Differential Equations

Worksheet 1: First Order Equations

November 17, 2019

The aims of this sheet are to (a) review elementary methods for analytically solving first order differential equations, (b) develop an understanding of the behaviour of the resulting solutions by drawing their graphs, (c) illustrate how qualitative understanding can be obtained without explicitly solving equations, (d) relate solution behaviour to that of physical quantities that are modelled by the equations, and (e) illustrate how qualitative understanding can be translated into rigorous results.

When asked to draw graphs of solutions of equations you should do this without the help of a computer, though you may then wish to compare your drawings with those produced by an appropriate package.

Exercise I: Exponential Growth The equation $\frac{dx}{dt} = rx$ can be interpreted as a simple model for the growth of a population with *net per capita growth rate* r.

1. Assume r is constant. If $r > 0$ how long does it take a solution of this equation to double in size? If $r < 0$, how long does it take a solution to (a) halve in size, and (b) decrease to e^{-1} times its initial value?

2. World Health Organization data published in 2018 give the life expectancy in Pakistan as 65.7 for males and 67.4 for females. On average each woman gives birth to 3.3 children, approximately half of whom are male and half female. Estimate the net per capita growth rate of the population of Pakistan? How long will it take for the population to double in size, assuming these birth rates and life expectancies remain constant?

Exercise II: Logistic Growth Most populations do not increase without bound - eventually they are restricted by limits on the resources that are necessary to support them. A simple model for this is obtained by assuming that the net per capita growth rate r decreases linearly with population size, $r = r(x) = r_0(1 - x/k)$, where r_0 and k are positive constants, resulting in the *logistic* differential equation:

$$\frac{dx}{dt} = r_0(1 - x/k)x.$$

Show that the solution of this equation with initial value $x = x_0$ is

$$x(t) = \frac{k}{1 - (1 - k/x_0)e^{-r_0 t}}.$$

Hence show that if $x_0 > 0$ then $x(t) \to k$ as $t \to \infty$. Draw graphs of some typical solutions with $0 < x_0 < k$ and $x_0 > k$. The parameter k is often referred to as the *carrying capacity* of the environment for the population.

Exercise III: Seasonal Growth The following equation can be regarded as modelling the growth of a population for which the net per capita growth rate varies periodically with time:

$$\frac{dx}{dt} = (r_0 + \gamma \cos t)x \qquad\qquad r_0 \geq 0.$$

1. Without solving the equation, sketch what you think graphs of typical solutions of this equation might look like for a range of values of γ.

2. Now solve the equation and describe the behaviour of its solutions as $t \to \infty$. How does this depend on r_0 and γ?

3. Draw graphs of some solutions (by hand *and* with a computer package) and compare the results with your initial sketches.

Exercise IV: Linear Equations

1. Show that the general solution of the equation

$$\frac{dx}{dt} + px = q, \tag{0.1}$$

where p and q are constants, is

$$x(t) = q/p + e^{-pt}(x(0) - q/p).$$

2. Draw graphs of some typical solutions of (0.1).

3. If p is positive how long does it take for the difference $|x(t) - q/p|$ to decay to e^{-1} times its initial value. This is called the *recovery time* of the system, and its reciprocal is the system's *resilience*. What happens if p is negative?

Exercise V: Recovery Time Example Consider a lake of constant volume V containing at time t a pollutant which is evenly distributed throughout the lake with a concentration $C(t)$. Assume that water containing a concentration C_I of pollutant enters the lake at a rate q, and that water leaves the lake at the same rate. Suppose that pollutants are also added directly to the lake at a constant rate P.

1. If at time $t = 0$ the concentration of pollutant is C_0, find an expression for the concentration $C(t)$ at time t. What is the limiting concentration as $t \to \infty$?

2. If the addition of pollutants to the lake is terminated ($C_I = 0$ and $P = 0$ for $t > 0$), determine the time period T that will elapse before the concentration of pollutant is reduced to (a) 50% and (b) 10% of its original value.

3. The table below contains data for several of the Great Lakes of North America. Using these data determine from part 2 the time T necessary to reduce the contamination of each of these lakes to 10% of the original value.

LAKE	V (km$^3 \times 10^3$)	q (km^3/year)
Superior	12.2	65.2
Michigan	4.9	158
Erie	0.46	175
Ontario	1.6	209

Equilibrium Points

An equilibrium point of the equation

$$\frac{dx}{dt} = f(x), \tag{0.2}$$

where $f(x)$ is an arbitrary differentiable function of x, is a solution of the form $x(t) = x_*$, where x_* is a constant. Near an equilibrium point x_* the differential equation (0.2) can be approximated by the linear equation

$$\frac{dy}{dt} = f'(x_*)y$$

where $y = x - x_*$ and $f'(x_*)$ is the derivative of f at x_*. Solutions of this linearised equation either grow or decay exponentially, depending on whether $f'(x_*)$ is positive or negative. If they decay the equilibrium is said to be *linearly stable* and if they grow it is *linearly unstable*. For linearly stable equilibria the quantity $T_R = |f'(x_*)|^{-1}$ is called the *recovery time* of the equilibrium, and its reciprocal T_R^{-1} is the *resilience* of the equilibrium.

Exercise VI: Recovery Times of Equilibria

1. Prove that if x_* is a constant then $x(t) = x_*$ is a solution of (0.2) if and only if $f(x_*) = 0$.

2. Show that the definitions of recovery time and resilience above agree with those used in Exercise IV.

3. Identify the linearly stable and unstable equilibria of the logistic differential equation in Exercise II. the recovery time of the linearly stable equilibrium depend on r_0 and k.

Exercise VII: Population Growth with Depensation A population growth rate is said to show *depensation* if the net per capita growth rate $r(x)$ decreases as the population decreases. The depensation is said to be *critical* if this growth rate is negative at very low populations. A model for the growth rate of a population $x(t)$ which shows depensation at low population densities and competition for resources at high population densities is given by:

$$r(x) \;=\; r_0(x - k_c)(1 - x/k) \tag{0.3}$$

and so

$$\frac{dx}{dt} \;=\; f(x) \;=\; r(x)x \;=\; r_0(x - k_c)(1 - x/k)x. \tag{0.4}$$

1. Sketch the graphs of $r(x)$ and $f(x)$ for the cases (a) $k_c < -k < 0$, (b) $-k < k_c < 0$ and (c) $0 < k_c < k$. In which of these cases does the growth rate show depensation? In which cases is the depensation critical?

2. For each of the cases (a), (b) and (c) find the non-negative equilibrium points x_* and determine which of them are linearly stable and which are linearly unstable.

3. Without explicitly solving the equation, sketch graphs of some typical solutions of equation (0.4) for each of the three cases (a), (b) and (c).

Exercise VIII: Constant Effort Harvesting If a population is subject to harvesting or predation (by humans or any other predator) its growth can be modelled by an equation of the form:

$$\frac{dx}{dt} = f(x) - h(x) \tag{0.5}$$

where $f(x)$ models the natural growth rate of the population and $h(x)$ the rate of reduction of the population due to harvesting. In general h is assumed to depend on the size of the population.

As in Exercise II assume that the natural growth rate $f(x)$ is logistic. *Constant effort harvesting* is modelled by taking $h(x) = ex$ where e measures the effort expended on harvesting. Then equation (0.5) becomes

$$\frac{dx}{dt} = r_0(1 - x/k)x - ex \tag{0.6}$$

where r_0, k and e are all assumed to be positive constants. The following should first be answered without using a computer, but you can then compare your drawings with those produced by an appropriate package.

1. Draw graphs of the right hand side of (0.6) as a function of x in the cases $e < r_0$ and $e > r_0$.

2. Draw graphs of the positive equilibrium points of (0.6) as functions of e. Indicate on your graphs which equilibrium points are linearly stable and which are linearly unstable. What happens to the population if e is increased above r_0?

3. Draw a graph of the recovery time of the linearly stable positive equilibrium of (0.6) as a function of e.

4. Draw a graph of the harvests $h(x_*)$ obtained from the linearly stable positive equilibria of (0.6) as a function of e. What value of e optimises the harvest? Show that at this value of e the linearly stable equilibrium population size is $k/2$, ie half the carrying capacity. This is known as the *maximum sustainable yield* of the population.

Rigorous Results

The following exercises are examples of how real analysis can be used to prove rigorous results about solutions of equation (0.2). Assume throughout that f is differentiable and has a continuous derivative. You may also assume the fact that, in this case, solutions of (0.2) with a given initial condition are unique. In other words, if $x_1(t)$ and $x_2(t)$ are two solutions with $x_1(0) = x_2(0)$ then $x_1(t) = x_2(t)$ for all values of t. This in turn implies that if $x_1(t)$ and $x_2(t)$ are two solutions such that $x_1(t_1) = x_2(t_2)$ then $x_2(t) = x_1(t + t_1 - t_2)$ for all t, ie the two solutions are 'phase shifts' of each other.

Exercise IX: Existence of Intermediate Equilibria Let x_1 and x_2 be equilibrium points of (0.2) with $x_1 < x_2$, $f'(x_1) < 0$ and $f'(x_2) < 0$. By using the definition of the derivative of $f(x)$ and the Intermediate Value Theorem, or otherwise, prove that there exists another equilibrium point x_3 of (0.2) with $x_1 < x_3 < x_2$. What can you say about $f'(x_3)$?

Exercise X: Two Lemmas Prove the following two results about solutions of (0.2). The definition of a derivative can be used in the proof of the first lemma, and the first lemma in the proof of the second lemma.

Lemma 0.1 *If x_* satisfies $f(x_*) > 0$ then there exists $\delta > 0$ such that if a solution satisfies $x(t) \in (x_* - \delta, x_*)$ then there exists $T > 0$ such that $x(t + T) > x_*$.*

Lemma 0.2 *Let $x(t)$ be a strictly increasing solution of (0.2) that is bounded above, ie:*

1. *if $t > s$ then $x(t) > x(s)$;*
2. *there exists M such that $x(t) \leq M$ for all t.*

Then $x(t) \to x_*$, where x_* is an equilibrium point of (0.2).

Exercise XI: Global Asymptotic Behaviour of Solutions Let x_1 and x_2 be equilibrium points with $x_1 < x_2$, $f'(x_1) > 0$, $f'(x_2) < 0$ and $f(x) \neq 0$ for all $x \in (x_1, x_2)$.

1. Sketch a graph of f in a neighbourhood of the interval $[x_1, x_2]$.
2. Use the result of Exercise X and the uniqueness of solutions of (0.2) to prove that if $x(0) \in (x_1, x_2)$ then the solution $x(t)$ satisfies $x(t) \to x_2$ as $t \to \infty$ and $x(t) \to x_1$ as $t \to -\infty$.

Definition 0.3

1. *An equilibrium point x_* of the equation (0.2) is said to be asymptotically stable if there exists an interval $I = (x_* - \delta, x_* + \delta)$, for some $\delta > 0$, such any solution $x(t)$ with $x(0) \in I$ satisfies $x(t) \to x_*$ as $t \to \infty$.*
2. *An equilibrium point x_* of equation (0.2) is said to be unstable if there exists $\Delta > 0$ such that for every δ satisfying $0 < \delta < \Delta$ there exists a solution $x(t)$ and a real number $T > 0$ such that $|x(0) - x_*| < \delta$, but $|x(T) - x_*| > \Delta$.*

Exercise XII: Linear Criteria for Stability

1. Let x_* be an equilibrium point of equation (0.2). Prove that x_* is asymptotically stable if it is linearly stable and unstable if it is linearly unstable. [Hint: Exercise X and the uniqueness of solutions of (0.2) can be used to prove this.]
2. Is the condition $f'(x_*) < 0$ necessary for x_* to be asymptotically stable? Is the condition $f'(x_*) > 0$ necessary for x_* to be unstable?

Differential Equations

Worksheet 2: Linear Systems of Differential Equations

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This worksheet reviews *linear systems* of differential equations:

$$\dot{\mathbf{x}} \;=\; \mathbf{A}\mathbf{x} \;+\; \mathbf{b}(t) \tag{0.1}$$

where $\mathbf{x} = (x_1, \ldots, x_n)^T$, $\mathbf{b}(t) = (b_1(t), \ldots, b_n(t))^T$, $\mathbf{A}$ is a constant $n \times n$ matrix, and a 'dot' over a variable indicates its derivative with respect to time. When $\mathbf{b}(t) \equiv 0$ the equation is said to be *homogeneous*:

$$\dot{\mathbf{x}} \;=\; \mathbf{A}\mathbf{x}. \tag{0.2}$$

Otherwise it is *inhomogeneous*. As in the rest of the course particular emphasis will be placed on asymptotic properties of the solutions as $t \to \infty$. Applications to pharmacokinetics and damped oscillators are used as illustrations.

Homogeneous Systems

Exercise I: Method of Solution of Homogeneous Linear Systems with Distinct Eigenvalues

1. Show that $\mathbf{x}(t) = e^{\mu t}\mathbf{x}_\mu$ is a solution of (0.2) if and only if $\mathbf{A}\mathbf{x}_\mu = \mu\mathbf{x}_\mu$, ie $\mathbf{x}_\mu$ is an eigenvector of $\mathbf{A}$ with eigenvalue μ.

2. Prove that if $\mu_1, \ldots, \mu_\ell$ are distinct eigenvalues then $\{\mathbf{x}_{\mu_1}, \ldots, \mathbf{x}_{\mu_\ell}\}$ is a linearly independent set of eigenvectors.

3. Deduce that, if the $n \times n$ matrix $\mathbf{A}$ has n distinct eigenvalues $\mu_1, \ldots, \mu_n$, then for every initial condition $\mathbf{x}(0)$ the solution of (0.2) can be written as

$$\mathbf{x}(t) \;=\; \sum_{i=1}^{n} \alpha_i e^{\mu_i t}\mathbf{x}_{\mu_i} \tag{0.3}$$

   where the 'arbitrary constants' α_i satisfy $\mathbf{x}(0) = \sum_{i=1}^{n} \alpha_i \mathbf{x}_{\mu_i}$. Thus (0.3) is the *general solution* of (0.2).

Exercise II: Examples of Homogeneous Linear Systems with Distinct Eigenvalues For each of the following matrices $\mathbf{A}$ and vectors $\mathbf{x}_0$, find (a) the general solution of (0.2), and (b) the solution with $\mathbf{x}(0) = \mathbf{x}_0$:

(i) $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad \mathbf{x}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\qquad$ (ii) $\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}$ $\quad \mathbf{x}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(iii) $\mathbf{A} = \begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix}$ $\quad \mathbf{x}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\qquad$ (iv) $\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}$ $\quad \mathbf{x}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

In each case use appropriate computer software to draw graphs of $x_1, \ldots, x_n$ against t for the solution with the given initial condition. The plots should all be on the same set of axes, so that you have a single diagram for each example.

Exercise III: A model from pharmacokinetics *Pharmacokinetics* describes how the concentration of a substance in the body, such as a drug, varies with time. Model the body as two *compartments*: plasma (ie blood) and tissue (organs etc). Let $p(t)$ and $q(t)$ denote the concentrations of the drug in the plasma and tissue, respectively, at time t. The drug is assumed to diffuse from the compartment with the higher concentration to that with the lower at a rate that is proportional to the difference between the concentrations. It is also assumed to be eliminated from the plasma (and hence the body) at a rate proportional to its concentration in the plasma. Finally it is assumed that the drug is injected into the plasma at a rate $I(t)$. These assumptions lead to the pair of differential equations:

$$\frac{dp}{dt} = -\alpha(p-q) - \gamma p + I(t) \tag{0.4}$$

$$\frac{dq}{dt} = \alpha(p-q) \tag{0.5}$$

where α and γ are positive constants.

1. Before solving these equations, use the form of the equations and your physical intuition to sketch graphs of $p(t)$ and $q(t)$ for $t > 0$ in the two cases:

   (a) $I(t) \equiv 0$ for $t \geq 0$; $\quad p(0) = p_0 > 0,\ q(0) = 0$;

   (b) $I(t) = I_0$, a constant; $\quad p(0) = 0,\ q(0) = 0$.

   Note that (a) models an instantaneous injection of the drug into the plasma at $t = 0$ while (b) models continuous injection, eg from a drip.

2. Rewrite the system (0.12, 0.13) in the form of (0.1). Show that the eigenvalues of the matrix $\mathbf{A}$ for this system are both negative. Solve the equations for case (a) of part 1. Use your solution in 3 and 4 below.

3. What happens to the concentrations of the drug in the plasma and tissue as $t \to \infty$? How long does it take for the concentration of the drug in the plasma to decay to e^{-1} times its initial value?

4. Use appropriate computer software to plot graphs of p and q against time (both in the same diagram), and compare the results with your sketches from part 1, case (a). Interpret your results in terms of the change in concentrations of the drug in the plasma and tissue. (You will be asked to do the same for case (b) in Exercise XI below.)

Exercise IV: Complex Eigenvalues

1. Show that if μ is an eigenvalue of $\mathbf{A}$ with eigenvector $\mathbf{x}_\mu$ then its complex conjugate, $\overline{\mu}$, is also an eigenvalue of $\mathbf{A}$, and that its eigenvector is the complex conjugate of the eigenvector with eigenvalue μ, ie $\mathbf{x}_{\overline{\mu}} = \overline{\mathbf{x}}_\mu$.

2. Prove that if μ and $\mathbf{x}_\mu$ are complex then $\mathbf{x}(t) = \alpha e^{\mu t}\mathbf{x}_\mu + \beta e^{\overline{\mu} t}\mathbf{x}_{\overline{\mu}}$ is real if and only if $\beta = \overline{\alpha}$.

3. Let $\mu = \sigma + i\omega$ and $\mathbf{x}_\mu = (r_1 e^{i\phi_1}, \ldots, r_n e^{i\phi_n})^T$. Show that if $\alpha = \frac{1}{2} r e^{i\theta}$ then

   $$\mathbf{x}(t) \;=\; \alpha e^{\mu t}\mathbf{x}_\mu + \overline{\alpha} e^{\overline{\mu} t}\mathbf{x}_{\overline{\mu}} \;=\; (x_1(t), \ldots, x_n(t))^T \qquad \text{where} \qquad x_j(t) \;=\; r r_j e^{\sigma t}\cos(\omega t + \theta + \phi_j).$$

   Here r and θ are two arbitrary constants that replace (the real and imaginary parts of) α. The constants σ and ω are determined by μ, and the r_j and ϕ_j by $\mathbf{x}_\mu$.

4. Draw typical graphs of $re^{\sigma t}\cos(\omega t + \psi)$, indicating clearly the roles played by r, σ, ω and ψ in determining their shapes. What happens as $\omega \to 0$, ie the complex conjugate pair of eigenvalues converge to a real eigenvalue?

Exercise V: Examples of Homogeneous Linear Systems with Complex Eigenvalues For each of the following matrices $\mathbf{A}$ and vectors $\mathbf{x}_0$, find (a) the real general solution of (0.2), and (b) the solution with $\mathbf{x}(0) = \mathbf{x}_0$:

(i) $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \mathbf{x}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (ii) $\mathbf{A} = \begin{pmatrix} -\frac{1}{2} & 1 \\ 1 & -\frac{1}{2} \end{pmatrix} \qquad \mathbf{x}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

(iii) $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix} \qquad \mathbf{x}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (iv) $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix} \qquad \mathbf{x}_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

(v) $\mathbf{A} = \begin{pmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{pmatrix} \qquad \mathbf{x}_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

In each case use appropriate computer software to draw graphs of $x_1, \ldots, x_n$ against t for the solution with the given initial condition. The plots should all be on the same set of axes, so that you have a single diagram for each example.

Exercise VI: The Damped Harmonic Oscillator Consider the damped harmonic oscillator:

$$\frac{d^2x}{dt^2} + \nu\frac{dx}{dt} + kx = 0 \tag{0.6}$$

where k is the *restoring force* and ν the *damping coefficient*, both per unit mass.

1. Show that (0.6) is equivalent to the following system of first order equations

$$\frac{dx}{dt} = y \tag{0.7}$$

$$\frac{dy}{dt} = -\nu y - kx. \tag{0.8}$$

   If x denotes the position of the mass then y is its velocity.

2. Rewrite the system (0.7, 0.8) in the form of (0.2) and then find the general solution of these equations, distinguishing between the cases:

   (a) $\nu > 2k^{\frac{1}{2}}$ (overdamped) (b) $\nu < 2k^{\frac{1}{2}}$ (underdamped) (c) $\nu = 0$ (undamped).

3. Draw graphs of the position $x(t)$ and velocity $y(t)$ (both on the same set of axes), in each of the cases (a) to (c), for initial conditions with $x(0) = 0, y(0) > 0$.

Stability

Exercise VII: Stability of Homogeneous Systems In the following assume that the matrix $\mathbf{A}$ in (0.2) has distinct eigenvalues (though the results hold in general).

1. Prove that every solution of (0.2) converges to 0 as $t \to \infty$ if and only if all the eigenvalues of $\mathbf{A}$ have strictly negative real parts. If this is the case $\mathbf{A}$ is said to be *asymptotically stable*.

2. Let $\mathbf{P}$ be an invertible $n \times n$ matrix. Show that the trace and determinant of $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ are equal to those of $\mathbf{A}$. (Hint: for trace you can use the fact that $\text{tr}(\mathbf{MN}) = \text{tr}(\mathbf{NM})$ for any pair of $n \times n$ matrices.) Deduce that the trace and determinant of $\mathbf{A}$ are, respectively, the sum and product of its eigenvalues. What are the signs of the trace and determinant of an asymptotically stable matrix?

3. *More generally,* prove that if $\mathbf{A}$ is asymptotically stable then all the coefficients of the characteristic polynomial $\chi(\mu) = \det(\mu\mathbb{I} - \mathbf{A})$ are strictly positive. Here $\mathbb{I}$ is the identity matrix.

4. Let T and D be the trace and determinant of a 2×2 matrix $\mathbf{A}$. Show that the eigenvalues of $\mathbf{A}$ are both real if $T^2 > 4D$ and are complex if $T^2 < 4D$. Show that $\mathbf{A}$ is asymptotically stable *if and only if* T is negative and D is positive.

5. Show that a 3×3 matrix A is asymptotically stable if and only if the coefficients of its characteristic polynomial $\chi(\mu) = \mu^3 + a_2\mu^2 + a_1\mu + a_0$ satisfy:

$$a_2 > 0; \quad a_0 > 0; \quad a_1a_2 - a_0 > 0.$$

   [Hint: you may find it useful to note that $a_1a_2 - a_0 = -(\mu_2 + \mu_3)\left(\mu_1^2 + (\mu_2 + \mu_3)\mu_1 + \mu_2\mu_3\right)$.]

6. Find an example of a 3×3 matrix for which all the coefficients of its characteristic polynomial are positive, but which is not asymptotically stable.

Exercise VIII: Stability Examples Without calculating their eigenvalues, determine which of the following matrices are asymptotically stable? Explain how you obtained your answers.

(i) $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 0 & -2 \end{pmatrix}$ (iv) $\mathbf{A} = \begin{pmatrix} 1 & 1 & 5 \\ -4 & 0 & 2 \\ -6 & 2 & 0 \end{pmatrix}$ (vii) $\mathbf{A} = \begin{pmatrix} -1 & 1 & 2 & -3 \\ 0 & -2 & -4 & 0 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

(ii) $\mathbf{A} = \begin{pmatrix} -2 & 1 \\ 1 & -3 \end{pmatrix}$ (v) $\mathbf{A} = \begin{pmatrix} -1 & 1 & 7 \\ 4 & -1 & -8 \\ 0 & 0 & 1 \end{pmatrix}$ (viii) $\mathbf{A} = \begin{pmatrix} -1 & 1 & 2 & -3 \\ -4 & -3 & -4 & 0 \\ 8 & 0 & 0 & -5 \\ -2 & 7 & 0 & -1 \end{pmatrix}$

(iii) $\mathbf{A} = \begin{pmatrix} -2 & 1 \\ 7 & -3 \end{pmatrix}$ (vi) $\mathbf{A} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix}$ (ix) $\mathbf{A} = \begin{pmatrix} -1 & 0 & 0 & -3 \\ 2 & -2 & -4 & 2 \\ 3 & 1 & 0 & -5 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

Recovery Times and Resilience

If μ is real, $\mathbf{x}(t) = e^{\mu t}\mathbf{x}_\mu$ decays to 0 if $\mu < 0$, and its recovery time is given by $|\mu|^{-1}$. Similarly, if μ is complex, $x(t) = re^{\sigma t}\cos(\omega t + \psi)$ decays to 0 if the real part of μ, $\sigma = \mathrm{Re}(\mu)$, is negative, and its recovery time is given by $|\sigma|^{-1}$. We define the *recovery time* of a general matrix $\mathbf{A}$ to be $T_R = (\min_j\{|\mathrm{Re}(\mu_j)|\})^{-1}$ where the minimization is over all eigenvalues of $\mathbf{A}$. We define the *resilience* of $\mathbf{A}$ to be T_R^{-1}, the reciprocal of the recovery time.

Exercise IX: Recovery Time Examples

1. Calculate the recovery time of each of the asymptotically stable matrices in Exercise VIII.
2. Referring back to Exercise VI, draw a graph of the resilience of the damped harmonic oscillator (0.7, 0.8) in considered as a function of $\nu/2k^{\frac{1}{2}}$. Which value of $\nu/2k^{\frac{1}{2}}$ minimizes the recovery time? This is known as *critical damping.*

Inhomogeneous Systems

Exercise X: General Solutions of Inhomogeneous Linear Systems

1. Show that if $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are two solutions of (0.1) then $(t) = \mathbf{x}_1(t) - \mathbf{x}_2(t)$ is a solution of (0.2).
2. Deduce that the general solution of (0.1) is $\mathbf{x}(t) = \mathbf{x}_p(t) + (t)$, where $x_p(t)$ is any *particular* solution of (0.1) and (t) is the general solution of (0.2). The solution (t) of (0.2) has n arbitrary constants which are determined by the initial conditions of the solution of (0.1).
3. Show that if $\mathbf{A}$ is asymptotically stable and $\mathbf{x}_p(t)$ is any particular solution of (0.1) then every solution, $\mathbf{x}(t)$, of (0.1) satisfies $|\mathbf{x}(t) - \mathbf{x}_p(t)| \to 0$ as $t \to \infty$. Thus $\mathbf{x}_p(t)$ is the *steady state response* of the system to the input $\mathbf{b}(t)$, while (t) gives the *transient* behaviour.
4. If $\mathbf{A}$ is asymptotically stable, how long does it take $|\mathbf{x}(t) - \mathbf{x}_p(t)|$ to decay to e^{-1} times its initial value $|\mathbf{x}(0) - \mathbf{x}_p(0)|$?

Exercise XI: The Pharmacokinetic Model Revisited Recall the two compartment pharmacokinetic model from Exercise III:

$$\frac{dp}{dt} = -\alpha(p - q) - \gamma p + I(t) \qquad (0.9)$$

$$\frac{dq}{dt} = \alpha(p - q) \qquad (0.10)$$

where α and γ are positive constants.

1. Assume $I(t) = I_0$ is a constant, as in case (b) of Exercise III. Find a particular solution of (0.12, 0.13) for which $p(t)$ and $q(t)$ are also constant. Hence find the general solution decomposed into the steady state response and transient part.
2. Find the solution of these equations in case (b) of Exercise III, ie with $I(t) = I_0$ and initial conditions $p(0) = 0 = q(0)$.
3. For this solution, what happens to the concentrations of the drug in the plasma and tissue as $t \to \infty$? How long does it take for the drug concentration in the tissue to reach 90% of its steady state value.
4. Use a computer package to plot graphs of p and q against time (both in the same diagram), and compare the results with your sketch for case (b) in part 1 of Exercise III. Interpret your results in terms of the change in concentrations of the drug in the plasma and tissue.

Phase Portraits of Linear Systems

Up until now we have visualised solutions of equations such as (0.1) by plotting graphs of each of $x_1(t), \ldots, x_n(t)$ against time. An alternative is to plot $(x_1(t), \ldots, x_n(t))$ as a curve in the *phase plane* $\mathbb{R}^n$. The advantage of this is that different solutions with different initial condition can be plotted in the same diagram, giving a complete picture of all the behaviour that can occur. These are known as *phase portraits.* For simplicity will focus on the case $n = 2$.

Exercise XII: Phase Portraits of Linear Systems with Real Eigenvalues

1. Show that the solutions of the linear system:

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \tag{0.11}$$

satisfy the equation $u^\mu v^{-\lambda} = u^\mu(0)v^{-\lambda}(0)$. Sketch graphs of these curves, and hence produce phase portraits of (0.11), in the cases:

(i) $\lambda < 0 < \mu$ (ii) $0 < \lambda < \mu$ (iii) $\lambda < \mu < 0$ (iv) $\lambda < \mu = 0$.

Compare your sketches with phase portraits produced using a computer package. Systems with eigenvalues satisfying (i), (ii) and (iii) are referred to as *saddles*, *unstable nodes* and *stable nodes*, respectively.

2. Sketch phase portraits of the two dimensional homogeneous linear systems defined by the following matrices:

(i) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (ii) $\begin{pmatrix} -2 & 1 \\ 0 & -3 \end{pmatrix}$ (iii) $\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$

and then compare your sketches with phase portraits produced using a computer package.

Exercise XIII: Our Pharmacokinetics Model Sketch phase portraits of the pharmacokinetics model from Exercises III and XI:

$$\frac{dp}{dt} = -\alpha(p-q) - \gamma p + I(t) \tag{0.12}$$

$$\frac{dq}{dt} = \alpha(p-q) \tag{0.13}$$

where α and γ are positive constants, in the cases (i) $I(t) \equiv 0$ and (ii) $I(t) = I_0 > 0$. Since p and q are concentrations of drugs your should restrict your final solution to the region $p \geq 0$, $q \geq 0$, though it is useful to consider other regions while you are working out what the phase portrait looks like. Compare your sketches with phase portraits produced using a computer package.

Exercise XIV: Phase Portraits of Linear Systems with Complex Eigenvalues

1. Show that the eigenvalues of the linear system:

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \sigma & -\omega \\ \omega & \sigma \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \tag{0.14}$$

are $\sigma \pm i\omega$.

2. Show that in polar coordinates $u = r\cos\theta$, $v = r\sin\theta$ these equations become:

$$\dot{r} = \sigma r \qquad \dot{\theta} = \omega. \tag{0.15}$$

3. Solve the equations (0.15) and hence sketch phase portraits of (0.14) in the cases:

(i) $\sigma < 0,\ \omega > 0$ (ii) $\sigma = 0,\ \omega > 0$ (iii) $\sigma > 0,\ \omega > 0$.

How do the solutions change for $\omega < 0$? Compare your sketches with phase portraits produced using a computer package.

4. Sketch phase portraits of the two dimensional homogeneous linear systems defined by the following matrices:

(i) $\begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$ (ii) $\begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$ (iii) $\begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}$

and then compare your sketches with phase portraits produced using a computer package.

Exercise XV: The Damped Harmonic Oscillator Sketch phase portraits of the two dimensional linear system corresponding to the damped harmonic oscillator studied in Exercise VI:

$$\frac{d^2x}{dt^2} + \nu\frac{dx}{dt} + kx = 0 \qquad \nu \geq 0, \quad k > 0$$

in the cases:

(i) $\nu^2 > 4k$ (overdamped) (ii) $\nu^2 < 4k$ (underdamped) (iii) $\nu = 0$ (undamped)

and then compare your sketches with phase portraits produced using a computer package.

Differential Equations

Worksheet 3: Nonlinear Systems, Linearization and Phase Portraits

November 21, 2019

This worksheet looks at two dimensional autonomous nonlinear systems of the form:

$$\dot{\mathbf{x}} = f(x,y) \tag{0.1}$$
$$\dot{\mathbf{y}} = g(x,y) \tag{0.2}$$

where $f(x,y)$ and $g(x,y)$ are differentiable functions of the real variables x and y. The main aims of the worksheet are to show how:

1. *Linearization* can be used to determine the stabilities of equilibria of systems of nonlinear equations;
2. *Phase portrait analysis* can determine the asymptotic behaviour of all solutions of the equations.

Stability of Equilibria

Stability of equilibrium points of systems of nonlinear equations can be defined in a similar way as for one dimensional systems (see Worksheet 1). Roughly speaking, an equilibrium point is asymptotically stable if all nearby solutions converge to it as $t \to \infty$, and it is unstable if there exist arbitrarily close initial conditions for which the trajectories move away from the equilibrium point and remain away from it as $t \to \infty$.

To determine the stability of an equilibrium point (x_*, y_*) of (0.1, 0.2) we approximate the dynamics of the nonlinear system near (x_*, y_*) by the homogeneous linear system for $u = x - x_*$, $v = y - y_*$ given by:

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x}(x_*, y_*) & \frac{\partial f}{\partial y}(x_*, y_*) \\ \frac{\partial g}{\partial x}(x_*, y_*) & \frac{\partial g}{\partial y}(x_*, y_*) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \tag{0.3}$$

where the entries in the matrix are the partial derivatives of f and g with respect to x and y evaluated at the equilibrium point. We call this the *linearization* of the nonlinear system at the equilibrium point. The key stability result is the following:

The equilibrium point (x_*, y_*) of the nonlinear system (0.1, 0.2) is asymptotically stable if all the eigenvalues of the linearization have strictly negative real parts, and it is unstable if there is at least one eigenvalue with strictly positive real part.

In other words, an equilibrium point is asymptotically stable if the linearization of the nonlinear system at the equilibrium point is asymptotically stable in the sense of Exercise VII of Worksheet 2. This is the analogue of the result for one dimensional systems explored in Worksheet 1. Note these are only *sufficient* conditions for asymptotic stability and instability. Nothing can be concluded from the result if all the eigenvalues have non-positive real parts, but some of the real parts are equal to 0.

Phase Portraits

Phase portraits of two dimensional nonlinear systems are typically constructed by combining two basic ideas:

1. The phase portrait of a nonlinear system near an equilibrium point 'looks like' that of the linearization of the system at the equilibrium.
2. The equation $f(x,y) = 0$ defines a curve in the phase plane along which $\dot{x} = 0$, and so the tangents to the solution curves at points on this curve must be parallel to the y-axis. Similarly, the tangents to solution curves at points on the curve $g(x,y) = 0$ must be parallel to the x-axis. These two curves are called *nullclines*. The equilibrium points occur at the intersection of the nullclines.

By analysing the equilibrium points of a nonlinear system and drawing their nullclines it is often possible to sketch a reasonable picture of the complete phase portrait.

(Most of) the following exercises concern two dimensional nonlinear systems that are examples of models from chemistry, epidemiology, ecology and natural resource management. In each case you are asked to determine the *types* of the equilibrium points, ie whether their linearisations are saddle points or stable or unstable nodes or foci, and how this depends on the parameters of the equation. You are then asked to sketch a phases portrait for each of the parameter regions and, finally, to discuss and interpret your results. For each of the models you may find it useful to first make a change of variables that suitably *non-dimensionalises* its parameters.

Exercise I: The SIR Epidemic Model The epidemic model introduced in Question 4 of Coursework I can be described as an SIS model because individuals go from being susceptible to infective and then back to susceptible. If instead of becoming susceptible again after the infection has passed they become immune, we obtain what is called an SIR model. Let S denote the number of susceptibles in the population, I the number of infectives, and R the number who have recovered and are immune. Assume also that members of the population are being born and die at the same per capita rate $r > 0$ so that the total population size $S + I + R = N$ is constant.

1. Show that this system can be modelled by the pair of equation:

$$\frac{dS}{dt} = rN - \alpha SI - rS \tag{0.4}$$

$$\frac{dI}{dt} = \alpha SI - (\beta + r)I \tag{0.5}$$

   (along with $R = N - S - I$) where α and β are non-negative parameters that you should interpret epidemiologically.

2. Sketch the nullclines for the system, noting carefully how they are qualitatively different in different parameter regions (which you should identify).
3. Determine the type of each of the equilibrium points of the scaled system in each of the parameter regions.
4. Sketch a phase portrait of the scaled system for each of the parameter regions.
5. Compare your sketches with phase portraits produced using a computer package.
6. Discuss your results. What do they say about the behaviour of the system and how it changes as the (original) parameters are varied?

Exercise II: A Coupled Autocatalytic Chemical Reaction System In an *autocatalytic* chemical reaction in a reaction vessel the rate of production of a substance X is proportional to the square of its concentration x. Assume that the substance is also being introduced into the vessel at a constant rate a and being removed at a rate proportional to its concentration. Then the rate of change of the substance in the vessel is

$$\frac{dx}{dt} = a + cx^2 - bx$$

where a, b and c are positive constants.

1. Using appropriate diagrams, describe the behaviour of the positive solutions of this equation, noting how they depend on a *non-dimensional* combination of the parameters a, b and c, and on the initial concentration $x(0)$.

Now suppose that two such reaction vessels are coupled in series, with the output from the first forming the input to the second. If the concentration of the substance in the first vessel is x, and that in the second is y then we obtain a coupled pair of equations:

$$\frac{dx}{dt} = a + c_1x^2 - b_1x \tag{0.6}$$

$$\frac{dy}{dt} = b_1x + c_2y^2 - b_2y. \tag{0.7}$$

The equilibrium points of this system are the solutions of the simultaneous pair of equations obtained by setting $\frac{dx}{dt} = 0 = \frac{dy}{dt}$.

2. Scale the variables to obtain a form of these equations with non-dimensional parameters.
3. Show that this system of equations can have up to 4 equilibrium points for which both x and y are non-negative. Determine the regions in (non-dimensional) parameter space in which the system has 0, 1, 2, 3 and 4 such positive equilibria.
4. Sketch the nullclines for the scaled system in each of the different parameter regions you obtained in part 3.
5. Determine the type of each of the equilibrium points of the scaled system in each of the parameter regions.
6. Sketch a phase portrait of the scaled system for each of the parameter regions.
7. Compare your sketches with phase portraits produced using a computer package.
8. Discuss your results. What do they say about the behaviour of the system and how it changes as the (original) parameters are varied?

Exercise III: Bioeconomic Models Consider again the model for logistic population growth with constant effort harvesting discussed in Exercise VIII of Worksheet 1:

$$\frac{dx}{dt} = r_0 \left(1 - \frac{x}{k}\right) x - ex \tag{0.8}$$

where x, e, r, k are all non-negative. Suppose the cost of harvesting per unit effort is c and the harvest is sold at a price p per unit catch, so the profit is $pex - ce$.

1. What is the maximum sustainable profit (or *maximum economic yield*) obtained by varying e?
2. What is the corresponding equilibrium size of the population, and how does it compare with that for the maximum sustainable yield?

Assume now that instead of e being controlled to maximize the profit there is a 'free market' and the harvesters increase their effort if the profit is positive and decrease it if it is negative. This suggests the equation:

$$\frac{de}{dt} = s(pex - ce) \tag{0.9}$$

where we take s to be a non-negative constant.

3. Scale the variables to obtain a form of the system (0.8, 0.9) with non-dimensional parameters.
4. Sketch the nullclines for the scaled system, noting carefully how they are qualitatively different in different parameter regions (which you should identify).
5. Determine the type of each of the equilibrium points of the scaled system in each of the parameter regions.
6. Sketch a phase portrait of the scaled system for each of the parameter regions.
7. Compare your sketches with phase portraits produced using a computer package.
8. Discuss your results. What do they say about the behaviour of the system and how it changes as the (original) parameters are varied?

Exercise IV: A Predator Prey System The harvesting model (0.8) can also be interpreted as a simple model for predation by taking e to be proportional to the size of the predator population. However the amount of prey taken by a predator (the *functional response* of the predator to the prey) typically does not increase indefinitely as the size of the prey population increases. It is more likely to 'level off' in some way. A common models for this is given by

$$h(x) = \frac{Ax}{x + B} \tag{0.10}$$

where A and B are positive constant parameters. Combining this with logistic growth of the prey species gives:

$$\frac{dx}{dt} = f(x) = r\left(1 - \frac{x}{k}\right) x - \frac{Ax}{x + B}. \tag{0.11}$$

1. Sketch a graph of the predation function $h(x)$, indicating the roles played by A and B.
2. Sketch graphs of (a non-dimensionalised form of) the function $f(x)$, showing how these differ in different parameter regions.
3. Describe how the positive equilibria of (0.11) vary with (non-dimensionalised) parameters. Illustrate your description by plotting graphs of the equilibrium values as functions of appropriate parameters.

In the above discussion the size of the predator population is implicitly assumed to be a constant proportional to the parameter A. If instead we assume that the predator population reproduces at a rate which is proportional to the rate at which it is consuming prey, and dies at a constant per capita rate we obtain the following system:

$$\frac{dx}{dt} = r\left(1 - \frac{x}{k}\right) x - \frac{axy}{x + B} \tag{0.12}$$

$$\frac{dy}{dt} = \frac{bxy}{x + B} - dy \tag{0.13}$$

where r, k, a, b and B are all positive constants.

4. Scale the variables to obtain a form of the system (0.12, 0.13) with non-dimensional parameters.

5. Sketch the nullclines for the scaled system, noting carefully how they are qualitatively different in different parameter regions (which you should identify).

6. Determine the type of each of the equilibrium points of the scaled system in each of the parameter regions.

7. Sketch a phase portrait of the scaled system for each of the parameter regions.

8. Compare your sketches with phase portraits produced using a computer package. What dynamical phenomenon do you see that we haven't met before in this course?

9. Discuss your results. What do they say about the behaviour of the system and how it changes as the (original) parameters are varied?

Exercise V: Hopf Bifurcation Consider the following system of differential equations:

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + a(u^2+v^2) \begin{pmatrix} u \\ v \end{pmatrix} \tag{0.14}$$

where σ, ω and a are parameters that can be positive or negative. Note that the only equilibrium point of this system is at $u = 0 = v$ and that the linearization of (0.14) at this equilibrium is the system discussed in Exercise XIV of Worksheet 2.

1. For what values of the parameters σ, ω and a is the equilibrium $u = 0 = v$ (a) asymptotically stable and (b) unstable?

2. Assume $a < 0$. Sketch typical phase portraits of this system when the equilibrium $u = 0 = v$ is (a) asymptotically stable and (b) unstable. [Hint: use the same change to polar coordinates that you used in Exercise XIV of Worksheet 2.]

3. In what way is the behaviour you observe in part 2 similar to that in Exercise IV?

4. Repeat part 2 for $a > 0$ and describe the differences between the two cases.

Exercise VI: A Relaxation Oscillator Consider the following system of differential equations:

$$\frac{dx}{dt} = x - \frac{1}{3}x^3 - y \tag{0.15}$$

$$\epsilon\frac{dy}{dt} = \epsilon(y + a) \tag{0.16}$$

where a and ϵ are positive constants. This is a generalisation of the *van der Pol* oscillator and a special case of the *Fitzhugh Nagumo* nerve impulse model.

1. Find the equilibrium points of this system and determine how their stability depends on the parameters a and ϵ.

2. What kind of behaviour would you expect to see if the parameters are varied so that the stability of the equilibrium point(s) changes? [Hint: compare with Exercises IV and V.]

3. Sketch the nullclines of the system (0.15, 0.16) for the cases $a > 1$ and $0 < a < 1$.

4. Sketch phase portraits of the system for $a > 1$ and $0 < a < 1$ assuming that ϵ is very small. [Hint: ϵ small implies that dy/dt is small and y is almost constant.]

5. Use you phase portrait to describe the behaviour of solutions as $t \to \infty$ in the cases $a > 1$ and $0 < a < 1$.

6. Use a computer package to produce a series of phase portraits for this system for a range of values of a between 1 and 0. Describe what happens in these phase portraits as ϵ is decreased to very small values.