\|_\infty$$

defines a "metric" on $\mathcal{C}(X)$

① $\rho(f,g) = 0 \iff f = g$ (from ⓐ)

② $\rho(f,g) = \rho(g,f)$

③ $\rho(f,h) \le \rho(f,g) + \rho(g,h)$ (from ⓒ)

→ Convergence in the metric ρ

$$\rho(f_n, f) \to 0 \quad \text{as } n \to \infty$$

$$\Rightarrow \|f_n - f\|_\infty \to 0 \quad \text{as } n \to \infty$$

$$\Rightarrow f_n \to f \text{ uniformly on } X \text{ as } n \to \infty.$$

☆ Proposition 1: If X is a Hausdorff comp. space, then $\mathcal{C}(X)$ is a complete metric space.

Proof: Let $\{f_n\}$ be a Cauchy sequence in $\mathcal{C}(X)$. Then

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty = \rho(f_n, f_m)$$

$\forall x \in X$.

→ So $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{C}$ $\forall x \in X$.

→ We define $f(x) := \lim_{n \to \infty} f_n(x) \quad \forall x \in X$

Goal: Prove $f \in \mathcal{C}(X)$ and $\|f_n - f\|_\infty \to 0$.

→ Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $n, m \geq N \Rightarrow \|f_n - f_m\|_\infty < \varepsilon$

→ For a fixed $x_0 \in X$, $\exists$ a neighborhood U of x_0 such that

$$|f_N(x_0) - f_N(x)| < \varepsilon$$

for $x \in U$.

→ $\therefore |f(x_0) - f(x)|$

$$\leq \lim_{n \to \infty} |f_n(x_0) - f_N(x_0)| + \lim_{n \to \infty} |f_N(x_0) - f_N(x)|$$

$$+ \lim_{n \to \infty} |f_N(x) - f_n(x)|$$

$\leq \varepsilon + \varepsilon + \varepsilon = 3\varepsilon$ whenever $x \in U$

$\Rightarrow f$ is continuous at x_0 (arbitrary)

$\Rightarrow f \in \mathcal{C}(X)$.

→ Finally for $n \geq N$ and $x \in X$, we have

$$|f_n(x) - f(x)| = |f_n(x) - \lim_{m \to \infty} f_m(x)|$$

$$= \lim_{m\to\infty} |f_n(x) - f_m(x)|$$

$$\leq \lim_{m\to\infty} \|f_n - f_m\|_\infty \leq \varepsilon$$

$$\Rightarrow \sup_{x\in X} |f_n(x) - f(x)| \leq \varepsilon \quad \text{for } n \geq N$$

$$\Rightarrow \|f_n - f\|_\infty \to 0 \quad \text{as} \quad n \to \infty$$

$\therefore$ $\mathcal{C}(X)$ is complete $\blacksquare$