DIFFERENTIAL GEOMETRY

HOME ASSIGNMENT 1

Please submit your work (by email or on paper) by 10:00 on Monday October 21 to any of the course lecturers.

Exercise 1 Let $k, n \geq 1$ be integers. Let $\sigma_{k,n}\colon [0, 2k\pi] \to \mathbb{R}^2$ be given by

$$\sigma_{k,n}(t) = \begin{pmatrix} (1+\frac{k}{n})\cos t - \frac{k}{n}\cos\left((1+\frac{n}{k})t\right) \\ (1+\frac{k}{n})\sin t - \frac{k}{n}\sin\left((1+\frac{n}{k})t\right) \end{pmatrix}.$$

(1) Prove that $\sigma_{k,n}$ is a closed curve.
(2) Compute the length of $\sigma_{k,n}$.

Hint: You may need to apply the identities
(A) $\cos(\alpha - \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$;
(B) $\sin^2\left(\frac{\alpha}{2}\right) = \frac{1-\cos\alpha}{2}$;
and split the integral over $[0, 2k\pi]$ into integrals over intervals where $\sin\left(\frac{nt}{2k}\right)$ is positive.

Exercise 2 Let $\sigma : [-2\pi, 2\pi] \to \mathbb{R}^3$ be the curve given by

$$\sigma(t) = (1 + \cos t, \sin t, 2\sin(t/2)).$$

(1) Prove that σ is regular.
(2) Prove that the support of σ is the intersection of the sphere of radius 2 centred at the origin with the cylinder having equation $(x-1)^2 + y^2 = 1$.

Exercise 3 Let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be defined by

$$\gamma(t) = (a\cosh t, b\sinh t, at).$$

(1) Calculate the arc length of γ.
(2) Calculate the curvature of γ.
(3) Calculate the torsion of γ.
(4) Prove that if $a = b = 1$ then the curvature is equal to the torsion for every value of the parameter $t \in \mathbb{R}$.

Hint: you may need to apply the following identity
(C) $\cosh\sinh^{-1}(s) = \sqrt{1+s^2}$

Exercise 4 Let $I \subset \mathbb{R}$ be an open interval and $\eta : I \to \mathbb{R}^3$ be a biregular curve parameterized by arc length. Let $\kappa(s)$ and $\tau(s)$ denote the curvature and the torsion of the curve η at s, respectively.

(1) Prove that $\kappa \equiv \pm\tau$ if and only if there exist a nonzero versor $\mathbf{v}$ such that $\langle \mathbf{t}, \mathbf{v}\rangle \equiv \langle \mathbf{b}, \mathbf{v}\rangle$.
(2) Furthermore, prove that if $\kappa \equiv \pm\tau$ then $\langle \mathbf{t}, \mathbf{v}\rangle$ is constant.

Date: 18 October 2019.

DIFFERENTIAL GEOMETRY

HOME ASSIGNMENT 1 (SOLUTIONS)

Exercise 1 Let $k, n \geq 1$ be integers. Let $\sigma_{k,n} \colon [0, 2k\pi] \to \mathbb{R}^2$ be given by

$$\sigma_{k,n}(t) = \begin{pmatrix} (1+\frac{k}{n})\cos t - \frac{k}{n}\cos\left((1+\frac{n}{k})t\right) \\ (1+\frac{k}{n})\sin t - \frac{k}{n}\sin\left((1+\frac{n}{k})t\right) \end{pmatrix}.$$

(1) Prove that $\sigma_{k,n}$ is a closed curve.
(2) Compute the length of $\sigma_{k,n}$.

Hint: You may need to apply the identities
(A) $\cos(\alpha - \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$;
(B) $\sin^2\left(\frac{\alpha}{2}\right) = \frac{1-\cos\alpha}{2}$;
and split the integral over $[0, 2k\pi]$ into integrals over intervals where $\sin\left(\frac{nt}{2k}\right)$ is positive.

Solution. For the first item, it is enough to check that $\sigma_{k,n}(0) = \sigma_{k,n}(2k\pi)$. The verification is immediate since $\frac{n}{k} = 2k\pi = 2n\pi$ is a multiple of 2π.

Let us compute the length of $\sigma_{k,n}$. First of all, we need to compute the norm of $\sigma'(t)$. Writing $\sigma_{k,n}(t) = (x(t), y(t))$ we have $\left\|\sigma'_{k,n}(t)\right\| = \sqrt{(x'(t))^2 + (y'(t))^2}$. A direct computation gives

$$\left\|\sigma'_{k,n}(t)\right\| = \left(1+\frac{k}{n}\right)\sqrt{2 - 2\sin t \sin\left(\left(1+\frac{n}{k}\right)t\right) - 2\cos t\cos\left(\left(1+\frac{n}{k}\right)t\right)}.$$

Notice that it is possible to directly sum terms of the form $\cos^2$ and $\sin^2$ to simplify the computations.

Using the identity (A) above with $\alpha = (1+\frac{n}{k})t$ and $\beta = t$ (and recalling that $-\sin t = \sin(-t)$ and $\cos t = \cos(-t)$) we get

$$\left\|\sigma'_{k,n}(t)\right\| = \left(1+\frac{k}{n}\right)\sqrt{2 - 2\cos\left(\frac{n}{k}t\right)}.$$

Applying identity (B) with $\alpha = \frac{nt}{k}$ then gives

$$\left\|\sigma'_{k,n}(t)\right\| = \left(1+\frac{k}{n}\right)2\sqrt{\sin\left(\frac{nt}{2k}\right)^2} = 2\left|\sin\left(\frac{nt}{2k}\right)\right|.$$

We thus get that the length $L(\sigma_{k,n})$ of $\sigma_{k,n}$ is given by

$$L(\sigma_{k,n}) = 2\left(1+\frac{k}{n}\right)\int_0^{2k\pi}\left|\sin\left(\frac{nt}{2k}\right)\right| dt.$$

The argument of sin in the last integral is periodic of period $\frac{2k}{n}2\pi = \frac{4k\pi}{n}$. Because of the absolute value, the argument is periodic of period $\frac{2k\pi}{n}$. Thus we can split the integral above in n integrals

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over $[0, \frac{2k\pi}{n}]$ and get

$$L(\sigma_{k,n}) = 2\left(1+\frac{k}{n}\right) n \int_0^{2k\pi/n} \sin\left(\frac{nt}{2k}\right) dt.$$

In the last equation we did non need to write the absolute value, since now sin is positive on the interval.

Computing the integral we finally get

$$L(\sigma_{k,n}) = 2\left(1+\frac{k}{n}\right) n \frac{2k}{n} (\cos(0) - \cos\pi) = 8k\left(1+\frac{k}{n}\right),$$

and the solution is complete. □

Exercise 2 Let $\sigma : [-2\pi, 2\pi] \to \mathbb{R}^3$ be the curve given by

$$\sigma(t) = (1+\cos t, \sin t, 2\sin(t/2)).$$

(1) Prove that σ is regular.
(2) Prove that the support of σ is the intersection of the sphere of radius 2 centred at the origin with the cylinder having equation $(x-1)^2 + y^2 = 1$.

Solution. (1) We have

$$\sigma'(t) = (-\sin(t), \cos t, \cos(t/2))$$

for every t. For every t, at least one of the first two coordinates is non zero. Thus, the vector $\sigma'(t)$ is never zero and the curve is regular.

(2) Let us denote by S the sphere and by C the cylinder given in the statement. We denote by $(x(t), y(t), z(t))$ the coordinates of $\sigma(t)$.

We first prove that the support of σ is included in $S \cap C$. For every t, we have

$$\begin{aligned} x(t)^2 + y(t)^2 + z(t)^2 &= (1+\cos(t))^2 + \sin(t)^2 + 4\sin(t/2)^2 \\ &= 2 + 2\cos(t) + 4\sin(t/2)^2 = 4 \end{aligned}$$

where in the last step we applied the identity $\cos(2\theta) = 1 - 2\sin(\theta)^2$ with $\theta = t/2$. This proves that the support of σ is included in S. We also have

$$(x(t)-1)^2 + y(t)^2 = (1+\cos(t)^2 - 1) + \sin(t)^2 = 1,$$

which proves that $\sigma(t) \in C$ for every t. Summing up, we have that the support of σ is contained in $S \cap C$, as desired.

Let us prove the opposite inclusion. Let $p = (x_0, y_0, z_0)$ be any point in $S \cap C$. We need to prove the existence of a $t \in [-2\pi, 2\pi]$ such that $\sigma(t) = p$. We start considering the system

$$\begin{cases} \cos t = x_0 - 1 \\ \sin t = y_0. \end{cases}$$

Since $p \in C$, we have $(x_0-1)^2 + y_0^2 = 1$. Thus it is possible to find a solution for the system above. Notice that we actually have two solutions in the interval of definition $[-2\pi, 2\pi]$ of σ. Let us denote them by $t_0 \in [0, 2\pi]$ and $t_1 = t_0 - 2\pi \in [-2\pi, 0]$. We need to show that one among $z(t_0) = 2\sin(t_0/2)$ and $z(t_1) = 2\sin(t_1/2)$ is equal to z_0. Notice that these two quantities have opposite sign.

Notice that, given x_0 and z_0, only two possibilities are allowed for z_0, and also these choices differ by the sign. More precisely, we have

$$z_0 = \pm\sqrt{4 - x_0^2 - y_0^2}$$

where $\pm$ denotes the two possible choices. The two possible choices $z(t_0)$ and $z(t_2)$ satisfy the above equality (and are the two possible choices) when substituting the values for x_0 and y_0 found above. The proof is completed.

□

Exercise 3 Let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be defined by

$$\gamma(t) = (a \cosh t, b \sinh t, at) .$$

(1) Calculate the arc length of γ.
(2) Calculate the curvature of γ.
(3) Calculate the torsion of γ.
(4) Prove that if $a = b = 1$ then the curvature is equal to the torsion for every value of the parameter $t \in \mathbb{R}$.

Hint: you may need to apply the following identity
(C) $\cosh \sinh^{-1}(s) = \sqrt{1 + s^2}$

Proof. (1) We have $\gamma'(t) = (a \sinh t, b \cosh t, a)$. Thus

$$\int_{s_0}^{s} \|\gamma'(t)\| \, dt = \int_{s_0}^{s} \sqrt{a^2 \sinh^2(t) + b^2 \cosh^2 t + a^2} dt$$
$$= \int_{s_0}^{s} \sqrt{(a^2 + b^2) \cosh^2(t)} dt = \sqrt{a^2 + b^2} \int_{s_0}^{s} |\cosh t| \, dt$$

We take s_0 and consider positive s (the opposite case is analogous). We get

$$\int_{s_0}^{s} \|\gamma'(t)\| \, dt = \sqrt{a^2 + b^2}(\sinh(s) - \sinh(s_0)).$$

(2) For the curvature we employ the formula

$$\kappa(t) = \frac{\|\gamma'(t) \wedge \gamma''(t)\|}{\|\gamma'(t)\|^3}.$$

We calculate that

$$\gamma''(t) = (a \cosh t, b \sinh t, 0),$$

and so

$$\gamma'(t) \wedge \gamma''(t) = (-ab \sinh t, a^2 \cosh t, -ab).$$

Thus

$$\|\gamma'(t) \wedge \gamma''(t)\| = \sqrt{a^2 b^2 \sinh^2 t + a^{4 2} t + a^2 b^2} = a\sqrt{a^2 + b^2} \cosh t.$$

Finally, we obtain

$$\kappa(t) = \frac{a}{(a^2 + b^2) \cosh^2 t}.$$

(3) For the torsion we employ the formula

$$\tau(t) = \frac{\langle \gamma'(t) \wedge \gamma''(t) \rangle}{\|\gamma'(t) \wedge \gamma''(t)\|^2}.$$

We calculate that

$$\gamma'''(t) = (a \sinh t, bt, 0).$$

Thus

$$\langle \gamma'(t) \wedge \gamma''(t) \rangle = a^2 b,$$

and we obtain

$$\tau(t) = \frac{b}{(a^2 + b^2)\cosh t}.$$

(4) Finally, we insert $a = b = 1$ in the expressions of the curvature and the torsion above to obtain

$$\kappa(t) = \frac{1}{2\cosh t} = \tau(t).$$

□

Exercise 4 Let $I \subset \mathbb{R}$ be an open interval and $\eta : I \to \mathbb{R}^3$ be a biregular curve parameterized by arc length. Let $\kappa(s)$ and $\tau(s)$ denote the curvature and the torsion of the curve η at s, respectively.

(1) Prove that $\kappa \equiv \pm\tau$ if and only if there exist a constant nonzero versor $\mathbf{v}$ such that $\langle \mathbf{t}, \mathbf{v} \rangle \equiv \langle \mathbf{b}, \mathbf{v} \rangle$.

Proof. First suppose that there exists a non-zero versor $\mathbf{v}$ which is constant such that

$$\langle \mathbf{t}, \mathbf{v} \rangle = \langle \mathbf{b}, \mathbf{v} \rangle.$$

Differentiating the above with respect to s and using that v is constant we obtain

$$\langle \dot{\mathbf{t}}, \mathbf{v} \rangle = \langle \dot{\mathbf{b}}, \mathbf{v} \rangle,$$

from which we obtain

$$\langle (\kappa + \tau)\mathbf{n}, \mathbf{v} \rangle = 0.$$

This implies that, for any given $s \in I$, (at least) one of the following identities must hold:

(1) $\tau(s) = -k(s)$,
(2) $\langle \mathbf{n}(s), \mathbf{v} \rangle = 0$

Let us prove that if the second holds on an interval $I' \subset I$ then $\tau(s) = k(s)$ on I'. The assertion then follows since the ratio $\tau(s)/k(s)$ must be continuous.

Recall that $\mathbf{t}, \mathbf{n}, \mathbf{b}$ form an orthonormal basis for all $s \in I$. Since $\langle \mathbf{n}, \mathbf{v} \rangle = 0$ on I', we must have $\mathbf{v} = A(s)\mathbf{t} + B(s)\mathbf{b}$ for some functions $A(s)$ and $B(s)$. Since $\mathbf{v}$ is constant, by differentiating we get

$$0 = \dot{A}(s)\mathbf{t} + A(s)\dot{\mathbf{t}} + \dot{B}(s)\mathbf{b} + B(s)\dot{\mathbf{b}} = \dot{A}(s)\mathbf{t} + \dot{B}(s)\mathbf{b} + (kA(s) - \tau B(s))\mathbf{n}.$$

Thus $kA(s) - \tau B(s) = 0$. But the assumption $\langle \mathbf{t}, \mathbf{v} \rangle = \langle \mathbf{b}, \mathbf{v} \rangle$ implies that $A(s) = B(s)$, so that $\tau(s) = k(s)$, as desired.

For the opposite inclusion, let us consider the two cases separately.

If $\tau \equiv k$, we have

$$\dot{\mathbf{t}} = k\mathbf{n} = \tau\mathbf{n} = -\dot{\mathbf{b}},$$

which means that the vector $\mathbf{v} := \mathbf{t} + \mathbf{b}$ is constant. Moreover, since $\langle \mathbf{t}, \mathbf{b} \rangle = 0$, we have

$$\langle \mathbf{t}, \mathbf{v} \rangle = \langle \mathbf{t}, \mathbf{t} + \mathbf{b} \rangle = 1 = \langle \mathbf{b}, \mathbf{t} + \mathbf{b} \rangle = \langle \mathbf{b}, \mathbf{v} \rangle$$

which proves the statement in this case.

Assume now that $\tau \equiv -k$. In this case, we get $\mathbf{t} - \mathbf{b} = 0$, so that the vector $\mathbf{w} := \mathbf{t} - \mathbf{b}$ is constant. It is then enough to take $\mathbf{v}$ orthogonal to $\mathbf{w}$. Indeed, we have $\langle \mathbf{t} - \mathbf{b}, \mathbf{v} \rangle = \langle \mathbf{w}, \mathbf{v} \rangle = 0$, as desired.

□

DIFFERENTIAL GEOMETRY

HOME ASSIGNMENT 2

Please submit your work by email by 12.00 on Monday October 28 to all the course lecturers.

Exercise 1 Let $X \subset \mathbb{R}^3$ be defined as

$$X := \left\{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 = z^2 + 1\right\}$$

and let $\phi \colon \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$\phi(t, h) := (\sqrt{h^2+1}\cos t, \sqrt{h^2+1}\sin t, h)$$

where $\sqrt{\cdot}$ denotes the positive square root. Let $p := (x_0, y_0, z_0)$ be any given point in X.

(1) Prove that X is a regular surface.
(2) Prove that there exists $U \subset \mathbb{R}^2$ such that the restriction of ϕ to U is a local parametrization of X in p.
(3) Prove or disprove the following: there exists $V \subset \mathbb{R}^2$ such that the restriction of ϕ to V is a *global* parametrization of X.
(4) Compute the tangent space of X at p.
(5) Compute the metric coefficients of the first fundamental form of X at p with respect to ϕ.
(6) Prove that a Gauss map $N \colon X \to S^2$ of X is well defined. Compute $N(p)$.
(7) Compute the form coefficients of the second fundamental form of X at p with respect to ϕ.

Exercise 2 Let $X \subset \mathbb{R}^3$ be defined as

$$X := \left\{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 = z^2 + 1\right\}.$$

Give an explicit diffeomorphism F between X and $\mathbb{R}^2 \setminus \{0\}$ (and prove in detail that F is indeed a diffeomorphism).

Exercise 3 Let $S \subset \mathbb{R}^3$ be a surface with a Gauss map $N \colon S \to S^2$. Prove that $dN \equiv 0$ if and only if S is contained in a plane.

Date: 25 October 2019.

DIFFERENTIAL GEOMETRY

HOME ASSIGNMENT 2 (SOLUTIONS)

Please submit your work by email by 12.00 on Tuesday October 29 to all the course lecturers.

Exercise 1 Let $X \subset \mathbb{R}^3$ be defined as

$$X := \{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 = z^2 + 1\}$$

and let $\phi \colon \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$\phi(t, h) := (\sqrt{h^2 + 1} \cos t, \sqrt{h^2 + 1} \sin t, h)$$

where $\sqrt{\cdot}$ denotes the positive square root. Let $p := (x_0, y_0, z_0)$ be any given point in X.

(1) Prove that X is a regular surface.
(2) Prove that there exists $U \subset \mathbb{R}^2$ such that the restriction of ϕ to U is a local parametrization of X in p.
(3) Prove or disprove the following: there exists $V \subset \mathbb{R}^2$ such that the restriction of ϕ to V is a *global* parametrization of X.
(4) Compute the tangent space of X at p.
(5) Compute the metric coefficients of the first fundamental form of X at p with respect to ϕ.
(6) Prove that a Gauss map $N \colon X \to S^2$ of X is well defined. Compute $N(p)$.
(7) Compute the form coefficients of the second fundamental form of X at p with respect to ϕ.

Solution. (1) X is the zero level of the function $G \colon \mathbb{R}^3 \to \mathbb{R}$ given by $G(x, y, z) = x^2+y^2-z^2-1$. Since the preimage of a regular value is a regular surface, it is enough to prove that 0 is a regular value for G. Recall that this means that no point q in the preimage of 0 is critical, i.e., satisfies $\nabla G = 0$. Let us compute the gradient of G. We have

$$\nabla G(x, y, z) = (2x, 2y, -2z).$$

The only point in $\mathbb{R}^3$ where ∇G vanishes is thus the origin. Since $G(O) = -1$, the value 0 is regular and X is a regular surface.

(2) As X the zero level set of the function G and 0 is regular value for G it suffices to check that $\phi(U) \subset X$, $\phi|_U$ injective and $d(\phi|_U)_{t,h}$ injective for all $(t, h) \in U$ in order to conclude that $\phi|_U$ is a local parametrization of X. It is immediate to check that, for all t, h, the image of ϕ is contained in X. Also, for every $p \in X$ there exists (t_0, h_0) such that $p = \phi(t_0, h_0)$. It is enough to take for U any sufficiently small neighbourhood U of (t_0, h_0) with the property that ϕ is injective on U. Notice that, for instance, we can take the set $U = (t_0-\pi, t_0+\pi)\times\mathbb{R}$. It is then straight forward to check that $d\phi_{(t,h)}$ is injective for each (t, h) in U. Calculating $d\phi_{(t,h)}$ we find that the third row of the second column is 1, then, as the first two rows of the first column cannot be simultaneously zero we may conclude that $d\phi_{(t,h)}$ is injective.

(3) We show that such a global parametrization cannot exist.

To do this, we show that on no such V the map ϕ can be a bijection (and thus, in particular, a homeomorphism with the image). Fix any h_0. Notice that the intersection of V with the line $h = h_0$ must be open (in the topology of the line). On the other hand, the image of the line $h = h_0$ is the circle $X \cap \{z = h\}$ in $\mathbb{R}^3$. Since there is no continuous bijection between (a union of) open intervals and the unit circle, the assertion follows.

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(4) We know that the tangent space at the point $p = (x_0, y_0, z_0) \in X$ is the orthogonal to the gradient at p of the map G above. Thus, the required tangent plane is given by $\langle \nabla G_p, (x, y, z) \rangle = 0$, and thus by

$$x_0 x + y_0 y - z_0 z = 0.$$

(5) Let us denote by ∂_t and ∂_h the tangent vectors induced by ϕ corresponding to the coordinates (t, h) on U, where $\phi \colon U \to X$ is a parametrization of X at p, see part (2). We have

$$\partial_t = \left(-\sqrt{h^2+1} \sin t, \sqrt{h^2+1} \cos t, 0 \right)$$

and

$$\partial_h = \left(\frac{h}{\sqrt{h^2+1}} \cos t, \frac{h}{\sqrt{h^2+1}} \sin t, 1 \right)$$

Thus, we deduce that

$$\begin{aligned} E &= \langle \partial_t, \partial_h \rangle_p = h^2 + 1 \left(= x_0^2 + y_0^2\right) \\ F &= \langle \partial_t, \partial_h \rangle_p = 0 \\ G &= \langle \partial_h, \partial_h \rangle_p = \frac{h^2}{h^2+1} + 1 = \frac{2h^2+1}{h^2+1} \left(= \frac{x_0^2 + y_0^2 + z_0^2}{x_0^2 + y_0^2} \right). \end{aligned}$$

(6) Since X is a regular surface given as level set of of a regular value of a smooth function, it is orientable. Thus the Gauss map is well defined. We show two ways to compute it.

A first way is to compute $\frac{\partial_t \wedge \partial_h}{\|\partial_t \wedge \partial_h\|}$ at the point $p = \phi(t_0, h_0)$. From the expression above of ∂_t and ∂_h we get

$$\partial_t \wedge \partial_h = \left(\sqrt{h^2+1} \cos t, \sqrt{h^2+1} \sin t, -h \right).$$

Substituting x_0, y_0, z_0 we find that, at $p = (x_0, y_0, z_0)$,

$$\partial_t \wedge \partial_h = (x_0, y_0, -z_0).$$

The norm of the above vector is $\sqrt{2h^2+1} = \sqrt{2z_0^2+1} = \sqrt{x_0^2 + y_0^2 + z_0^2}$. Thus the choice of N induced by this basis is

$$N(p) = N(x_0, y_0, z_0) = \frac{(x_0, y_0, -z_0)}{\sqrt{x_0^2 + y_0^2 + z_0^2}}$$

or

$$N(p) = N\big(\phi(t_0, h_0)\big) = \frac{\left(\sqrt{h^2+1} \cos t, \sqrt{h^2+1} \sin t, -h \right)}{\sqrt{2h^2+1}},$$

and the computation is complete.

Another way to compute the Gauss map is to remember that it is given by the (normalized) gradient of the function G above. Since $\nabla G = (2x, 2y, -2z)$, whose norm is $2\sqrt{x^2 + y^2 + z^2}$, at the point $p = (x_0, y_0, z_0) \in X$ we get

$$N(p) = \frac{(x_0, y_0, -z_0)}{\sqrt{x_0^2 + y_0^2 + z_0^2}},$$

which coincides with the result found above (notice that also taking the opposite sign would have given a correct solution).

(7) We need to compute the second derivatives of the coordinates of ϕ. As vectors, we get

$$\frac{\partial^2 \phi}{\partial^2 t} = \left(-\sqrt{h^2+1}\cos t, -\sqrt{h^2+1}\sin t, 0\right)$$
$$\frac{\partial^2 \phi}{\partial t \partial h} = \left(-\frac{h}{\sqrt{h^2+1}}\sin t, \frac{h}{\sqrt{h^2+1}}\cos t, 0\right)$$
$$\frac{\partial^2 \phi}{\partial^2 h} = \left(\frac{1}{(h^2+1)^{3/2}}\cos t, \frac{1}{(h^2+1)^{3/2}}\sin t, 0\right)$$

and so, with the expression of $N \circ \phi$ found in item (6), we get

$$e = \langle N \circ \phi, \frac{\partial^2 \phi}{\partial^2 t}\rangle = -\frac{h^2+1}{\sqrt{2h^2+1}}$$
$$f = \langle N \circ \phi, \frac{\partial^2 \phi}{\partial t \partial h}\rangle = 0$$
$$g = \langle N \circ \phi, \frac{\partial^2 \phi}{\partial^2 h}\rangle = \frac{1}{(h^2+1)\sqrt{2h^2+1}}.$$

□

Exercise 2 Let $X \subset \mathbb{R}^3$ be defined as

$$X := \left\{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 = z^2 + 1\right\}.$$

Give an explicit diffeomorphism F between X and $\mathbb{R}^2 \setminus \{0\}$ (and prove in detail that F is indeed a diffeomorphism).

Solution. Consider the restriction to X of the map $\tilde{F} \colon \mathbb{R}^3 \setminus \{x = y = 0\} \to \mathbb{R}^2$ given by

$$\tilde{F}(x, y, z) = \left(\frac{x}{\sqrt{x^2+y^2}}e^z, \frac{y}{\sqrt{x^2+y^2}}e^z\right).$$

The image of X is included in $\mathbb{R}^2 \setminus \{0\}$. We denote by $F : X \to \mathbb{R}^2 \setminus \{0\}$ the restriction of $\tilde{F}$ to X. We are going to prove that F gives the desired diffeomorphism.

- F is bijective. This can be proved by a direct computation. One can also observe that, for a given h_0, the restriction of F to the set $X \cap \{z = h_0\}$ is a bijection with the circle of center the origin and radius e^{h_0}. Changing h_0 one sees that F covers bijectively all circles of positive radius centered at the origin.
- Let us prove that F is C^∞. If $\phi \colon U \to X$ is a local parametrization obtained as in Exercise 1, we have

$$F \circ \phi(t, u) = (\cos t, \sin t, e^h),$$

which is C^∞ as required (we use here the identity map as a parametrization on the image).
- Let us prove that the inverse of F is C^∞. We can use polar coordinates on (an open subset of) $\mathbb{R}^2 \setminus \{0\}$. This means considering the parametrization ψ of open balls in $\mathbb{R}^2 \setminus \{0\}$ given by $\psi(\theta, r) = (r \cos t, r \sin t)$. Using this parametrization, we need to check that the map

$$\phi^{-1} \circ F^{-1} \circ \psi(\theta, r)$$

is C^∞. But we have

$$\phi^{-1} \circ F^{-1} \circ \psi(\theta, r) = \phi^{-1} \circ F^{-1}(r \cos\theta, r \sin\theta). \tag{1}$$

In order to compute $F^{-1}(r\cos\theta, r\sin\theta)$ by the definition of F we need to solve the system

$$\begin{cases} \frac{x}{\sqrt{x^2+y^2}}e^z = r\cos\theta \\ \frac{y}{\sqrt{x^2+y^2}}e^z = r\sin\theta \\ x^2+y^2 = z^2-1, \end{cases}$$

where the last identities correspond to requiring that the solution must give a point of X.

Taking the sum of the square of the first two identities, we get $e^{2z} = r^2$, which gives $z = \log r$. Substituting this and the last identity in the first two, we get

$$\begin{cases} \frac{x}{\sqrt{(\log r)^2+1}} = \cos\theta \\ \frac{y}{\sqrt{(\log r)^2+1}} = \sin\theta. \end{cases}$$

Finally, we get $(x, y, z) = \left(\sqrt{(\log r)^2+1}\cos t, \sqrt{(\log r)^2+1}\sin t, \log r\right)$. Thus, we can continue the computation in (1), getting

$$\phi^{-1} \circ F^{-1} \circ \psi(\theta, r) = \phi^{-1}\left(\sqrt{(\log r)^2+1}\cos\theta, \sqrt{(\log r)^2+1}\sin\theta, \log r\right).$$

From the definition of ϕ is now follows that

$$\phi^{-1} \circ F^{-1} \circ \psi(\theta, r) = (\theta, \log r)$$

which is C^∞. This concludes the proof that F^{-1} is C^∞.

□

Exercise 3 Let $S \subset \mathbb{R}^3$ be a surface with a Gauss map $N\colon S \to S^2$. Prove that $dN \equiv 0$ if and only if S is contained in a plane.

Proof. The condition $dN \equiv 0$ means that the Gauss map N is constant. Since the value $N(p)$ at a point $p \in S$ gives the orthogonal direction to the tangent plane, we see that $dN \equiv 0$ if and only if the tangent plane is constant.

If S is contained in a plane π, all curves in S are contained in π, and thus all tangent vectors are contained in the plane π_0 parallel to π passing through the origin. As a consequence, the tangent plane T_p is included in π_0 for all $p \in S$, and thus the tangent plane is constant.

Conversely, assume that the tangent plane is constantly equal to a plane π_0. We assume for simplicity that the origin of $\mathbb{R}^3$ is contained in the surface S (the general case follows by just a translation of the problem). Moreover, we can assume that $\pi_0 = \{z = 0\}$, since again the general case just follows by applying a rotation.

We want to show that $S \subseteq \pi_0$. It is enough to show the following: for every p, q in the same parametrizing chart, any curve $\gamma\colon [0,1] \to R^3$, with support contained in S, and such that $\sigma(0) = p$ and $\sigma(1) = q$, lies π_0 (the general case follows by joining any two points in the surface by a finite number of curves as above). We can again assume for simplicity that p is the origin. For every $t \in [0,1]$ we have

$$\sigma(t) = \int_0^t \sigma'(s)ds.$$

Now, since $T_{\sigma(s)} = \pi_0$, the vector $\sigma'(s)$ has no component in the vertical direction, and it follows from the identity above that $\sigma(t) \in \pi_0$ for all $t \in [0,1]$, as required. □