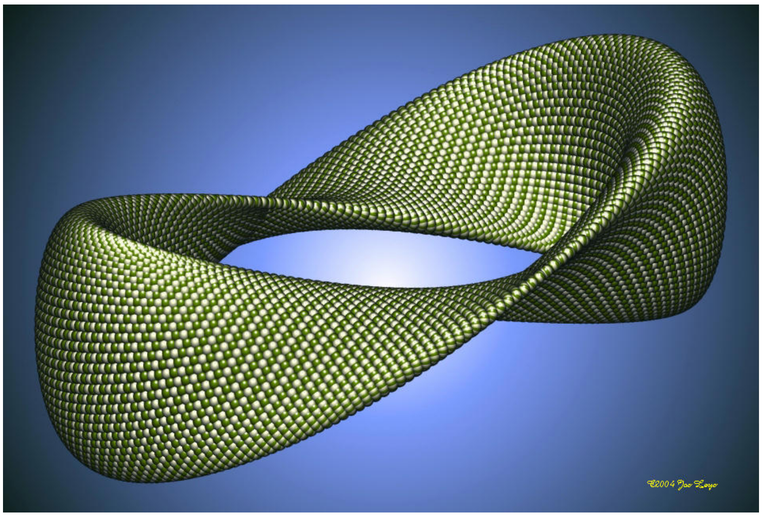
TOPOLOGY

An Introduction



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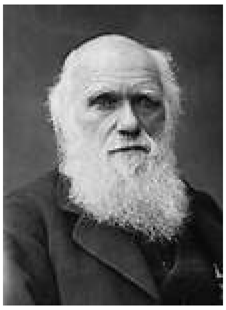
http://math.unice.fr

"The concept of dog does not bark."

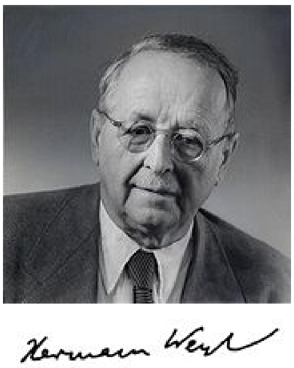
Spinoza



Charles Darwin 1809 – 1882

A mathematician is a blind man in a dark room looking for a black cat which isn't there

Charles Darwin



Hermann Weyl 1885 – 1955

In these days the angel of topology and the devil of abstract algebra fight for the soul of every individual discipline of mathematics.

Hermann Weyl

Solomon Lefschetz 1884 – 1972

If it's just turning the crank it's algebra, but if it's got an idea in it, it's topology.

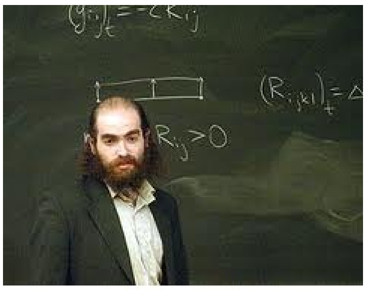
Solomon Lefschetz

Leonhard Euler, 1707-1783

Leonhard Euler was a pioneering Swiss mathematician and physicist. He made important discoveries in fields as diverse as infinitesimal calculus, analysis, graph theory, number theory, applied mathematics, astronomy, physics, and logic. In 1736, Euler solved the problem known as the Seven Bridges of Königsberg. Euler also discovered the formula $V - E + F = 2$ relating the number of vertices, edges, and faces of a convex polyhedron, and hence of a planar graph. The constant in this formula is now known as the Euler characteristic for the graph (or other mathematical object), and is related to the genus of the object. The study and generalization of this formula, specifically by Cauchy and L'Huillier, is at the origin of topology.

Jules Henri Poincaré was a French mathematician, theoretical physicist, engineer, and a philosopher of science. He is often described as a polymath, and in mathematics as The Last Universalist, since he excelled in all fields of the discipline as it existed during his lifetime. As a mathematician and physicist, he made many original fundamental contributions to pure and applied mathematics, mathematical physics, and celestial mechanics. He was responsible for formulating the Poincaré conjecture, which was one of the most famous unsolved problems in mathematics until it was solved in 2002-2003. In his research on the three-body problem, Poincaré became the first person to discover a chaotic deterministic system which laid the foundations of modern chaos theory. He is also considered to be one of the founders of the field of topology.

Henri Poincaré, 1854-1912



Gregory Perelman - 13 June 1966

Grigori Yakovlevich Perelman is a Russian mathematician who has made landmark contributions to Riemannian geometry and geometric topology. In 2003, he proved Thurston's geometrization conjecture. This consequently solved in the affirmative the Poincaré conjecture, posed in 1904, which before its solution was viewed as one of the most important and difficult open problems in topology. In August 2006, Perelman was awarded the Fields Medal. In March 2010, the Clay Mathematics Institute hereby awards the Millennium Prize for resolution of the Poincaré conjecture to Grigoriy Perelman.

Contents

Introduction

Although it exists many textbooks and lecture notes on Topology, we decided to write one more. The origin was back to 2005 when a master of mathematics was established in Cambodia. Among other courses, the need for a basic course in Topology seemed to be necessary. These lecture notes are a very extended version of the courses given in this Cambodian master of mathematics.

The content is classical, it covers, in the first part, the main topics of the general topology or so called point-set topology. In a second part, we introduce the notion of continuous deformations and covering spaces which is an introduction to homotopy theory through the fundamental group also called Poincaré group.
Topology needs an ability with the notions of the set theory and we recall the basic notions with some focus on the duality. Duality plays an important role in these lecture notes. This concept is not clearly defined but it is introduced through many examples. The reader has not to confuse duality and negation. Moreover, duality is a way to get a better global view.
Even if the set theory plays an important role in topology, it is just a tool. The basic idea is to express the notion of nearness. In the metric spaces, nearness is measured by the distance, but there are many situations where there is no such a measure. The nearness is characterized by the open sets and the set theory is the tool.

General Topology was a very active domain of researches during the first half of the last century. Many mathematicians[2] contributed to construct this beautiful theory and the reader will meet some of them along this pages. We have selected three mathematicians to describe the evolution of the Topology. Euler as a precursor, Poincaré who started the very active time of researches and Perelman who solved in the early years 2000, the most famous conjecture in Topology due to Poincaré one century earlier .

Every simply connected, closed 3-manifold is homeomorphic to the 3-sphere.

General Topology is an important and necessary tool in almost all the other fields of mathematics. It is not a domain of research, mathematicians agreed to say that this theory is completed. However, due to the need to understand many problems arising from our world being more and more complex, new active domains of researches appeared recently: computational topology and geometry. Roughly speaking, it consists to study continuous objects with discrete processes.
Topology has also some surprising applications and we cannot resist to mention the Nobel Prize in Physics 2016. The Nobel Prize in Physics 2016 was divided, one half awarded to David J.

[2] partial biographies from wikipedia

Thouless, the other half jointly to F. Duncan M. Haldane and J. Michael Kosterlitz "*for theoretical discoveries of topological phase transitions and topological phases of matter*".
The three Laureates' use of topological concepts in physics was decisive for their discoveries. Topology is a branch of mathematics that describes properties that change step-wise. With modern topology as a tool, this year's Laureates presented surprising results, which have opened up new fields of research and led to the creation of new and important concepts within several areas of physics.
Topology describes the properties that remain intact when an object is stretched, twisted or deformed, but not if it is torn apart. Topologically, a sphere and a bowl belong to the same category, because a spherical lump of clay can be transformed into a bowl. However, a bagel with a hole in the middle and a coffee cup with a hole in the handle belong to another category; they can also be remodelled to form each other's shapes. Topological objects can thus contain one hole, or two, or three, or four... but this number has to be an integer. This turned out to be useful in describing the electrical conductance found in the quantum Hall effect, which only changes in steps that are exact multiples of an integer.

The first part of these lecture notes will concern the so-called general topology, i.e. open and closed sets, separabilities axioms, limits, continuous maps, topological constructions, topological properties and complete metric spaces. However, topology needs to be familiar with set theory. So, we start by giving a short but important summary of set theory.
In a second part, we introduce some notions of continuous deformations, i.e. homotopy and we define the fundamental group, also called Poincaré group, of a topological space. The Seifert-Van Kampen theorem will be proved and we will give a presentation of the covering spaces.

Here are some books that are my references for these lecture notes. This is far from a bibliography.

- Bourbaki. *Topologie Générale* Ch. 1-4. Springer, 2007.
- J. Dugundji. *Topology.* Allyn and Bacon, Boston, 1966.
- J. L. Kelley. *General Topology* Graduate Texts in Maths. 27, Springer, 1955.
- J.R. Munkres. *Topology.* Pearson Prentice Hall, 2003.
- Ryszard Engelking *General Topology* Heldermann Verlag Berlin, 1989.

To make shorter, the author uses the following abbreviations:

- "w.r.t." for "with respect to"
- "iff" for "if and only if".

I use the notation $]a, b[$ for the open interval (a, b).
Even if many misprints were corrected, it certainly remains many of them. All your remarks, comments are welcomed.

Chapter 1

"Naive" Set Theory

This is a short introduction of the set theory from the "naive" point of view, i.e. we don't define the notions of elements and sets. In other words, we don't construct the set theory from axioms.

1.1 Preliminary

The fundamental principle consists to say that, given a proposition A, then either A is true or the negation, non$A$, of A is true, and only one is possible. It is called the **law of excluded middle**.
Let A and B be two propositions, we define the propositions

- (A or B) which is true only if either A is true or B is true.
- (A and B) which is true only if both A and B are true.

The proposition $A \Longrightarrow B$ means (non$A$ or B). If A is true, then B is true. But if non$A$ is true, then either B can be true or non$B$ can be true.
Boolean Algebra. In Boolean algebra, a proposition A is true iff its value is 1 and 0 iff it is false. The basic operations of Boolean calculus are as follows:

A	non$A$
0	1
1	0

A	B	A ou B
0	0	0
0	1	1
1	0	1
1	1	1

A	B	A et B
0	0	0
0	1	0
1	0	0
1	1	1

A	B	$A \Longrightarrow B$
0	0	1
0	1	1
1	0	0
1	1	1

1.2 Sets. Elements

Modern topology depends strongly on the ideas of set theory, developed by Georg Cantor.[1]
We will only introduce a naive point of view.

[1]Georg Ferdinand Ludwig Philipp Cantor, (1845-1918), was a German mathematician, best known as the inventor of set theory. Cantor established the importance of one-to-one correspondence between the members of two sets, defined infinite and well-ordered sets. He defined the cardinal and ordinal numbers and their arithmetic. Cantor's work is of great philosophical interest, a fact of which he was well aware.

The first words are **set** and **element**.
A set is a collection of elements (Notice that this is not a definition). We denote $x \in X$ to say that x is an element of the set X.
Here are some well known examples: $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$.
A set can be described either by the list of all its elements, $X = \{x, y, \ldots\}$ or by a property P, $X = \{x \mid P(x)\}$.
Example and definition: Let X be a set. Define $\emptyset_X = \{x \in X \mid x \neq x\}$ which is called the **empty set** defined by X. Notice that $\emptyset_X$ has no element.
Let X be a set. A subset Y of X denoted $Y \subset X$ is the set defined by $(x \in Y \Rightarrow x \in X)$.
Equality of sets. $X = Y$ iff $X \subset Y$ and $Y \subset X$.

Remark 1.2.1.

- *$\emptyset_X \subset X$ for any set X.*
- *$\emptyset_X = \emptyset_Y$ for any sets X, Y, $(x \in \emptyset_X \Rightarrow P(x)$ for any x and any property $P)$.*
  Thus we can write $\emptyset$ for the unique empty set. So there exists a unique empty set $\emptyset$ and $\emptyset \subset X$ for any set X.

What can we say about the collection of all sets?
Suppose there exists a set X whose elements are all the sets. Then $Y = \{A \in X \mid A \notin A\}$ is a subset of X so a set. Hence, either $Y \in Y$, so $Y \notin Y$ by definition of Y, or $Y \notin Y$, so $Y \in Y$, also by definition of Y which is a contradiction.
We can summarize by saying that there exist two types of collections: classes and sets. The class X is called set if there is a class $\mathcal{X}$ such that $X \in \mathcal{X}$. This approach is intuitive and naive. An exposition of Set Theory requires more precision. There are several different axiomatic set theories, each having technical advantages and shortcomings.

1.2.1 Intersection & Union

Let X and Y be two sets. Then we define their **union**

$$X \cup Y = \{a \mid a \in X \text{ or } a \in Y\}$$

and their **intersection**

$$X \cap Y = \{a \mid a \in X \text{ and } a \in Y\}.$$

More generally, let $X_i, i \in I$ be a family of sets. We define their union as

$$\bigcup_{i \in I} A_i = \{x \mid \exists i \in I \text{ such that } x \in A_i\}$$

and their intersection as

$$\bigcap_{i \in I} A_i = \{x \mid \forall i \in I, x \in A_i\}.$$

Distributivity (De Morgan's laws[2]):

$$(X \cap Y) \cup Z = (X \cup Z) \cap (Y \cup Z) \text{ and } (X \cup Y) \cap Z = (X \cap Z) \cup (Y \cap Z).$$

Remark 1.2.2. *Notice the "similarity" between the definitions and the properties of union and intersection. It will be called* **duality***.*
To permute $\cup$ and $\cap$ means to permute "or" and "and", $\exists$ and $\forall$.

[2] Augustus De Morgan (27 June 1806 - 18 March 1871) was a British mathematician and logician. He formulated De Morgan's laws and introduced the term mathematical induction, making its idea rigorous

1.2.2 Difference

We define the **difference** of two sets X and Y as $X \setminus Y = \{x \mid x \in X \text{ and } x \notin Y\}$. In particular, if $Y \subset X$ then the set $X \setminus Y$ is called the **complement** of Y in X.

Exercice 1.2.3. *Let X be a set and let Y and Y' be two subsets of X.*

1. $X \setminus (Y \cup Y') = (X \setminus Y) \cap (X \setminus Y')$ *and* $X \setminus (Y \cap Y') = (X \setminus Y) \cup (X \setminus Y')$.

2. $X \setminus (X \setminus Y) = Y$.

1.2.3 Symmetric Difference

Let X and Y be two sets. Then we define their **symmetric difference** denoted as $X\Delta Y$ and defined as follows: $X\Delta Y = (X \setminus Y) \cup (Y \setminus X)$.

Exercice 1.2.4. *Compare $X\Delta Y$ and $(X \cup Y) \setminus (X \cap Y)$.*

1.2.4 Equivalence Relation. Partition

A **partition** of a set X is a family of subsets $(A_i)_{i\in I}$ of X such that $X = \bigcup_{i\in I} A_i$ and for all $i, j \in I,\ i \neq j,\ A_i \cap A_j = \emptyset$.
An **equivalence relation** $\mathcal{R}$ in the set X satisfies the three properties:

1. $\forall x \in X$ then $x\mathcal{R}x$. (Reflexivity)

2. $\forall x, y \in X$ then $(x\mathcal{R}y) \Longrightarrow (y\mathcal{R}x)$. (Symmetry)

3. $\forall x, y, z \in X$ then $(x\mathcal{R}y$ and $y\mathcal{R}z) \Longrightarrow (x\mathcal{R}z)$. (Transitivity)

The equivalence class of $x \in X$ is the set denoted $\overline{x}$ and defined as follows: $\overline{x} = \{y \in X \mid y\mathcal{R}x\}$.
The set of all equivalence classes is called the **quotient set** and denoted $X/\mathcal{R}$.
Notice that an equivalence relation on the set X defines a partition of X consisting of the equivalence classes.
Given a partition $X = \bigcup_{i\in I} A_i$ of the set X, defines an equivalence relation $\mathcal{R}$ on X where $x\mathcal{R}y$ iff there exists i such that x and $y \in A_i$.

1.2.5 Order Relation. Poset. Upper Bound. Supremum. Lower Bound. Infimum.

An **order relation** $\leq$ on the set X satisfies the three properties:

1. $\forall x \in X, x \leq x$. (*Reflexitivity*)

2. $\forall x, y \in X, (x \leq y$ and $y \leq x) \Longrightarrow (x = y)$. (*Skew symmetry*)

3. $\forall x, y, z \in X, (x \leq y$ and $y \leq z) \Longrightarrow (x \leq z)$. (*Transitivity*)

A **poset** is a set X equipped with an order relation $\leq$.
An **upper bound** of a subset S of some poset $(X, \leq)$ is an element of X which is greater than or equal to every element of S.
The term **lower bound** is defined dually as an element of X which is lesser than or equal to every element of S.
A set with an upper bound is said to be **bounded from above** by that bound, a set with a lower bound is said to be bounded from below by that bound.
The **supremum** (sup) of S, if it exists, is the least element of X that is greater than or equal to each element of S. Consequently, the supremum is also referred to as the least upper bound (lub). If the supremum exists, it may or may not belong to S. If the supremum exists, it is unique.
Infimum (inf) (also referred to as the greatest lower bound) (glb), is in a precise sense dual to the concept of a supremum.

Example 1.2.5.
- *Let $\mathbb{R}$ be the real set of numbers with the usual order $\leq$. Then it is a linear ordered set, i.e. any two real numbers are comparable.*
- *Let $\mathbb{N}$ be the set of positive integers. Define the order relation $\preceq$ by $m \preceq n$ if m divides n. Then $(\mathbb{N}, \preceq)$ is a poset.*

  In the following, all the sets are subsets of $\mathbb{R}$ with the usual order.
- Let $\{1,2,3\} \subset \mathbb{R}$, then $\sup\{1,2,3\} = 3$;
- $\sup\{x \in \mathbb{R} \mid 0 < x < 1\} = \sup\{x \in \mathbb{R} \mid 0 \leq x \leq 1\} = 1$;
- $\sup\left\{(-1)^n - \dfrac{1}{n} \mid n \in \mathbb{N}^*\right\} = 1$;
- Let $A, B \subset X$, and $(X, \leq)$ be a poset, then $\sup\{a + b \mid a \in A, b \in B\} = \sup(A) + \sup(B)$.
- $\sup\{x \in \mathbb{Q} \mid x^2 < 2\}$ does not exist in $\mathbb{Q}$.
- But $\sup\{x \in \mathbb{R} \mid x^2 < 2\} = \sqrt{2}$.
- $\sup\{x \in \mathbb{Q} \mid x^3 < 2\}$ does not exist in $\mathbb{Q}$.
- But $\inf\{x \in \mathbb{R} \mid x^3 > 2\} = \sqrt[3]{2}$;
- $\inf\left\{(-1)^n + \dfrac{1}{n} \mid n \in \mathbb{N}^*\right\} = -1$.

1.3 Maps

1.3.1 Definition

Definition 1.3.1. *A* **map** *f from a set X to a set Y, denoted $f : X \longrightarrow Y$ assignes to each element $x \in X$ a unique element $f(x) \in Y$.*
Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$, then we define the map $g \circ f : X \longrightarrow Z$ by $g \circ f(x) = g\big(f(x)\big)$.

The set of all maps from X to Y is denoted Y^X. In particular, the set of all subsets of X is 2^X.
We will understand this notation in the section Cardinality.
The notion of duality has full meaning with both the sets and the maps between the sets.
As it will be noticed in the section "Categories", and as it will be clear in this section, the notions of sets and maps have to be worked together.

1.3.2 Surjection. Injection. Bijection

Definition 1.3.2. • *The map* $f : X \longrightarrow Y$ *is called* **surjection** *if* $f(X) = Y$, *i.e. for any* $y \in Y$, *there exist* $x \in X$ *such that* $f(x) = y$.

- *The map* $f : X \longrightarrow Y$ *is called* **injection** *if for any* $x_1, x_2 \in X, x_1 \neq x_2 \Longrightarrow f(x_1) \neq f(x_2)$.
- *The map* $f : X \longrightarrow Y$ *is called* **bijection** *if* f *is both an injection and a surjection.*

The map f is an injection iff $f(x_1) = f(x_2) \Longrightarrow x_1 = x_2$.
Notice that there is a "duality" between these two first properties as well in the third one which does not appear in the definitions of injection and surjection. This duality can be expressed as follows: Let $f : X \longrightarrow Y$ be a map such that

- 1. for any set Z and any two maps $g_1, g_2 : Y \longrightarrow Z$ satisfying $g_1 \circ f = g_2 \circ f$, then $g_1 = g_2$.
- 2. for any set Z and any two maps $g_1, g_2 : Z \longrightarrow X$ satisfying $f \circ g_1 = f \circ g_2$, then $g_1 = g_2$.

$$X \xrightarrow{f} Y \underset{g_2}{\overset{g_1}{\rightrightarrows}} Z \qquad || \qquad Z \underset{g_2}{\overset{g_1}{\rightrightarrows}} X \xrightarrow{f} Y$$

The properties 1. and 2. are clearly dual. Moreover, 1. characterizes the map f as surjection and 2. as injection.
Suppose f surjection, then $f(X) = Y$. Let $y \in Y$, then there exists $x \in X$ such that $f(x) = y$. So $g_1(y) = g_1 \circ f(x) = g_2 \circ f(x) = g_2(y)$.
Suppose $(g_1 \circ f = g_2 \circ f) \Longrightarrow (g_1 = g_2)$, If f is not a surjection, there exists $y \in Y$ such that $y \notin f(X)$. There exist g_1 and g_2 such that $g_1(y) \neq g_2(y)$ and $g_1(f(x)) = g_2(f(x))$ for any $x \in X$. Then $g_1 \neq g_2$ which is a contradiction and f is a surjection.
Suppose f injection. Let $z \in Z$, Then $f(g_1(z)) = f(g_2(z))$, so $g_1(z) = g_2(z)$, and $g_1 = g_2$.
Suppose $(f \circ g_1 = f \circ g_2) \Longrightarrow (g_1 = g_2)$.
Then $f(g_1(z)) = f(x_1) = f(g_2(z)) = f(x_2) \Longrightarrow (g_1(z) = g_2(z))$ and f is an injection.

1.3.3 Inverse Map

Let $f : X \longrightarrow Y$ be a map, f may be not invertible. Let define the set of subsets of X, (resp. Y), as $\mathcal{P}(X) = \{A \mid A \subset X\}$, (resp. $\mathcal{P}(Y) = \{B \mid B \subset Y\}$). The map f induces two maps,

- the first one also denoted $f : \mathcal{P}(X) \longrightarrow \mathcal{P}(Y)$ such that $f(A) = \{f(x) \mid x \in A, A \in \mathcal{P}(X)\}$
- and the **inverse map** as the map $f^{-1} : \mathcal{P}(Y) \longrightarrow \mathcal{P}(X)$ as follows. Let $B \subset Y$, i.e. $B \in \mathcal{P}(Y)$, then $f^{-1}(B) = \{x \in X \mid f(x) \in B\} \in \mathcal{P}(X)$.

Notice that the map denoted f^{-1} is not the inverse of the map f, i.e. f^{-1} is not a map from Y to X, and we don't have $f \circ f^{-1} = Id_{\mathcal{P}(X)}$ and $f^{-1} \circ f = Id_{\mathcal{P}(Y)}$.
If f is a bijection, then $f^{-1}(\{y\}) = \{x\}$ for any $y \in Y$, so f^{-1} can be viewed as a map $Y \longrightarrow X$ and $f \circ f^{-1} = Id_Y, f^{-1} \circ f = Id_X$.

Properties of these two maps

$$f\left(\bigcup_{i \in I} A_i\right) = \bigcup_{i \in I} f(A_i), \ \forall A_i \subset X$$
$$f(\emptyset) = \emptyset$$

In general, $f(A_1 \bigcap A_2) \neq f(A_1) \bigcap f(A_2)$.
We also have $f(X) \neq Y$ and $f(X \setminus A) \neq f(X) \setminus f(A)$.

$$f^{-1}\left(\bigcup_{i\in I} B_i\right) = \bigcup_{i\in I} f^{-1}(B_i),\ \forall B_i \subset Y$$
$$f^{-1}\left(\bigcap_{i\in I} B_i\right) = \bigcap_{i\in I} f^{-1}(B_i),\ \forall B_i \subset Y$$
$$f^{-1}(\emptyset) = \emptyset$$
$$f^{-1}(Y \setminus B) \neq f^{-1}(Y) \setminus f^{-1}(B)$$

Notice that there is not a perfect duality between f and f^{-1}.

1.3.4 Universal Mapping Property

1. Let A be a subset of X. Then there exists a **canonical injection** called the inclusion $\iota : A \hookrightarrow X$ such that $\iota(x) = x$ for any $x \in A$.
   Moreover, for any set Y and any map $f : Y \longrightarrow X$ such that $f(Y) \subset A$, there exists a unique map $g : Y \longrightarrow A$ such that $f = \iota \circ g$. It is called **Universal Mapping Property**[3].

$$\begin{tikzcd} Y \arrow[d, dotted, "g"'] \arrow[dr, "f"] & \\ A \arrow[r, hook, "\iota"'] & X \end{tikzcd}$$

2. Let $\mathcal{R}$ be an equivalence relation on the set X. Then there exists a **canonical surjection** $p : X \longrightarrow X/\mathcal{R}$ such that $p(x) = \overline{x} = \{x' \in X \mid x'\mathcal{R}x\}$ for any $x \in X$.
   Moreover, we have the following **Universal Mapping Property**. For any set Y and any map $f : X \longrightarrow Y$ such that $x\mathcal{R}x' \implies f(x) = f(x')$, there exists a unique map $g : X/\mathcal{R} \longrightarrow Y$ such that $f = g \circ p$.

$$\begin{tikzcd} & Y \\ X \arrow[ur, "f"] \arrow[r, "p"'] & X/\mathcal{R} \arrow[u, dotted, "g"'] \end{tikzcd}$$

Notice the duality between these two constructions. The diagrams correspond by changing the direction of the arrows, by interchanging canonical injection and surjection. We can say that the subset (A, ι) satisfies a *left universal mapping property* and the quotient set $(X/\mathcal{R}, p)$ a *right universal mapping property*.

Let $f : X \longrightarrow Y$ be a map. In general, f is neither injective nor surjective.

1. The map f can be "made" surjective as follows:

$$\begin{tikzcd} X \arrow[r, "\widehat{f}"] \arrow[rr, bend right, "f"'] & f(X) \arrow[r, hook, "\iota"] & Y \end{tikzcd}$$

[3] The map ι sends the element a onto $\iota(a) = a$. It is the same element, but a is element of the set A and $\iota(a)$ is element of the set X.
We have $g(z) = f(z)$ for any $z \in Z$, so g and f are equal, but g is going to A and f to X.

where $\widehat{f}$ is the surjective map $x \longmapsto f(x)$ and ι is the canonical injection.

2. "Making" the map f injective seems more difficult. However, using duality, we have

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[rr, bend right, "f"'] & X/\mathcal{R} \arrow[r, "\overline{f}"] & Y \end{tikzcd}$$

where $\mathcal{R}$ is the equivalence relation on X: $x\mathcal{R}x'$ if $f(x) = f(x')$, $\overline{f}$ is the injective map $[x] \longmapsto f(x)$, and p is the canonical surjection.
1. the map f is the composition of a surjection and a (canonical) injection, and 2. of an injection and a (canonical) surjection.

Exercice 1.3.3. *1. Show that there exists a bijection between the sets $X \times Y$ and $Y \times X$. Is this bijection unique?*

2. Let E be the set of all even numbers in $\mathbb{N}$. Show there exists a bijection between E and $\mathbb{N}$.

3. Show there exists a bijection between $\mathbb{N}$ and $\mathbb{Z}$.

1.3.5 Product of Sets

Definition 1.3.4. *Let X and Y be two sets. We define their* **product** *as*

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}$$

More generally, let n sets $X_1, \ldots, X_n$ we define the product

$$X_1 \times \cdots \times X_n = \prod_{1 \leq i \leq n} X_i = \{(x_1, \ldots, x_n) \mid x_i \in X_i, \ i = 1, \ldots, n\}$$

Whenever $X_i = X$ for any $i = 1, \ldots, n$, the product $\prod\limits_{1 \leq i \leq n} X_i$ is denoted X^n.

How can we define the product of any infinite family of sets? For example, (X_i) where $i \in \mathbb{R}$. We cannot write an element of the product as a sequence, what would be the consecutive element of x_0, or x_π? Such a product will be defined in 1.3.9.
The product of sets satisfies a "universal property" as follows:
Let X and Y be two sets and let $X \times Y$ be their product. Then there exist two canonical maps called **projections** $p_X : X \times Y \longrightarrow X$ and $p_Y : X \times Y \longrightarrow Y$ such that $p_X\big((x, y)\big) = x$ and $p_Y\big((x, y)\big) = y$.
Moreover, we have the following (left) **Universal Mapping Property**. For any set Z and any maps $f_X : Z \longrightarrow X$ and $f_Y : Z \longrightarrow Y$, there exists a **unique** map $h : Z \longrightarrow X \times Y$ such that $p_X \circ h = f_X$ and $p_Y \circ h = f_Y$. For any $z \in Z, h(z) = (f_X(z), f_Y(z))$.

$$\begin{tikzcd} & Z \arrow[dl, "f_X"'] \arrow[d, dotted, "h"] \arrow[dr, "f_Y"] & \\ X & X \times Y \arrow[l, "p_X"] \arrow[r, "p_Y"'] & Y \end{tikzcd}$$

More generally, we have the following (left) **Universal Mapping Property**,

Let $(X_i)_{i\in I}$ be a family of sets and $p_j : \prod_i X_i \longrightarrow X_j, j \in I$. For any set Z and any map $f_j : Z \longrightarrow X_j$, there exists a unique map $h : Z \longrightarrow \prod_i X_i$ such that $f_j = p_j \circ h$.

$$\begin{tikzcd}
& Z \arrow[dl, "f_j"'] \arrow[d, dotted, "h"] \\
X_j & \prod_i X_i \arrow[l, "p_j"]
\end{tikzcd}$$

1.3.6 Sum of Sets

Definition 1.3.5. *Let X and Y be two sets. Then we define their* **sum**, *also called* **coproduct** *as follows:*

$$X \coprod Y = (\{0\} \times X) \cup (\{1\} \times Y)$$

We can identify X (resp. Y) with the subset $(\{0\} \times X)$ (resp. $(\{1\} \times Y)$. The sum $X \coprod Y$ is the disjoint union of X and Y, it is the union of two copies of X and Y which are disjoint. Notice that it is defined up to some bijections. If $X \cap Y = \emptyset$, we can identify $X \coprod Y$ with $X \cup Y$.
If $X = Y$, then $X \coprod Y$ consists of two copies of X.

Exercice 1.3.6. *Define the disjoint union of a family $(X_i)_{i\in I}$ of sets.*

The sum of sets satisfies the (right) **Universal Mapping Property**:
Let X and Y be two sets and let $X \coprod Y$ be their disjoint sum. Then there exist two canonical maps $i_X : X \longrightarrow X \coprod Y$ and $i_Y : Y \longrightarrow X \coprod Y$ such that $i_X(x) = (0, x)$ and $i_Y(y) = (1, y)$. Moreover, for any set Z and any maps $f_X : X \longrightarrow Z$ and $f_Y : Y \longrightarrow Z$ there exists a **unique** map $h : X \coprod Y \longrightarrow Z$ such that $h \circ i_X = f_X$ and $h \circ i_Y = f_Y$. We have $h(0, x) = f_X(x)$, $h(1, y) = f_Y(y)$.

$$\begin{tikzcd}
& Z & \\
X \arrow[ur, "f_X"] \arrow[r, "i_X"'] & X \coprod Y \arrow[u, dotted, "h"] & Y \arrow[ul, "f_Y"'] \arrow[l, "i_Y"]
\end{tikzcd}$$

More generally, the following (right) **Universal Mapping Property** is satisfied,

Let $\coprod_i X_i = \bigcup_i \{i\} \times X_i$. Let $(X_i)_{i\in I}$ be a family of sets and $\iota_j : X_j \longrightarrow \coprod_i X_i, j \in I, x_j \longmapsto (j, x_j)$. For any set Z and any map $f_j : X_j \longrightarrow Z$, there exists a unique map $h : \coprod_i X_i \longrightarrow Z$ such that $f_j = h \circ \iota_j$.

$$\begin{tikzcd}
& Z \\
X_j \arrow[ur, "f_j"] \arrow[r, "\iota_j"'] & \coprod_i X_i \arrow[u, dotted, "h"']
\end{tikzcd}$$

Notice the duality "sum" $\longleftrightarrow$ "product".
These two examples give the complete definition of the product and the sum of two sets.
These constructions of sum and product of two sets can be defined for any family of sets.

1.3.7 Fibered Product. Fibered Sum

These notions are some important generalizations of product and sum of sets.

Fibered Product

[4] Let B and C be two sets, $p_B : B \times C \longrightarrow B, p_C : B \times C \longrightarrow C$ the two canonical surjections $p_B(b, c) = b, p_C(b, c) = c$.

[4] also called Cartesian square or pullback

Let D be an arbitrary set and $f_B : D \longrightarrow B, f_C : D \longrightarrow C$ two arbitrary maps. Then there is a unique map $h_x : D \longrightarrow B \times C$ such that $f_B = p_B \circ h_x$ and $f_C = p_C \circ h_x$ (Universal mapping property of the product set).

$$\begin{tikzcd}
D \arrow[drr, bend left, "f_C"] \arrow[dr, "h_x"'] \arrow[ddr, bend right, "f_B"'] & & \\
& B \times C \arrow[r, "p_C"'] \arrow[d, "p_B"] & C \\
& B &
\end{tikzcd}$$

Consider a set A and two maps $g_B : B \longrightarrow A,\ g_C : C \longrightarrow A$ such that $g_B \circ p_B = g_C \circ p_C$. Define

$$B \prod_A C = \{(b,c) \in B \times C \mid g_B(p_B(b,c)) = g_C(p_C(b,c))\} = \{(b,c) \in B \times C \mid g_B(b) = g_C(c)\}$$

For any set D and any maps $f'_B : D \longrightarrow B, f'_C : D \longrightarrow C$ such that $g_B \circ f'_B = g_C \circ f'_C$, there exists a unique map $h : D \longrightarrow B \prod_A C$ such that the following diagram is commutative (universal mapping property)

$$\begin{tikzcd}
D \arrow[r, "h"] \arrow[drrr, bend left, "f'_C"] \arrow[drr, "h'_x"'] \arrow[ddrr, bend right, "f'_B"'] & B \prod_A C \arrow[dr, hook, dotted, "\iota"] & & \\
& & B \times C \arrow[r, "p_C"'] \arrow[d, "p_B"] & C \arrow[d, "g_C"'] \\
& & B \arrow[r, "g_B"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
D \arrow[dr, "h"] \arrow[drr, bend left, "f_C"] \arrow[ddr, bend right, "f_B"'] & & \\
& B \prod_A C \arrow[r, "\iota \circ p_C"'] \arrow[d, "\iota \circ p_B"] & C \arrow[d, "g_C"'] \\
& B \arrow[r, "g_B"] & A
\end{tikzcd}$$

The map $h'_x = \iota \circ h : D \longrightarrow B \prod C$ satisfies $f'_B = p_B \circ h'_x$ and $f'_C = p_C \circ h'_x$ where ι is the inclusion.

Definition 1.3.7. *The* **fibered product** *of* (g_B, g_C) *is the triple* $\left(B \prod_A C, g_B, g_C\right)$, *where* $B \prod_A C = \{(b,c) \in B \times C \mid g_B(b) = g_C(c)\}$.

Example 1.3.8. *1. Let $g_B : B \longrightarrow A$ be a map and $C \subset A$. Denote g_C the inclusion of C into A. Then there is a bijection from $B \prod_A C$ onto $g_B^{-1}(C) \subset B$.*

2. If $A = \{a\}$, then $B \prod_A C = B \times C$.

Fibered Sum

[5] Let B and C be two sets, $\iota_B : B \longrightarrow B\coprod C,\ \iota_C : C \longrightarrow B\coprod C$ the two canonical injections, $\iota_B(b) = (0, b), \iota_C(c) = (1, c)$.
Let D ba an arbitrary set and $f_B : B \longrightarrow D,\ f_C : C \longrightarrow D$. Then there exists a unique map $h_s : B\coprod C \longrightarrow D$ such that $f_B = h_s \circ \iota_B,\ f_C = h_s \circ \iota_C$ (Universal mapping property of the coproduct set).

$$\begin{tikzcd}
& B \arrow[d, "\iota_B"] \arrow[ddr, bend left, "f_B"] & \\
C \arrow[r, "\iota_C"] \arrow[drr, bend right, "f_C"'] & B\coprod C \arrow[dr, dotted, "h_s"] & \\
& & D
\end{tikzcd}$$

Consider a set A and two maps $g_B : A \longrightarrow B,\ g_C : A \longrightarrow C$ such that $\iota_B \circ g_B = \iota_C \circ g_C$.
Let $\sim$ be the equivalence relation generated by the relations $g_B(a) \sim g_C(a)$ for all $a \in A$. Let $p : B\coprod C \longrightarrow B\coprod C/\sim$ be the canonical surjection.
Then there exists a unique map $h : B\coprod C/\sim \longrightarrow D$ making the following diagram commutative (universal mapping property)

$$\begin{tikzcd}
A \arrow[r, "g_B"] \arrow[d, "g_C"'] & B \arrow[d, "\iota_B"] \arrow[ddrr, bend left, "f'_B"] & & \\
C \arrow[r, "\iota_C"] \arrow[drrr, bend right, "f'_C"'] & B\coprod C \arrow[r, "p"] \arrow[drr, "h'_s"] & B\coprod_A C \arrow[dr, dotted, "h"] & \\
& & & D
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "g_B"] \arrow[d, "g_C"'] & B \arrow[d, "p\circ\iota_B"] \arrow[ddr, bend left, "f_B"] & \\
C \arrow[r, "p\circ\iota_C"] \arrow[drr, bend right, "f_C"'] & B\coprod_A C \arrow[dr, dotted, "h"] & \\
& & D
\end{tikzcd}$$

The map $h'_s = h \circ p : B\coprod C \longrightarrow D$ satisfies $f'_B = h'_s \circ \iota_B$ and $f'_C = h'_s \circ \iota_C$.

Definition 1.3.9. *The* **fibered sum** *of* (g_B, g_C) *is the triple denoted* $\left(B\coprod_A C, g_B, g_C\right)$ *where*
$$B\coprod_A C = \left\{[u] \mid u \in B\coprod C, u = \iota_B(g_B(a)) = \iota_C(g_C(a)), a \in A\right\}.$$

Example 1.3.10.

1. *Let $A \subset B$, let ι be the inclusion from A into B and let c be the unique map from A onto a singleton $\{*\}$. Then there is a bijection from $X\coprod_A\{*\}$ and the quotient set B/A.*

2. *If $A = \emptyset$, and B, C two sets with the trivial maps $\emptyset \longrightarrow B, \emptyset \longrightarrow C$, then $B\coprod_\emptyset C = B\coprod C$.*

[5]also called cocartesian square or pushout or fibered coproduct

1.3.8 Cardinality

Let us say that the two sets X and Y have the same **cardinality** if there exists a bijection $f : X \longrightarrow Y$. It is said that X and Y are **equipotent** and we denote card(X) =card(Y). card(X) is the "collection" of all sets equipotent to the set X. Notice that card(X) is also denoted $\aleph(X)$.
If $X \subset Y$, then card$(X) \leq$card(Y). In particular, we denote card$(\emptyset) = 0$ and card$(\{1, 2, \dots, n\} = n$.
A set is **countable infinite** if it has the cardinality of the set $\mathbb{N}$, i.e. card$(\mathbb{N}) = \aleph_0$.
If a set X is finite or countable infinite, we say that it is **countable** and $\aleph(X) \leq \aleph_0$.
A set is said to be **uncountable** if it is not countable. $\mathbb{N} \subset \mathbb{R}$ and $\mathbb{R}$ is not equipotent to $\mathbb{N}$, so card$(\mathbb{N})$ <card$(\mathbb{R})$.

Exercice 1.3.11. *Let X and Y be two finite sets. Determine the cardinality of Y^X.*

Remark 1.3.12. *Cardinality satisfies the three following properties: reflexivity, symmetry and transitivity. But the collection of all sets is not a set. However we can define an equivalence relation in the class of all sets and, as such, equipotence is an equivalence relation and $\aleph(X)$ is the equivalence class of X. We have* card$(X) = \aleph(X)$.

Properties

- Every subset X of a countable set is countable.
- The union of countable many countable sets is countable.
- $\aleph(X \cup Y) \leq \aleph(X) + \aleph(Y)$ with equality if $X \cap Y = \emptyset$.
- $\aleph(X \coprod Y) = \aleph(X) + \aleph(Y)$
- $\aleph(X \times Y) = \aleph(X).\aleph(Y)$.
- If $\aleph(X) \geq \aleph_0$ then $\aleph(X).\aleph(X) = \aleph(X)$. This result is not valid for finite sets.
- The closed unit interval $I = [0, 1] \subset \mathbb{R}$ is uncountable and $\mathbb{R}$ is uncountable, i.e. $\aleph(\mathbb{R}) > \aleph_0$.
- $\aleph(X) < \aleph(\mathcal{P}(X))$. So there is no largest cardinal.

Cardinal Arithmetic

- Addition: Let X and Y be two disjoint sets, then $\aleph(X) + \aleph(Y) = \aleph(X \cup Y)$.
  - $(\aleph(X) + \aleph(Y)) + \aleph(Z) = \aleph(X) + (\aleph(Y) + \aleph(Z))$.
  - $\aleph(X) + \aleph(Y) = \aleph(Y) + \aleph(X)$.
  - $\aleph(X) + \aleph(\emptyset) = \aleph(\emptyset) + \aleph(X) = \aleph(X), \aleph(\emptyset) = 0$.
  - If $\aleph(X) \leq \aleph(Y)$, then $\aleph(X) + \aleph(Z) \leq \aleph(Y) + \aleph(Z)$.
  - $\aleph_0 + \aleph_0 = \aleph_0$.
  - $\aleph(\mathbb{R}) + \aleph_0 = \aleph(\mathbb{R})$.
  - $\aleph(\mathbb{R}) + \aleph(\mathbb{R}) = \aleph(\mathbb{R})$.
- Multiplication: $\aleph(X \times Y) = \aleph(X).\aleph(Y)$.
  - $(\aleph(X).\aleph(Y)).\aleph(Z) = \aleph(X).(\aleph(Y).\aleph(Z))$.

- $\aleph(X).\aleph(Y) = \aleph(Y).\aleph(X)$.
- $\aleph(X).\aleph(\{*\}) = \aleph(\{*\}).\aleph(X) = \aleph(X)$.
- $\aleph(X).\aleph(\emptyset) = \aleph(\emptyset).\aleph(X) = \aleph(\emptyset)$.
- $\aleph(X).(\aleph(Y) + \aleph(Z)) = \aleph(X).\aleph(Y)) + \aleph(X).\aleph(Z))$.
- If $\aleph(X) \leq \aleph(Y)$, then $\aleph(X).\aleph(Z) \leq \aleph(Y).\aleph(Z)$.
- $\aleph_0.\aleph_0 = \aleph_0$.
- $\aleph(\mathbb{R}).\aleph_0 = \aleph(\mathbb{R})$.
- $\aleph(\mathbb{R}).\aleph(\mathbb{R}) = \aleph(\mathbb{R})$.

- The set of all maps from Y to X is denoted X^Y and $\aleph(X^Y) = \aleph(X)^{\aleph(Y)}$.
  - $\aleph(X)^{\aleph(Y)+\aleph(Z)} = \aleph(X)^{\aleph(Y)}.\aleph(X)^{\aleph(Z)}$.
  - $(\aleph(X).\aleph(Y))^{\aleph(Z)} = \aleph(X)^{\aleph(Z)}.\aleph(Y)^{\aleph(Z)}$.
  - $\aleph(X) \leq \aleph(Y) \Longrightarrow \aleph(X)^{\aleph(Z)} \leq \aleph(Y)^{\aleph(Z)}$.
  - $\aleph(X) \leq \aleph(Y) \Longrightarrow \aleph(Z)^{\aleph(X)} \leq \aleph(Z)^{\aleph(Y)}$.
  - $\aleph_0^{\aleph_0} = \aleph(\mathbb{R})$ and $2^{\aleph_0} = \aleph(\mathbb{R})$.

1.3.9 Cantor Set

Let $x \in [0,1]$, there exists a sequence $s_1, s_2, \ldots$ where $s_i \in \{0,1,2\}$ for each i and such that $x = \sum\limits_{i \geq 1} \frac{s_i}{3^i}$.
We denote $x = s_1 s_2 s_3 \ldots$ and called it **triadic expansion** of x.
Notice that some x has two such expansions. For example $2000\ldots$ and $1222\ldots$ represent the same number $\frac{2}{3}$ since $\frac{2}{3} = \frac{2}{3} + \frac{0}{3^2} + \frac{0}{3^3} + \cdots = \frac{1}{3} + \frac{2}{3^2} + \frac{2}{3^3} + \cdots$.
This situation occurs when one of the expansions repeats 0's and the other repeats 2's from some point on. No number can be written in more than one way without using the digit 1.

Definition 1.3.13. *The* **Cantor set** C *is the set of all those* $x \in [0,1]$ *that have a triadic expansion in which the digit* 1 *does not occur, i.e.* $x = s_1 s_2 s_3 \ldots$ *where* $s_i \neq 1$.

Geometrical Interpretation: Let $F_1 = [0,1]$, $F_2 = [0,1] \backslash]\frac{1}{3}, \frac{2}{3}[= [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$.
Note that the "middle part" $]\frac{1}{3}, \frac{2}{3}[$ consists precisely of those x's whose triadic expansion must have 1 in the first digit. So, $F_2 = \{s_1 s_2 s_3 \ldots \mid s_1 \neq 1\}$.
Now delete from F_2 the middle part of each of the two intervals, i.e. $F_3 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$.
Note that the middle part which is deleted consists precisely of the elements $s_1 s_2 s_3 \ldots$ such that $s_1 \neq 1$ and $s_2 = 1$. Thus, $F_3 = \{s_1 s_2 s_3 \ldots \mid s_1 \neq 1,\ s_2 \neq 1\}$.
Then, continue the similar process inductively, at each stage deleting the open middle third of each closed interval remaining from the previous stage.
We obtain a descending sequence

$$F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots$$

of subsets of $[0,1]$, each of which is a finite union of disjoint closed intervals.
The Cantor set if $\bigcap\limits_{n \geq 1} F_n$.

Proposition 1.3.14. *The Cantor set* C *cannot contain an interval, and it is uncountable.*

Proof : The sum of the lengths of all open intervals removed from $[0,1]$ to form C is 1 since $\frac{1}{3}+\frac{2}{9}+\frac{4}{27}+\cdots=\frac{1}{3}\sum_{n\geq 0}\left(\frac{2}{3}\right)^n=1$. It follows that C cannot contains an interval.
For each $x \in [0,1]$, let $b_1b_2b_3\ldots$ be a binary expansion for x, i.e. $b_i = \ ,1$ for each i. Thus $x=\sum_{i\geq 1}\frac{b_i}{2^i}$. For each i, let $s_i = 2b_i$, and let $f(x)$ be the point in $[0,1]$ whose triadic expansion is $s_1s_2s_3\ldots$. Then f is one-to-one, $f(x) \in C$ for each $x \neq 1$, so f is surjective. Then C is uncountable. ■

1.3.10 Infinite Products

Let $\{X_i\}_{i\in I}$ be an infinite family of sets. If I is countable, we can write $\prod_{i\in I} X_i = \{(x_i) \mid x_i \in X_i\}$ as a "natural" extension of the finite product. But what is the meaning of (x_i) if I is uncountable? What about if I is not an ordered set?
To solve this problem, we introduce the bijection

$$\{(x_1,x_2)\in X_1\times X_2\} \longrightarrow \{f:\{1,2\}\to X_1\cup X_2 \mid f(1)\in X_1, f(2)\in X_2\}$$

For example, let $X_1 = X_2 = X$, and $f(1) = f(2) = x$, then f is denoted (x,x).
The right hand side gives a presentation of the product which does not depend on the order, and it can be extended as follows:

$$\prod_{i\in I} X_i := \left\{ f : I \to \bigcup_{i\in I} X_i \mid f(i)\in X_i \right\}$$

The map $p_i : \prod_{i\in I} X_i \longrightarrow X_i$ such that $p_i(f) = f(i)$ is called the ith-projection map and $f(i)$ is the ith-coordinate of f.
If I is finite, there is a bijection

$$\{(x_1,\ldots,x_n)\in X_1\times\cdots\times X_n\} \longrightarrow \{f:\{1,\ldots n\}\to X_1\cup\cdots\cup X_n \mid f(1)\in X_1,\ldots,f(n)\in X_n\}$$

If I is countable, there is a bijection

$$\left\{ (x_i)_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} X_i \right\} \longrightarrow \left\{ f:\mathbb{N}\to \bigcup_{i\in\mathbb{N}} X_i \mid f(i)\in X_i, i\in\mathbb{N} \right\}$$

An element of $\prod_{i\in I} X_i$ is a map that "chooses" a coordinate from each set in the family. It is a "choice map" which assigns to any set A of a nonempty family of nonempty sets, an element of A. The Axiom of choice is the following:
For any set A of a nonempty family of nonempty sets, there exists a choice map defined on A.
Then the Axiom of Choice is equivalent to the statement that each product of nonempty sets is nonempty.

1.3.11 Exercises

1. What is the cardinality of the set $\{\emptyset\}$?

2. Is it true that $\emptyset \in \{\emptyset, \{\emptyset\}\}$?
3. A set of cardinality 1 is called a *singleton*. What is card$\{\{\emptyset\}\}$?
4. Prove that $\mathbb{Z}$ and $\mathbb{Q}$ are countable.
5. Prove that $\mathbb{R}\backslash\mathbb{Q}$ is uncountable.
6. Is it possible to have $A \notin A$?
7. Prove that $A \setminus (A \setminus B) = A \cap B$ for any sets A and B?
8. Prove that $A \subset B$ iff $A \setminus B = \emptyset$.
9. Define the disjoint union of the sets $X_1, X_2, \ldots, X_n$.
10. Prove that $X \Delta Y = (X \cup Y) \setminus (X \cap Y)$.
11. Let $A, A' \subset X$ and $B, B' \subset Y$. Prove the following:
    (a) $(A \cup A') \times (B \cup B') = (A \times B) \cup (A \times B') \cup (A' \times B) \cup (A' \times B')$
    (b) $(A \times B) \cap (A' \times B') = (A \cap A') \times (B \cap B')$
    (c) $(A \times B) \setminus (A' \times B') = ((A \setminus A') \times B) \cup (A \times (B \setminus B'))$
12. Let $f : X \longrightarrow Y$ be a map. Then
    (a) f injective $\Longrightarrow f^{-1}\big(f(A)\big) = A,\ \forall\ A \subset X$. Show that the equality is not true if f is not injective.
    (b) f surjective $\Longrightarrow f\big(f^{-1}(B)\big) = B,\ \forall\ B \subset Y$. Show that the equality is not true if f is not surjective.

1.4 Complements: Categories. Functors

Category theory is an area of study in mathematics that deals in an abstract way with mathematical structures and relationships between them: it abstracts from sets and functions respectively to objects linked in diagrams by morphisms or arrows.
A **category** C consists of the following three mathematical entities:

- A class $ob(C)$, whose elements are called **objects**;
- A class $\hom(C)$, whose elements are called **morphisms** or maps or arrows. Each morphism f has a unique source object a and target object b. We write $f : a \longrightarrow b$, and we say f is a morphism from a to b. We write $\hom(a, b)$ (or $\mathrm{Hom}(a, b)$, or $\mathrm{Mor}(a, b)$) to denote the hom-class of all morphisms from a to b.
- A binary operation $\circ$, called composition of morphisms, such that for any three objects a, b, and c, we have $\hom(a, b) \times hom(b, c) \longrightarrow hom(a, c)$. The composition of $f : a \longrightarrow b$ and $g : b \longrightarrow c$ is written as $g \circ f$or gf, governed by two axioms:
  - Associativity: If $f : a \longrightarrow b, g : b \longrightarrow c$ and $h : c \longrightarrow d$ then $h \circ (g \circ f) = (h \circ g) \circ f$,
  - Identity: For every object x, there exists a morphism $1_x : x \longrightarrow x$ called the identity morphism for x, such that for every morphism $f : a \longrightarrow b$, we have $1_b \circ f = f = f \circ 1_a$.

Example 1.4.1.

- *The category of sets, denoted as $\mathcal{S}ets$ whose objects are the sets and the morphisms are the maps.*
- *The category of groups, denoted as $\mathcal{G}rps$ whose objects are the groups and the morphisms are the homomorphisms between groups.*
- *The category of topological spaces, denoted as $\mathcal{T}op$ whose objects are the topological spaces and the morphisms are the continuous maps as we will defined in the following.*
- *A poset is a category where the objects are the elements of the poset, the morphisms are arrows pointing from x to y when $x \leq y$.*
- *Any group can be seen as a category with a single object in which every morphism is invertible (for every morphism f there is a morphism g that is both left and right inverse to f under composition) by viewing the group as acting on itself by left multiplication. A morphism which is invertible in this sense is called an isomorphism.*

Any category C can itself be considered as a new category in a different way: the objects are the same as those in the original category but the morphisms are those of the original category reversed. This is called the **dual** or **opposite category** and is denoted C^{op}.
We defined the sum of two (a family of) sets. The categories give a general setting for the sum, as follows.
Let $(a_i)_{i \in I}$ be a family of objects in a category C, the **sum** of this family is given by an object a of C and for every $i \in I$ a morphism $f_i : a_i \longrightarrow a$ such that
For any object b and any morphisms $g_i : a_i \longrightarrow b$ of C, there is a unique morphism $f : a \longrightarrow b$ if C such that $g_i = f \circ f_i$ for any $i \in I$ (cf. left side figure and the dual in the right side figure, i.e. in the category C^{op}).

$$\begin{tikzcd} & b \\ a_i \arrow[ur, "f_i"] \arrow[r, "g_i"'] & a \arrow[u, dotted, "f"'] \end{tikzcd} \qquad \begin{tikzcd} b \arrow[d, dotted, "f"'] \arrow[dr, "f_i"] & \\ a \arrow[r, "g_i"'] & a_i \end{tikzcd}$$

If the category C is the category of sets, we get the sum of sets as we defined above.
As exercise, define the product, fibered sum, fibered product in a category.

A **functor** F from the category C_1 to the category C_2 associated to

- any object a of C_1, the object $F(a)$ of C_2.
- any morphism $f : a \longrightarrow b$ of C_1, the morphism
  - $F(f) : F(a) \longrightarrow F(b)$ such that $F(g \circ f) = F(g) \circ F(f)$. The functor F is said to be **covariant**.
  - $F(f) : F(b) \longrightarrow F(a)$ such that $F(g \circ f) = F(f) \circ F(g)$. The functor F is said to be **contravariant**.

1.5 More Examples of Duality: Retractions. Sections

We have seen the duality between injection and surjection as morphisms in the category of sets.
It is a particular case of the most general notions of retraction and section.
Let C be a category.

A **retraction** for a morphism $f : A \longrightarrow B$ is a morphism $r : B \longrightarrow A$ such that $r \circ f = Id_A$ A retraction is injective	A **section** for a morphism $f : A \longrightarrow B$ is a morphism $s : B \longrightarrow A$ such that $f \circ s = Id_B$ A section is surjective

$$\begin{tikzcd} & B \arrow[dr, "r"] & \\ A \arrow[ur, "f"] \arrow[rr, "Id_A"] & & A \end{tikzcd}$$

$$\begin{tikzcd} & A \arrow[dr, "f"] & \\ B \arrow[ur, "s"] \arrow[rr, "Id_B"] & & B \end{tikzcd}$$

If the morphism $f : A \longrightarrow B$ has a retraction, then for any object C and any morphisms $g_i : C \longrightarrow A$, $i = 1, 2$ such that $f \circ g_1 = f \circ g_2$, then $g_1 = g_2$	If the morphism $f : A \longrightarrow B$ has a section, then for any object C and any morphisms $g_i : B \longrightarrow C$, $i = 1, 2$ such that $g_1 \circ f = g_2 \circ f$, then $g_1 = g_2$

If the morphism $f : A \longrightarrow B$ has a retraction r then for any object C and any morphism $g : A \longrightarrow C$, there exists a morphism $h : B \longrightarrow C$ such that $h \circ f = g$	If the morphism $f : A \longrightarrow B$ has a section s then for any object C and any morphism $g : C \longrightarrow B$, there exists a morphism $h : C \longrightarrow A$ such that $f \circ h = g$

$$\begin{tikzcd} & B \arrow[dr, "r"] & \\ A \arrow[ur, "f"] \arrow[rr, "Id_A"] & & A \end{tikzcd}$$

$$\Downarrow$$

$$\begin{tikzcd} & B \arrow[dr, dotted, "h"] & \\ A \arrow[ur, "f"] \arrow[rr, "g"] & & C \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g"'] & B \arrow[r, dotted, "r"] \arrow[d, dotted, "h"] & A \arrow[dl, "g"] \\ & C & \end{tikzcd}$$

$$\begin{tikzcd} & A \arrow[dr, "f"] & \\ B \arrow[ur, "s"] \arrow[rr, "Id_B"] & & B \end{tikzcd}$$

$$\Downarrow$$

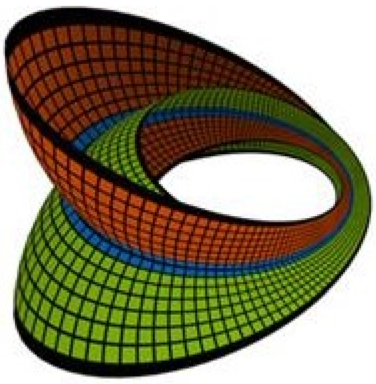
$$\begin{tikzcd} & A \arrow[dr, "f"] & \\ C \arrow[ur, dotted, "h"] \arrow[rr, "g"] & & B \end{tikzcd}$$

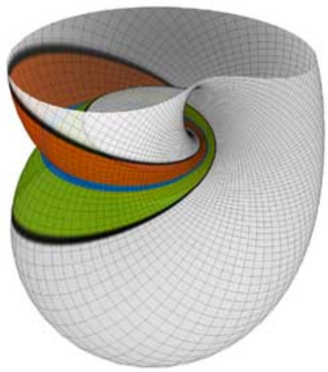
$$\begin{tikzcd} A \arrow[r, dotted, "s"] & B \arrow[r, "f"] & A \\ & C \arrow[ul, "g"] \arrow[u, dotted, "h"] \arrow[ur, "g"'] & \end{tikzcd}$$

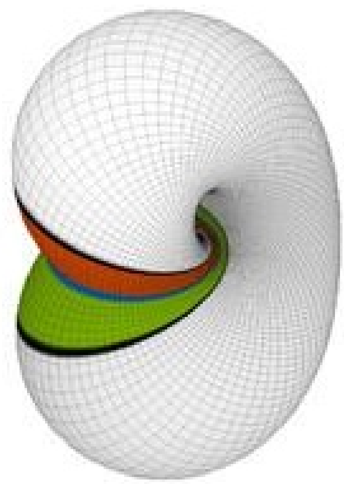
Part I

General Topology

Lawson-Klein bottle







H.B. Lawson came across an elegant realization of a Klein bottle in the 3-sphere $\mathbb{S}^3$ among a family of helicoidal - staircase-like - surfaces. The figure on the right side shows a projection of the surface into ordinary 3-dimensional space, $\mathbb{R}^3$, where the top part of the surface is clipped away to enable an inside view. Along the blue central circle you can see a red and a green Möbius band intersection.

Chapter 2

Topological Spaces

$$\boxed{\tau\acute{o}\pi o\zeta\ \lambda\acute{o}\gamma o\varsigma}$$

Topology is derived from Greek: $\begin{cases} \text{Topos} & \longleftrightarrow & \text{place/location} \\ \text{Logos} & \longleftrightarrow & \text{discourse/reason/logic} \end{cases}$

Topology was also called **Analysis Situs**.

2.1 Open Sets. Topology

Let x be any point of the open interval $]a, b[= \{x \mid a < x < b\}$, then we can move x both sides without leaving the interval, i.e. there is a "small" interval inside $]a, b[$ centered at x, although in the interval $[a, b] = \{x \mid a \leq x \leq b\}$, the points a and b of the interval don't satisfy this property. Moving on the left from a, we get points outside the interval $[a, b]$, and similarly on the right of the point b.

Let us recall that a subset O of $\mathbb{R}$ is said to be *open* if, for each $x_0 \in O$, there is some $r > 0$ such that the open interval $]x_0 - r, x_0 + r[$ is contained entirely in O. We can say that the set O is a *neighbourhood* of each of its points. This property is valuable for all points of O. In particular, the intervals $]a, b[,\ a < b$ of $\mathbb{R}$ are open.

Notice that, for example, any subset of the form $[a, b] \subset \mathbb{R}$ does not satisfy this property.

Let us infer some properties of the open intervals:

- $\emptyset$ and $\mathbb{R}$ are open in $\mathbb{R}$.
- Any union of open sets of $\mathbb{R}$ is open in $\mathbb{R}$.
- Any intersection of finitely many open sets of $\mathbb{R}$ is open in $\mathbb{R}$.

Notice that the intersection of infinitely many open sets can be not open. For example, consider the family of open sets $\left]-\frac{1}{n}, \frac{1}{n}\right[$ whose the intersection is $\{0\}$ which is not open.

Now, using this example, we can give the next definition.

Definition 2.1.1. *A* **topological space** *consists of a pair* (X, τ) *where* X *is a set and* $\tau = \{O_i \subseteq X, i \in I\}$, *called* **open sets** *satisfying the following conditions:*

1. $\emptyset$ *and* $X \in \tau$
2. *The union of any collection of open sets is an open set,*
$$\left(\forall O_{i_j} \in \tau, j \in J \subseteq I, \bigcup_{j \in J} O_{i_j} \in \tau\right)$$
3. *The intersection of finitely many open sets is an open set,*
$$\left(\forall O_{i_j} \in \tau, j = 1, \ldots, n, \forall n \in \mathbb{N}, \bigcap_{1 \leq j \leq n} O_{i_j} \in \tau\right)$$

If the topology τ *is clear from the context, we can omit it and we denote* X *for the topological space* (*to distinguish from* X *is a set*).

In a set, all the points have the same role and they cannot be distinguished. The topology says how "nearby" are the points of the set. For example, let $\mathbb{R}$ with the usual distance $d(x, y) = |x-y|$. Then, the points 1 and 2 are nearer than the points 1 and 3. The topology, i.e. the open sets, makes precise the notion of nearness, it is a mathematical definition of nearness. Given a point x_0 of (X, τ), an open set containing x_0 could be view as a set of points "near" x_0; smaller is the open set w.r.t. inclusion, containing the point x_0, "nearer" from x_0 are the points. However, notice that all the points of the topological space (X, τ) are "near" since $X \in \tau$. It means that all the considered points are in the same space.
Let x, y, z be three distinct points of the topological space (X, τ). If there exists $O \in \tau$ such that $x, y \in O$ and $z \notin O$, if for any $O' \in \tau$ such that $x, z \in O'$, then $y \in O'$, we can say that x and y are nearer that x and z. But it is not a measure of the nearness. Such a measure is effective in the metric spaces.

Example 2.1.2.

1. (X, τ) *where* $\tau = \mathcal{P}(X)$, *the set of all subsets of* X, *is a topological space.* τ *is called* **discrete** *topology on* X. *Notice that for any* $x \in X, \{x\} \in \tau$ *so* $\{x\}$ *is open. Any two points are not "very near", same nearness .*
2. (X, τ) *where* $\tau = \{\emptyset, X\}$ *is a topological space.* τ *is called* **trivial** *topology on* X. *Any two points are "near", same nearness, which can explain the word "trivial" to qualify the topology.*
3. *Let* $X = \{a, b, c\}$ *and* $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ *is a topology on* X. (*The points* a *and* b *belong to the open set* $\{a, b\}$, *and they are nearer than* a *and* c, *or* b *and* c.
Notice that the topologies on finite sets have a limited interest, and it is mainly used to find counteresamples. W. Thurston[1] *said "An oddball topic that can lend good insight to a variety of questions."*
4. $(\mathbb{R}, \tau)$ *where* τ *is the set of open sets as we defined at the beginning of this section, is a topology called the* **standard** *or* **canonical** *topology on* $\mathbb{R}$. *In the following, we will simply write* $\mathbb{R}$ *when the topology is the standard one.*

[1] William Paul Thurston (October 30, 1946 - August 21, 2012) was an American mathematician. He was a pioneer in the field of low-dimensional topology. In 1982, he was awarded the Fields Medal for his contributions to the study of 3-manifolds. From 2003 until his death he was a professor of mathematics and computer science at Cornell University.

5. *Let X be a subset of the n-dimensional Euclidean space $\mathbb{R}^n$(with the distance, $d(x,y)$ for $x,y \in \mathbb{R}^n$), defined later in 2.3.2.. A subset O of X is called open (i.e. $O \in \tau$), if given a point $x_0 \in O$, there exists $\delta > 0$ such that $\{x \in X \mid d(x,x_0) < \delta\} \subset O$. Notice that $\emptyset$ is considered as open.*

2.1.1 Exercises

1. For a fixed point $a \in X$, the set of all subsets $O \subset X$ consisting of $\emptyset$ and all the sets containing a, is a topology.

2. For a fixed point $a \in X$, the set of all subsets $O \subset X$ not containing a is not a topology.

3. Let X be an infinite set. The family of subsets consisting of X and all finite subsets of X is not a topology.

4. Let X be an infinite set. The family of all subsets of X consisting of $\emptyset$ and all subsets such the complement is countable is a topology.

2.2 Basis for Topology

The discrete topology on an infinite set consists of all the subsets. All the one-point sets are open and any open set is a union of one-point sets. We can say that the one-point sets "generate" the discrete topology via the union, i.e. any open set O is a union of the one-point sets given by the elements of O.
Let $\mathbb{R}$ be the space where the topology is the standard one. Then a subset is open if for any element of the subset, there exists an open interval containing this element and contained in the subset. So, the open intervals suffice to determine the open sets.
The purpose is to look for some family of subsets from which all the open sets are obtained via the union.

Definition 2.2.1. *Given a topology τ on X, a* **basis** *Σ for τ is a collection of open sets in τ such that every element in τ can be written as a union of elements of Σ. In this case, we say that Σ* **generates** *τ.*

If Σ is a basis for the topology τ, then any Σ' such that $\Sigma \subset \Sigma' \subseteq \tau$ is also a basis for τ.
So, the cardinality of a basis is meaningless.

Remark 2.2.2. *Let Σ be a set of subsets of X. Then, it is possible that there is no topology having Σ as a basis.*

Example 2.2.3. *Let $X = \{a,b,c\}$ and $\Sigma = \{\{a\},\{a,b\},\{b,c\}\}$.
Then $\tau = \{\emptyset, X, \{a\},\{a,b\},\{b,c\}\}$ which is not a topology on X.*

So, we need some criteria to determine whether a subset Σ is a topology.

Proposition 2.2.4. *A family Σ of open sets of X is a basis for the topology iff for every open set O and every point $x \in O$, there is a set $V \in \Sigma$ such that $x \in V \subset O$.*

Proof: $\Longrightarrow$) Let Σ be a basis for τ and O open set. Then $O = \cup_i V_i$ where $V_i \in \Sigma$. So, for $x \in O$, there exists some i such that $x \in V_i \subset O$.
$\Longleftarrow$) Given $O \in \tau, x \in O$, there exists $V_x \in \Sigma$ such that $x \in V_x \subset O$. So $O = \cup_{x \in O} V_x$, hence Σ is a basis for τ. ∎

The basis Σ generates the topology τ if for $O \in \tau$ and any $x \in O$, there exists $V \in \Sigma$ such that $x \in V$.
This is a topology which is the intersection of all topologies on X containing Σ. (exercise).

Proposition 2.2.5. *A family Σ of subsets of X is a basis for a topology on X iff $X = \cup_{O\in\Sigma} O$ and the intersection of any two sets in Σ is the union of sets in Σ.*

Proof: $\Longrightarrow$) Let Σ be a basis for the topology τ on X. Then X is open and the intersection of two open sets is open, so it is a union of sets in Σ.
$\Longleftarrow$) We have to prove that the set of all unions of sets of Σ is a topology τ.
It is clear that $X \in \tau$ and $\emptyset \in \tau$.
It remains to prove that the intersection of two elements of τ is an element of τ. (exercise). ■

Let Σ be a set of subsets of X satisfying one of the equivalent conditions of the propositions above, then there is a unique topology τ on X for which Σ is a basis.

Definition 2.2.6. *A family $\mathcal{S}$ of open sets is called a* **subbasis** *for a topological space (X, τ) if the union equal X and the family of all finite intersections $O_1 \cap O_2 \cap \ldots \cap O_k$ where $O_i \in \mathcal{S}$ for $i = 1, 2, \ldots, k$ is a basis for (X, τ).*
A family $\mathcal{B}(x)$ of open sets containing $x \in X$ is called **basis for a topological space** (X, τ) **at the point** x *if for any open set U containing x, there exists some $O \in \mathcal{B}(x)$ such that $x \in O \subset U$.*

2.2.1 Exercises

1. Show that a possible choice of basis for $\mathbb{R}$ with the usual (also called standard) topology, would be the set of all open intervals, plus the empty set.

2. Show that the set of all open intervals $]a, b[\subset \mathbb{R}, a, b \in \mathbb{Q}$ is a basis for the standard topology on $\mathbb{R}$.

3. Let $X = \mathbb{R}$ and let τ be generated by the collection of half-open intervals $[a, b[$, for all $a < b \in \mathbb{R}$. Show that this is also a topological space, called the **half-open topology**.

4. Find all the bases of topology of a discrete space, a trivial space.

5. Let X be a set and P a partition of X. Show that $\{A \mid A \in P\} \cup \{\emptyset, X\}$ is a basis for a topology on X. Show that $O \subset X$ is open iff $O = \cup_i A_i$, for $A \in P$. Show that the complement of any open set is also open.

6. Show that the set of all intervals $[a, \infty[$, for all $a \in \mathbb{R}$, along with the empty set, is a basis for a topology on $\mathbb{R}$, called the **half-infinite topology**.

2.3 Metric spaces

2.3.1 Definition and Examples

The metric spaces are very important and they can be considered as an introduction to topology.
But, what is a metric (or a distance)? Here are some examples of metrics.
Let a and b be two cities. What is the distance between a and b?
It could be either the distance in kilometers by the roads,
or the distance in kilometers for a bird following the geodesic,
or the time, in minutes, of the shortest journey,

or the cost, in euros, of the cheapest journey by train,
or ...
All these distances are different, but they all share the same following properties:

1. Distances are positive.
2. Two points are zero distance apart if and only if they are the same point.
3. The distance from a to b is the same as the distance from b to a.
4. The distance from a to b via c is at least as great as the distance from a to b.

Definition 2.3.1. *A* **metric space** *is a pair* (X, d) *where* X *is a set and a map*

$$d : X \times X \longrightarrow \mathbb{R}$$

such that

1. $d(x, y) \geq 0$ *for all* $x, y \in X$
2. $d(x, y) = 0$ *iff* $x = y$
3. $d(x, y) = d(y, x)$ *for all* $x, y \in X$
4. $d(x, z) \leq d(x, y) + d(y, z)$ *for all* $x, y, z \in X$ *(Triangle Inequality)*

The map d *is called a* **metric** *(or a* **distance function***) on* X*.*

The map d measures the nearness between two points.

Example 2.3.2. 1. *Let* $\mathbb{R}^n = \{s = \{x_1, \ldots, x_n\} \mid x_i \in \mathbb{R}, \forall i = 1, \ldots, n\}$ *be the* n*-dimensional Euclidean space (with the distance* $d(x, y) = \|x - y\| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$*, for* $x, y \in \mathbb{R}^n$*). In the following, we will simply write* $\mathbb{R}^n$ *for the Euclidean space. Notice that for* $n = 1$*, this distance is the usual one on* $\mathbb{R}$*:* $d(x, y) = \|x - y\| = |x - y|$*.*
Notice that in the Euclidean plane, the Euclidean metric is nothing else than Pythagoras theorem.

2. *Let* X *be an arbitrary set. Define for any* $x, y \in X$

$$d(x, y) = \begin{cases} 1 & \text{if} \quad x \neq y \\ 0 & \text{if} \quad x = y \end{cases}$$

Then d *is a metric that induces the discrete topology on* X*.*

3. *Let* $X = \{x_1, \ldots, x_n\}$ *be a finite set. Let* $A = (a_{ij})_{i,j=1,\ldots,n}$ *be a symmetric* $n \times n$*-matrix with positive values and zeroes on the main diagonal and such that for any* $i, j, k = 1, \ldots, n$*,* $a_{ik} \leq a_{ij} + a_{jk}$*. Define, for any* i, j*,* $d(x_i, x_j) = a_{ij}$*. For example, let* $X = \{x_1, x_2, x_3, x_4\} = \{(0, 0), (1, 0), (1, -1), (-4, 3)\} \subset \mathbb{R}^2$*. And let*

$$A = \begin{pmatrix} 0 & 1 & \sqrt{2} & 5 \\ 1 & 0 & 1 & \sqrt{34} \\ \sqrt{2} & 1 & 0 & \sqrt{41} \\ 5 & \sqrt{34} & \sqrt{41} & 0 \end{pmatrix}$$

d *is a distance which is the restriction to* X *of the Euclidean distance on* $\mathbb{R}^2$*.*

4. *On* $\mathbb{R}$*, there are many different metrics as, for example,* $d_1(a, b) = |a - b|$*,* $d_2(a, b) = 2|a - b|$ *and* $d_3(a, b) = |a^3 - b^3|$ *but* $d(a, b) = |a^2 - b^2|$ *is not a metric.*

2.3.2 Metric Topology

Definition 2.3.3. *Let (X, d) be a metric space. Given $x \in X$ and $r > 0$, the* **open ball** $B(x; r)$ *of radius r and center x is defined by*

$$B(x; r) = \{y \in X \mid d(x, y) < r\}$$

The open balls are the cornerstones of the topology who will be defined from the metric.

Definition 2.3.4. *Let (X, d) be a metric space. Then $O \subset X$ is said to be an* **open set** *if $\forall x \in O, \exists \delta > 0$ such that $B(x; \delta) \subset O$.*

Proposition 2.3.5. *Let (X, d) be a metric space. Then the collection of open sets in X satisfies the following properties:*

1. *$\emptyset$ and $X \in \tau$*
2. *The union of any collection of open sets is an open set.*
3. *The intersection of finitely many open sets is an open set.*

Proof: (exercise) ■

Lemma 2.3.6. *Let (X, d) be a metric space and let $x_0 \in X$. Then, for any $r > 0$, the ball $B(x_0; r)$ is an open set of X.*

Proof: Let $x \in B(x_0; r)$. We have to show that $\exists \delta > 0$ such that $B(x; \delta) \subset B(x_0; r)$. Let $\delta > 0$ where $\delta = r - d(x, x_0)$ so $d(x, x_0) < r$. Suppose $x' \in B(x; \delta)$, then

$$d(x', x_0) \leq d(x', x) + d(x, x_0) < \delta + d(x, x_0) = r$$

Hence $x' \in B(x_0; r)$ and $B(x; \delta) \subset B(x_0; r)$. ■

The metric space (X, d) is a topological space (X, τ_d) where the topology τ_d is the collection of all open sets defined by the metric. It is called **metric topology**.
The set of all balls in a metric space is a basis for the metric topology.
We recall that the metric topology is defined as follows:

$$\boxed{\mathbf{O \subset X \text{ is open iff for any } x \in O, \text{ there exists } \varepsilon_x > 0 \text{ and a ball } B(x; \varepsilon_x) \subset O}}$$

So a metric space is a topological space but the converse is not true.

Example 2.3.7. *Let X be a non empty set which is not a one-point set, with the trivial topology. Then it does not exist metric on X which induced the trivial topology on it.*
(Hint: Suppose there is a metric d which determines the trivial topology on X. Let $x, y \in X$, $x \neq y$, then $d(x, y) = r > 0$. Consider the open ball $B(x; r)$, then $x \in B(x; r)$ and $y \notin B(x; r)$, so $B(x; r)$ is an open set which is distinct of X and no empty. The topology is trivial, so it is a contradiction)

We shall show other examples of topological spaces which are not metric. Moreover, we will give a characterization of topological spaces such that the topology is coming from a metric (5.11), i.e. it will be called **metrization** of topological spaces.

2.3.3 Metric Topologically Equivalent

A natural question is: is it possible that two distances d_1 and d_2 on the same set define the same topology?

Definition 2.3.8. *Let d_1 and d_2 be two distances on the same set X. We say that d_1 and d_2 are* **topologically equivalent** *if distances d_1 and d_2 define the same topology on X.*

Proposition 2.3.9. *Let (X, d_1) and (X, d_2) be two metric spaces on the same set. Then the following assertions are equivalent*

1. *The d_1 and d_2 are topologically equivalent.*
2. *$A \subset X$ is open for d_1 iff A is open for d_2.*
3. *For any $x \in X$ and any $r > 0$, there exist $r' > 0, r'' > 0$ such that $B_{d_1}(x; r') \subseteq B_{d_2}(x; r)$ and $B_{d_2}(x; r'') \subseteq B_{d_1}(x; r)$.*

Proof: (exercise) ∎

The next result is a sufficient (but not necessary) condition for the topological equivalence of metrics.

Proposition 2.3.10. *Two distances d_1 and d_2 are equivalent if for any $x \in X$, there exist positive constants a and b such that, for every point $y \in X$*

$$a.d_1(x, y) \leq d_2(x, y) \leq b.d_1(x, y)$$

Proof: (exercise) ∎

Metrizability of a space depends only on the topology of the space, but properties that involve a specific metric for the space, in general do not. For example, let (X, d) be a metric space. The **diameter** of a subset A of X is $\sup\{d(x, y) \mid x, y \in A\}$. It is defined if A is bounded. Boundedness is not a topological property. Consider $\mathbb{R}$ equipped with the metric d_1 defined by $d_1(x, y) = |x - y|$. Then $\mathbb{R}$ is not bounded. Let d_2 be the metric defined by $d_2(x, y) = \inf\{d_1(x, y), 1\}$. Then $\mathbb{R}$ is bounded for this metric d_2. It is easy to verify that the two metric d_1 and d_2 are equivalent. This example proves that the condition of the previous proposition is not necessary.

There exists a stronger version of equivalence of distances.

Definition 2.3.11. *Let d_1 and d_2 be two distances on the same set X. We say that d_1 and d_2 are* **strongly-equivalent** *if there exist two constants $A, A' > 0$ such that*

$$d_1(x, y) \leq A.d_2(x, y) \text{ and } d_2(x, y) \leq A'.d_1(x, y) \text{ for all } x, y \in X$$

If the two metrics d_1 and d_2 are strongly-equivalent, then they are equivalent. An intuitive reason why topological equivalence does not imply strong equivalence is that bounded sets under one metric are also bounded under a strongly equivalent metric, but not necessarily under a topologically equivalent metric.

Example 2.3.12. *Let $X = \{x \in \mathbb{R} \mid x > 0\}$ and let $d_1(x, y) = |x - y|$, $d_2(x, y) = \left|\frac{1}{x} - \frac{1}{y}\right|$. Then d_1 and d_2 are not strongly-equivalent. However, they define the same metric topology. (exercise). Suppose $d_1(x, y) \longrightarrow 0$, then we have to prove that $d_2(x, y) \longrightarrow 0$ is not always true. Choose $x = \frac{1}{n}$ and $y = \frac{1}{2n}$. Then $\left|\frac{1}{n} - \frac{1}{2n}\right| = \frac{1}{2n} \to 0$ when $n \to \infty$, on the other hand $\left|\frac{1}{\frac{1}{n}} - \frac{1}{\frac{1}{2n}}\right| = n \to \infty$ when $n \to \infty$*

Recall that a sequence $(x_n)_{n\geq 1}$ in a metric space (X, d) converges to $x \in X$ if $\forall \varepsilon > 0, \exists N > 0$ such that $d(x_n, x) < \varepsilon, \forall n > N$.

Proposition 2.3.13. *If the distances d_1 and d_2 are strongly-equivalent, then $x_n \xrightarrow{d_1} x$ iff $x_n \xrightarrow{d_2} x$.*

Proof: (exercise) ■

Proposition 2.3.14. *If the distances d_1 and d_2 are strongly-equivalent, then $d_1(x_n, y_n) \xrightarrow{d_1} 0$ iff $d_2(x_n, y_n) \xrightarrow{d_2} 0$.*

Proof: (exercise) ■

Proposition 2.3.15. *Let $X = \mathbb{R}^n$ and define the three following distances on X :*

- $d_1(x, y) = \|x - y\|_1$ *where* $\|x\|_1 = |x_1| + \cdots + |x_n|$
- $d_2(x, y) = \|x - y\|_2$ *where* $\|x\|_2 = \sqrt{\displaystyle\sum_{i=1}^{n} x_i^2}$
- $d_\infty(x, y) = \|x - y\|_\infty$ *where* $\|x\|_\infty = \max\{|x_1|, \ldots, |x_n|\}$

Then the distances d_1, d_2 and d_∞ are strongly-equivalent.

Proof: (exercise) ■

Remark 2.3.16. *Not all the metrics are strongly-equivalent.*

Example 2.3.17. *Let $X = \mathcal{C}([0, 1]; \mathbb{R})$ (the space of all continuous functions from the unit interval in $\mathbb{R}$). Let define the two distances (exercise) on X:*

$d_1(f, g) = \displaystyle\int_0^1 |f(x) - g(x)|dx$

$d_2(f, g) = \max_x\{|f(x) - g(x)|\}$

Then d_1 and d_2 are not strongly-equivalent. (exercise). (Hint: Consider the maps f and g such that $g = 0$ and given $0 < \varepsilon \ll 1, f(x) = 0, x \geq \varepsilon$ and for $0 \leq x \leq \varepsilon, f(x) = ax + b$ where $f(0) = 1$ and $f(\varepsilon) = 0$).

Proposition 2.3.18. *Let (X, d_1) and (X, d_2) be two metric spaces on the same set where the distances d_1 and d_2 are strongly-equivalent. Then they define the same topology, i.e. they are equivalent, i.e. $\tau_1 = \tau_2$.*

Proof: It is enough to notice that $d_1(x, y) \leq A.d_2(x, y)$ implies $B_{d_2}(x_0; r) \subset B_{d_1}(x_0; A.r)$ and $d_2(x, y) \leq A'.d_1(x, y)$ implies $B_{d_1}(x_0; r) \subset B_{d_2}(x_0; A'.r)$. ■

2.3.4 Exercises

1. Let $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$ and the following maps $d_1, d_2, d_\infty, d_d, d_R$:

   (a) $d_1(x, y) = |x_1 - y_1| + |x_2 - y_2|$.

   (b) $d_2(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$.

(c) $d_\infty(x,y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$.

(d) $d_d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$ This is called *discrete metric.*

(e) $d_R(x,y) = \begin{cases} d_2(x,y) & \text{if } x, y, 0 \text{ are collinear} \\ d_2(x,0) + d_2(0,y) & \text{if not} \end{cases}$

Which of the above maps are metrics on $\mathbb{R}^2$.

2. Let $d : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the map defined as follows:
For $x = (x_1, x_2)$ and $y = (y_1, y_2)$, $d(x,y) = (x_1 - y_1)^2 + (x_2 - y_2)^2$. Is d a metric?

3. Describe the subsets $\{x \in \mathbb{R}^2 \mid d(x,0) = 1\}$ for $d = d_1, d_2, d_\infty, d_d, d_R$.
Let $x = \left(\frac{1}{2}, 0\right), y = \left(\frac{1}{2}, \varepsilon\right)$ where $\varepsilon > 0$, and small enough.
Determine $d_2(x,y)$ and $d_R(x,y)$.

4. Let (X,d) be a metric space and $A \subset X$. Show that the metric d induces a metric d_A on A making (A, d_A) a metric space. Describe the open balls in (A, d_A).

5. Let X be an arbitrary set. Define for $x, y \in X$

$$d(x,y) = \begin{cases} 1 & \text{if} \quad x \neq y \\ 0 & \text{if} \quad x = y \end{cases}$$

Show that d is a metric on X and describe the topology induced on X.

6. Let $X = \mathbb{R}^2$ and for $z = (x,y), z' = (x', y')$ in X, define

$$d(z,z') = \begin{cases} |y - y'| & \text{if} \quad x = x' \\ |y| + |y'| + |x - x'| & \text{if} \quad x \neq x' \end{cases}$$

Show that d is a metric on X and describe the path with the shortest distance between two points.

7. Let (X,d) be a metric space.
Show that the map d_b defined by $d_b(x,y) = \dfrac{d(x,y)}{1 + d(x,y)}$ is a metric on X. Deduce that for any $x, y \in X, d_b(x,y) \leq 1$.
Show that d and d_b are two equivalent metrics.

8. Let d be the Euclidean metric on $\mathbb{R}^2$ and consider the metric space $(\mathbb{R}^2, d_R)$ where

$$\begin{aligned} d_R : \mathbb{R} \times \mathbb{R} &\longrightarrow \mathbb{R}_{\geq 0} \\ (x,y) &\longmapsto \begin{cases} d(x,y) & \text{if} \quad x, y, 0 \quad \text{collinear} \\ d(x,0) + d(0,y) & \text{if not} \end{cases} \end{aligned}$$

(a) Let $p_0 \in \mathbb{R}^2$ and $r > 0$. Determine the ball $B(p_0, r) = \{q \in \mathbb{R}^2 \mid d_R(p_0, q) < r\}$ and $\mathrm{Cl}(B(p_0,r)) = \{q \in \mathbb{R}^2 \mid d_R(p_0,q) \leq r\}$, where r is going from 0 to $+\infty$. (Hint: Consider the cases $r < d(o, p_0)$ and $r \geq d(o, p_0)$).

(b) Given $q \in B(p_0, r)$, show that there exist a ball $B(q, r') \subset B(p_0, r)$.

9. Let $\mathbb{Z}$ be the set of integers and let $p > 0, p$ prime. Given two integers $m, n \in \mathbb{Z}$, there exists a unique $t \in \mathbb{Z}$ such that $m - n = p^t.k$ where $k \in \mathbb{Z}$, k not divisible by p. Let define

$$\begin{aligned} d : \mathbb{Z} \times \mathbb{Z} &\longrightarrow \mathbb{R} \\ (m, n) &\longmapsto 0 \text{ if } m = n \\ (m, n) &\longmapsto \frac{1}{p^t} \text{ if not} \end{aligned}$$

   (a) Prove that d is a distance and $\mathbb{Z}$ is a metric space.

   (b) Let $p = 3$. Describe the open ball $B(0; 1)$.

10. Let $\mathbb{R}^2$ the Euclidean space and let a and b two distinct points of $\mathbb{R}^2$. Define the segment $[a, b]$ determined by the two points a and b. Let m be the middle of $[a, b]$. Show that m is the only point of $\mathbb{R}^2$ such that $d(a, m) = d(m, b) = \frac{1}{2}d(a, b)$.

11. Let A be a non-empty subset of the metric space (Y, d). The **diameter** of A is $\sup\{d(a, b) \mid a, b \in A\}$. It is denoted $\delta(A)$. It is said that A is **bounded** if it is no empty and its diameter is finite.
    Let X be an arbitrary topological space. Let $C(X, Y; d)$ be the set of bounded continuous map of X into Y, i.e.

$$C(X, Y; d) = \{f \in Y^X \mid f \text{ continuous and } \delta f(X) < \infty\}$$

    Define $D(f, g) = \sup\{d\big(f(x), g(x)\big) \mid x \in X\}$ for any $f, g \in C(X, Y; d)$. Show that D is a distance on $C(X, Y; d)$.

12. Let $\mathbb{R}^2$ be the real plane equipped with the two distances d_1, d_2 defined as follows: $d_1(x, y) = |x_1 - y_1| + |x_2 - y_2|$ and $d_2(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$.
    Let $a = (1, 1),\ b = (-1, 1),\ c = (1, -1),\ e = (-1, -1)$ and $o = (0, 0)$ be five points. Compute $d_i(a, o), d_i(b, o), d_i(c, o), d_i(e, o), d_i(a, b), d_i(a, c), d_i(b, e), d_i(c, e)$ and $d_i(a, e), i = 1, 2$.
    Let $a = (a_1, a_2),\ b = (b_1, b_2)$ be two distinct points of $\mathbb{R}^2$. The segment $[a, b]$ is the set $\{((1-t)a_1 + tb_1, (1-t)a_2 + tb_2) \mid 0 \le t \le 1\}$. The middle of $[a, b]$ is the point where $t = \frac{1}{2}$.
    Determine the middle of the segments $[a, e]$, and $[b, c]$.
    Is it possible to characterize the middle of a segment in term of the distance $d_i, i = 1, 2$?

2.4 Poset of Topologies

Let (X, τ_1) and (X, τ_2) be two topological spaces on the same set X.

Definition 2.4.1. *If $\tau_1 \subset \tau_2$, then the topology τ_2 is said to be* **finer***, i.e. (***stronger***,* **larger***) than the topology τ_1. And τ_1 is said to be* **coarser***, i.e. (***weaker***,* **smaller***) than the topology τ_2.*

The relation "finer" between the topologies on the set X is an order relation.

Example 2.4.2. *1. The discrete topology on X is the finest (i.e. strongest or largest) one.*

2. The trivial topology on X is the coarsest (i.e. weakest or smallest) one.

Remark 2.4.3. *Two topologies (X, τ_1) and (X, τ_2) on the same set X are not necessarily comparable. In other words, the set of all topologies on the set X is not a linear order set. It is a poset.*

Example 2.4.4. *Let $X = \{a, b\}$ and $\tau_1 = \{\emptyset, \{a\}, X\}$, $\tau_2 = \{\emptyset, \{b\}, X\}$. Then neither $\tau_1 \subset \tau_2$ nor $\tau_2 \subset \tau_1$. So these two topologies are not comparable.*

2.4.1 Exercises

1. Define all the topologies on the set $\{a, b\}$ and describe their poset.

2. (a) Let τ_1 consist of the empty set together with all subsets of $\mathbb{R}$ whose complement is finite. Prove that τ_1 is a topology on $\mathbb{R}$. Prove that every set in τ_1 is open in the usual topology, but not conversely.

   (b) Let τ_2 consist of all sets O such that for each $x \in O$, there is an interval $[a, b[$ with $x \in [a, b[\subset O$. Prove that τ_2 is a topology on $\mathbb{R}$.

   (c) Compare the three topologies τ_1, τ_2 and the standard one.

3. Let define the topology τ_ℓ on the set of real numbers $\mathbb{R}$ generated by the family of all half-open intervals of the form $[a, b[= \{x \mid a \leq x < b\}$, where $a < b$.
   Let define the topology τ_K on the set of real numbers $\mathbb{R}$ generated by the family of all open intervals $]a, b[$ where $a < b$ along with all sets of the form $]a, b[\setminus K$ where $K = \left\{\frac{1}{n} \mid n \in \mathbb{N}_{>0}\right\}$.
   Show that the topologies τ_ℓ and τ_K are strictly finer than the standard topology on $\mathbb{R}$, but are not comparable with one another.

2.5 Closed Sets

2.5.1 Definition and Properties

Definition 2.5.1. *Let (X, τ) be a topological space. A subset $C \subset X$ is said to be* **closed** *if $X \setminus C$ is open.*

The set $C \subset X$ is closed iff the complement $X \setminus C$ is open and the set $O \subset X$ is open iff the complement $X \setminus O$ is closed.
Notice that the property of being closed is not the negation of the property of being open.

Exercice 2.5.2. *Find examples of a topological space (X, τ) and subsets such that*

1. *They are both open and closed*

2. *They are neither open nor closed.*

Proposition 2.5.3. *Let (X, τ) be a topological space. Then the collection of closed sets has the following properties*

1. *$\emptyset$ and X are closed.*

2. *The intersection of any collection of closed sets is a closed set.*

3. *The union of finitely many closed sets is a closed set.*

Proof: (exercise) ■

Remark 2.5.4. *A topology can be defined in terms of closed sets. It is the duality "open-closed".*

OPEN	CLOSED
$O = X \setminus C$	$C = X \setminus O$
$\emptyset = X \setminus X$	$X = X \setminus \emptyset$
$X = X \setminus \emptyset$	$\emptyset = X \setminus X$
$\bigcup\limits_{i\in I} O_i = X \setminus \bigcap\limits_{i\in I} C_i = \bigcup\limits_{i\in I}(O_i \setminus C_i)$	$\bigcap\limits_{i\in I} C_i = X \setminus \bigcup\limits_{i\in I} O_i = \bigcap\limits_{i\in I}(X \setminus O_i)$
$\bigcap\limits_{i\leq k} O_i = X \setminus \bigcup\limits_{i\leq k} C_i = \bigcap\limits_{i\leq k}(O_i \setminus C_i)$	$\bigcup\limits_{i\leq k} C_i = X \setminus \bigcap\limits_{i\leq k} O_i = \bigcup\limits_{i\leq k}(X \setminus O_i)$

Example 2.5.5.
- *Let $\mathbb{R}$ with the standard topology. Then for any $a, b \in \mathbb{R}$, $a \leq b$, the interval $[a, b]$ is closed. In particular, the one-point set $\{a\}$ is closed for any $a \in \mathbb{R}$.*
- *Let (X, τ) be a topological space where τ is the discrete topology. Then any subset $A \subseteq X$ is both open and closed.*

Remark 2.5.6. *Being open and closed are not mutually exclusive. In fact, subsets that are both open and closed often exist, and play a special role as we will see in 5.2.*

Example 2.5.7. *Let $a, b \in \mathbb{Z}$, $a \neq 0$. Denote $N_{a,b} = \{\cdots, b-2a, b-a, b, b+a, b+2a, \cdots\} = \{an + b \mid n \in \mathbb{Z}\}$. The set of all $N_{a,b}, a, b \in \mathbb{Z}, a \neq 0$ is a basis for a topology on $\mathbb{Z}$.*
- $\mathbb{Z} = \bigcup_{a,b} N_{a,b} = N_{1,0}$.
- For any $n \in \mathbb{Z}, n \in N_{a_1,b_1} \cap N_{a_2,b_2} \Longrightarrow n \in N_{a,b}$ where $a = lcm(a_1, a_2), b = n$.
Notice that $N_{a_1,b_1} \cap N_{a_2,b_2}$ can be empty. Take $a_1 = a_2, b_1 \neq b_2$ and $b_1 \neq a_1$.
Let $n \in N_{a_1,b_1} \cap N_{a_2,b_2}$, then $n = a_1p + b_1 = a_2q + b_2$. Let α be the commun factor of a_1 and a_2, i.e. $a_1 = \alpha.a'_1, a_2 = \alpha.a'_2$. Define $a = a_1a'_2 = a_2.a'_1$. Then

$$a_1p + b_1 = a_2q + b_2 \Longrightarrow a.k + a_1p + b_1 = a.k + a_2q + b_2 \quad \textit{for some } k \in \mathbb{Z}$$

and

$$a_1(p + a'_2k) + b_1 = a_2(q + a'_1k) + b_2 = ak + n$$

so for any $k \in \mathbb{Z}$, these integers belongs to $N_{a_1,b_1} \cap N_{a_2,b_2}$.
Moreover, we have the two following properties.
1. Every open set contains some $N_{a,b}$. Then a finite set is never open and its complement is never closed.
2. $N_{a,b}$ is both open and closed. It is closed: $N_{a,b} = \mathbb{Z} \setminus \bigcup\limits_{j=1}^{a-1} N_{a,b+j}$.

As an amazing consequence, we deduce that the set of prime numbers is infinite, (topological proof of a question in Number theory).
The only integers which are no multiple of a prime are ± 1, i.e. each integer has a decomposition in a product of prime numbers, so $\bigcup\limits_{p\ prime} N_{p,0} = \mathbb{Z} \setminus \{-1, +1\}$ which, from 1., is not closed, so it is not a finite union of closed sets. From 2., $N_{p,0}$ is closed and $\bigcup\limits_{p\ prime} N_{p,0}$ is not a finite union, so the set of prime numbers is infinite.

2.5.2 Basis for Closed Sets

A topology is defined either by the open sets or dually by the closed sets. We defined a basis for the open sets in 2.2. We can define, dually, a basis for the closed sets.

Definition 2.5.8. *A* **basis** *for the closed sets for the topological space X, is a collection Ξ of closed sets such that every closed set is an intersection of elements of Ξ.*

Proposition 2.5.9. *A set Ξ of closed sets is a basis for the topology of X iff for any closed set C and any point $x \notin C$, there exist an element $B \in \Xi$ containing C and not containing the point x.*

Proof: Exercise (It is the dual of prop. 2.2.4) ■

2.5.3 Zariski Topology

Let $\mathbb{F}$ be a commutative field, and let $\mathcal{A}_n(\mathbb{F}) = \mathbb{F}^n$.
A **Zariski**[2] set of $\mathcal{A}_n(\mathbb{F})$ is $V = \{x := (x_1, \ldots, x_n) \in \mathbb{F}^n \mid P_i(x) = 0, P_i \in \mathbb{F}[X_1, \ldots, X_n], i \in I\}$.

Proposition 2.5.10. *The set of all Zariski sets is the set of closed sets for a topology on $\mathcal{A}_n(\mathbb{F})$, called* **Zariski topology**.

Proof: Notice that V is determined by the ideal $(\{P_i(X_1, \ldots, X_n) \mid i \in I\})$.

- $\emptyset$ is a Zariski set; let $P = 1$.
- Let V, W be two Zariski sets defined by $(P_i)_{i \in I}$, resp. $(Q_j)_{j \in J}$. We denote $(x_1, \ldots, x_n) = x$. Then $V \cup W = \{x \in \mathbb{F}^n \mid P_i Q_j(x) = 0, i \in I, j \in J\}$.
- Let $V_j, j \in J$, be a family of Zariski sets defined by the family of polynomials $(P_{ij})_{i \in I_j}$ for $j \in J$. Then $\bigcap_{j \in J} V_j = \{x \in \mathbb{F}^n \mid P_{ij}(x) = 0, j \in J, i \in I\}$.

Example 2.5.11. *Every ideal of $\mathbb{C}[X]$ is principal, so every Zariski (closed) set in $\mathcal{A}_1(\mathbb{C})$ is the set of zeros of one polynomial $P(X)$. $\mathbb{C}$ is an algebraically closed field, so every polynomial factorizes, $P(X) = a(X - a_1)^{n_1} \cdots (X - a_k)^{n_k}$, and $V(P) = \{a_1, \ldots, a_k\}$.*
Then the closed set for the Zariski topology on $\mathcal{A}_1(\mathbb{C})$ are $\emptyset$, the finite subsets, and $\mathcal{A}_1(\mathbb{C})$.

Remark 2.5.12. *Determine the Zariski topology on $\mathcal{A}_1(\mathbb{R})$.*
If $\mathbb{F}$ is a finite field, then the Zariski topology on $\mathcal{A}_1(\mathbb{F})$ is the discrete topology.

Notice that we define similarly the Zariski topology on the projective spaces.

2.5.4 Exercises

1. Find a family of closed subsets of $\mathbb{R}$ whose union is not closed.
2. Let (X, d) be a metric space. Let $x_0 \in X$ and let $r > 0$. Show that $\{y \in X \mid d(x_0, y) \leq r\}$ is closed. It is called *closed ball* of center x_0 and radius r.
3. Let $P(x_0, x_1, \ldots, x_n)$ be a homogeneous polynomial of degree k, i.e. $P(\lambda x_0, \lambda x_1, \ldots, \lambda x_n) = \lambda^k P(x_0, x_1, \ldots, x_n)$. Let $\mathbf{P}_n(\mathbb{F}) = \mathbb{F}^{n+1} \setminus \{0\}/\sim$ where $z \sim z'$ if $z' = \lambda z, \lambda \neq 0$. Denote $\bar{z}$ the equivalence class of z.
   Define $V_P = \{\bar{z} \in \mathbf{P}_n(\mathbb{F}) \mid P(z) = 0\}$.
   Define the coarsest topology on $\mathbf{P}_n(\mathbb{F})$ in which the V_P are the closed sets. This is the so called Zariski topology.
   Show that this topology is not Hausdorff and every nonempty nopen set is dense in $\mathbf{P}_n(\mathbb{F})$.
   Determine the closed sets in $\mathbf{P}_1(\mathbb{R})$.

[2]Oscar Zariski (April 24, 1899 ? July 4, 1986) was a Russian-born American mathematician and one of the most influential algebraic geometers of the 20th century.

2.6 Neighbourhoods

Definition 2.6.1. *Let (X, τ) be a topological space and $x \in X$. A subset $N \subset X$ is said to be a* **neighbourhood** *of x if $x \in N$ and there exists an open set O containing x and contained in N.*

Sometimes, a neighbourhood of x is defined as an open set containing x. If so, we call it an open neighbourhood.
A closed neighbourhood of x is a closed set containing a open neighbourhood of x.
A neighbourhood of the point x is the set of all the points "near" x, according the common meaning. It does not need to be either open or closed.

Lemma 2.6.2. *Let (X, τ) be a topological space. A subset $N \subset X$ is open iff N is a neighbourhood of each of its point.*

Proof:

$\Longrightarrow$) An open set N is a neighbourhood of each of its points, by definition of neighbourhood.

$\Longleftarrow$) Conversely, suppose $N \subset X$ is a neighbourhood of each of its points. Let $x \in N$, then there exists an open set O_x containing x and contained in N. Thus N is open since it is the union of all the open sets O_x where $x \in N$. ∎

$\boxed{\mathbf{O \subset X \text{ is open iff } \forall x \in O, \exists N_x \text{ open neighbourhood of } x \text{ such that } N_x \subset O}}$.

In 2.8. we will give a characterization of closed sets by neighbourhoods, i.e. the "dual" of the charaterization of open sets.

2.6.3. Properties of neighbourhoods.
Let (X, τ) be a topological space.

1. *Let N_1 and $N_2 \subset X$ be two neighbourhoods of $x \in X$, then $N_1 \cap N_2$ is a neighbourhood of x.*

2. *Let $N \subset X$ be a neighbourhood of $x \in X$ and $M \supset N$, then M is also a neighbourhood of x.*

2.6.1 Exercises

1. Let X be a topological space. Show that the set $\mathcal{N}$ of all neighbourhoods satisfy the following properties:

   - Every neighbourhood of x contains x.
   - Any finite intersection of neighbourhoods is a neighbourhood.
   - Let $N(x)$ be a neighbourhood of x. Then there exists a neighbourhood $N'(x) \subset N(x)$ of x such that $N(x)$ is a neighbourhood of each point of $N'(x)$.

2. Give an explicit description of all neighbourhoods of a point in

   - a discrete space.
   - a trivial space.

2.7 Separability I

2.7.1 Hausdorff Spaces

Definition 2.7.1. *A topological space (X, τ) is said to be* **Hausdorff**[3] *if any distinct points $p \neq q$ of X have disjoint neighbourhoods.*
($\forall\, p, q \in X, p \neq q, \exists\; O_p$, open neighbourhood of p and $\exists\; O_q$, open neighbourhood of q such that $O_p \cap O_q = \emptyset$).

Hausdorff spaces are named for Felix Hausdorff, one of the founders of topology. Hausdorff's original definition of a topological space (in 1914) included the Hausdorff condition as an axiom.

Example 2.7.2. *1. $\mathbb{R}$, with the standard topology, is Hausdorff.*

2. A topological space, with the discrete topology, is Hausdorff.

3. A metric space is Hausdorff.

4. The affine line $\mathcal{A}_1(\mathbb{R})$ is not Hausdorff for the Zariski topology. More generally, if the field $\mathbb{F}$ is infinite, the affine line $\mathcal{A}_1(\mathbb{F})$ is not Hausdorff.

Remark 2.7.3. *A non Hausdorff space cannot be a metric space.*

Exercice 2.7.4. *Let $(\mathbb{R}, \tau)$ be the topological space where $\tau = \{A \subseteq \mathbb{R} \mid \mathbb{R} \setminus A \text{ finite}\} \cup \{\emptyset\} \cup \{\mathbb{R}\}$.*

1. Show that O open for τ is open for the standard topology.

2. Show that τ is not a metric topology.

Proposition 2.7.5. *In a Hausdorff space, any one-point set is closed.*

Proof: Consider the one-point set $\{p\}$. For any $q \neq p$, there exist an open set O_q not containing p. The union $\cup O_q, q \neq p$ is open and is equal to $X \setminus \{p\}$, so $\{p\}$ is closed. ■

Proposition 2.7.6. *Let X be a topological space. The followings are equivalent*

1. X is Hausdorff.

2. The intersection of closed neighbourhoods of any point is the one-point set.

3. The diagonal $\Delta = \{(x, x) \mid x \in X\}$ is closed in $X \times X$.

Proof:

- $1 \implies 2$. Let $x, y \in X, x \neq y$, then there exist two disjoint open neighbourhoods O_x, O_y. Then $O_y \subset X \setminus \{x\}$ and $X \setminus \{x\}$ is open, hence the one-point set $\{x\}$ is closed.
- $2 \implies 1$. Let $y \neq x$ and O_x an open neighbourhood of x. Let $C_{O_{x,i}}, i \in I$ denote the family of closed neighbourhood of x containing O_x. The intersection of all closed sets containing O_x is a closed neighbourhood of x. By assumption, the intersection of these closed neighbourhoods for all O_x is $\bigcap\limits_{O_x} C_{O_x} = \{x\}$, i.e. there is some C_{O_x} such that $y \notin C_{O_x}$, so $y \in X \setminus C_{O_x}$ open set that is disjoint from O_x open neighbourhood of x.

[3] Felix Hausdorff is a German mathematician (1868-1942)

- $1 \Longrightarrow 3$. Let $(x, y) \notin \Delta$, i.e. $x \neq y$, so there exist disjoint open neighbourhoods O_x, O_y. Then $O_x \times O_y$ is an open neighbourhood of $(x, y) \in X \times X$ that does not meet the diagonal Δ. Then Δ is closed in $X \times X$.
- $3 \Longrightarrow 1$. Let $(x, y) \notin \Delta$, i.e. $x \neq y$. Then there exists an open neighbourhood of the point x, $O_{(x,y)} = O_x \times O_y$ where O_x, (resp. O_y) is an open neighbourhood of x (resp. y), such that $O_{(x,y)} \cap \Delta = \emptyset$, i.e. $O_x \cap O_y = \emptyset$ and X is Hausdorff.

■

2.7.2 Separability Conditions

Hausdorff spaces are the most common separated spaces in topology, but there exist some other separability conditions, and in the following, we give some of them.
Let (X, τ) be a topological space, and two distinct points $p, q \in X, p \neq q$.

Definition 2.7.7.
- $\mathbf{T}_0$**-spaces** *also called* **Kolmogorov spaces**[4]
  There exists $O \in \tau$ such that O contains only one of the two points p, q.
- $\mathbf{T}_1$**-spaces** *also called* **Frechet spaces**[5].
  There exists $O \in \tau$ such that $p \in O,\ q \notin O$, and there exists $O' \in \tau$ such that $p \notin O',\ q \in O'$.
  X is a $\mathbf{T}_1$-spaces iff $\forall p \in X, \{p\}$ is closed.
- $\mathbf{T}_2$**-spaces** *also called* **Hausdorff spaces**.
- $\mathbf{T}_3$**-spaces**[6]
  $\forall p, \forall F'$ closed set, $p \notin F', \exists O, O' \in \tau$ such that $p \in O, F' \subseteq O', O \cap O' = \emptyset$.
  X is a $\mathbf{T}_3$-spaces iff any point has a fundamental system of closed neighbourhoods.
- $\mathbf{T}_4$**-spaces**[7]
  X is a $\mathbf{T}_1$-space and given two disjoint closed sets F, F', there exist two disjoint open sets O, O' such that $F \subset O$ and $F' \subset O'$.

Example 2.7.8. *Let $X = \{0, 1\}$ and $\tau = \{\emptyset, \{0\}, X\}$. Then X is $\mathbf{T}_0$ and not $\mathbf{T}_1$.*
Let X be an infinite set. Define the topology τ on X where $\tau = \{\emptyset\} \cup \{O \mid\ X \setminus O$ is finite$\}$. Then X is $\mathbf{T}_1$ but not Hausdorff.

Definition 2.7.9. *Suppose that the one-point sets are closed, i.e. the space is $\mathbf{T}_1$.*

- *A space satisfying $\mathbf{T}_3$ is called* **regular**.
- *A space satisfying $\mathbf{T}_4$ is called* **normal**.

[4] Andrey Nikolaevich Kolmogorov, (25 April 1903-20 October 1987) was a Russian mathematician who made significant contributions to the mathematics of probability theory, topology, intuitionistic logic, turbulence, classical mechanics, algorithmic information theory and computational complexity.

[5] Maurice Fréchet (September 2, 1878, June 4, 1973) was a French mathematician. He made major contributions to the topology of point sets and introduced the entire concept of metric spaces. He also made several important contributions to the field of statistics and probability, as well as calculus.

[6] Leopold Vietoris (4 June 1891-9 April 2002), was an Austrian mathematician known for his contribution to topology.

[7] Heinrich Franz Tietze (31 August 1880 - 17 February 1964) was an Austrian mathematician, famous for the Tietze extension theorem on functions from topological spaces to real numbers. He also developed the Tietze transformations for group presentations, and was the first to pose the group isomorphism problem. Tietze's graph is also named after him; it describes the boundary of a subdivision of the Möbius strip into six mutually-adjacent regions, found by Tietze as part of an extension of the four color theorem to non-orientable surfaces.

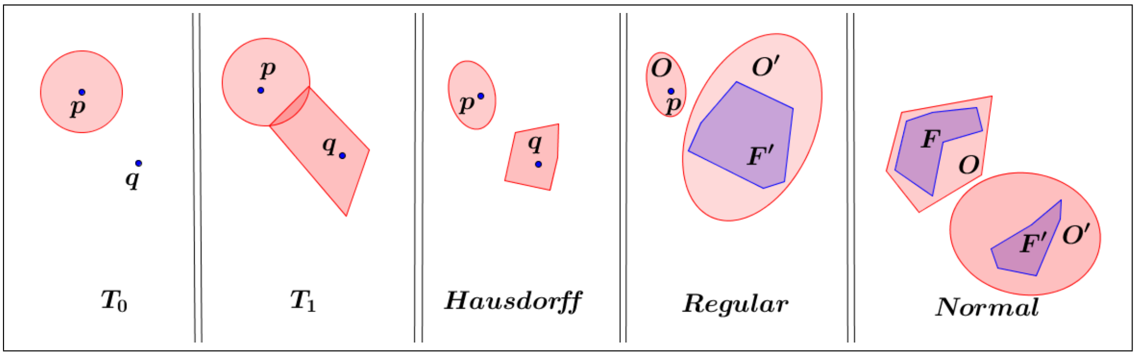


Figure 2.1

Proposition 2.7.10. *A normal space is regular and Hausdorff.*
A regular space is a Hausdorff space.

Proof: The proof is clear from the definitions. ■

2.7.3 Exercises

1. Let $\mathcal{A}_1(\mathbb{F})$, where $\mathbb{F}$ infinite, equipped with the Zariski topology. Then, for any $p, q \in \mathcal{A}_1(\mathbb{F}), p \neq q$, show that there exists an open neighbourhood N_p of p such that $q \notin N_p$.

2. A topological space, with the trivial topology, is not Hausdorff (if it is not a one-point set).

3. Let X be a Hausdorff space and let $x \in X$. Then $\{x\}$ is closed.
If X is a $\mathbf{T}_1$-space, show that every one-point set is closed.

4. Let $(\mathbb{R}, \tau)$ be the topological space where ($O \in \tau$ if either $O = \emptyset$ or $\mathbb{R} \backslash O$ is a finite set). Show that $\mathbb{R}$ with this topology is not Hausdorff.

5. Let $(\mathbb{N}, \tau)$ be the topological space where ($O \in \tau$ if either $O = \emptyset$ or $\mathbb{N} \backslash O$ is a finite set). Show that $\mathbb{N}$ with this topology is not Hausdorff.

6. Let $X = [0, +\infty[$ and $\tau = \{\emptyset, X,]a, +\infty[$, for any $a \in X\}$. Prove that τ is a topology on X which is not Hausdorff.

7. We defined the **half-infinite topology** on $\mathbb{R}$ to be generated by the set of all intervals $[a, \infty[$, for all $a \in \mathbb{R}$, along with the empty set. Is it a Hausdorff space?

8. Let G be a topological group[8] which is a T_1-space is Hausdorff.

2.8 Interior. Closure. Dense. Frontier. Adherent Points. Accumulation Points

In this section, openness and closeness are dually characterized.

[8] A topological group is a group $(G, e, *)$ and a topological space (G, τ) such that the mappings $(g, h) \longmapsto g * h$ and $g \longmapsto g^{-1}$ are continuous

2.8.1 Interior. Closure

Let (X, τ) be a topological space and let $A \subset X$.

Definition 2.8.1. 1. *The* **interior** *of A is the greatest (w.r.t. inclusion) open set contained in A, i.e. the union of all the open sets contained in A. It is denoted* Int*(A).*

2. *The* **closure** *of A is the smallest (w.r.t. inclusion) closed set containing A, i.e. the intersection of all the closed sets containing A. It is denoted* $\mathrm{Cl}(A)$.

We saw how nearness between two points is defined in terms of open sets. This notion that we can also call "closeness", can be defined in terms of closed sets, i.e. a point is "close" to a set if it belongs to the closure of the set.

Remark 2.8.2. *Notice that* $\mathrm{Int}(A)$ *is an open set as union of open sets and* Cl*(A) is a closed set as intersection of closed sets.*

$$A \textit{ open iff } A = \mathrm{Int}(A)$$

$$A \textit{ closed iff } A = \mathrm{Cl}(A)$$

The definitions of interior and closure are "dual" under open $\longleftrightarrow$ closed, greatest $\longleftrightarrow$ smallest, union $\longleftrightarrow$ intersection, contained $\longleftrightarrow$ containing.
We recall that open and closed are not the negation from each other.

Proposition 2.8.3. $x \in \mathrm{Cl}(A)$ *iff for any neighbourhood N_x of x, $N_x \cap A \neq \emptyset$.*
$x \in \mathrm{Int}(A)$ *iff there exists some neighbourhood N_x of x such that $N_x \subset A$.*

Proof:

$$\begin{aligned} x \in \mathrm{Cl}(A) \quad &\text{iff} \quad x \in F \text{ for any closed set } F \supset A \\ &\text{iff} \quad x \notin X \setminus F \text{ for any closed set } F \supset A \\ &\text{iff} \quad x \notin O \text{ for any open set } O \subset X \setminus A \\ &\text{iff} \quad \text{for any neighbourhood } N_x \text{ of } x,\ N_x \cap A \neq \emptyset \end{aligned}$$

The second assertion follows from the definition of the interior: $x \in \mathrm{Int}(A)$ iff $x \in O$ for some open set O such that $O \subset A$. (As an exercise, verify it by using duality). ■

Remark 2.8.4. *They are characterizations of open sets and closed sets.*

In particular, let (X, d) be a metric space, A a subset of X. Then $\mathrm{Cl}(A) = \{x \in X \mid d(x, A) = 0\}$. (exercise)

Example 2.8.5. *Let $\mathbb{R}$, with the standard topology.*

1. $\mathrm{Int}([0, 1[) =]0, 1[$.

2. $\mathrm{Int}(\mathbb{Q}) = \emptyset$ *and* $\mathrm{Cl}(\mathbb{Q}) = \mathbb{R}$.(*Hint: Any open interval in $\mathbb{R}$ contains irrational points*).

3. $\mathrm{Int}(\mathbb{R}\setminus\mathbb{Q}) = \emptyset$ *and* $\mathrm{Cl}(\mathbb{R}\setminus\mathbb{Q}) = \mathbb{R}$.(*Hint: Any open interval in $\mathbb{R}$ contains rational points*).

Exercice 2.8.6. *Prove these equalities.*

Lemma 2.8.7. *Let (X, τ) be a topological space and let $A \subset X$ and $B \subset X$.*

1. $A \subset B \Longrightarrow \mathrm{Int}(A) \subset \mathrm{Int}(B)$ and $\mathrm{Cl}(A) \subset \mathrm{Cl}(B)$.

2. $\mathrm{Int}(X \setminus A) = X \setminus \mathrm{Cl}(A)$ and $\mathrm{Cl}(X \setminus A) = X \setminus \mathrm{Int}(A)$.

3. $\mathrm{Int}(A) \cap \mathrm{Int}(B) = \mathrm{Int}(A \cap B)$ and $\mathrm{Cl}(A) \cup \mathrm{Cl}(B) = \mathrm{Cl}(A \cup B)$

Proof: (exercise) ■

Remark 2.8.8. *The second item shows that* Int(A) *can be defined as the negation of* Cl(X \ A).
$x \in \mathrm{Int(A)}$ *iff* $\exists N_x$ *such that* $N_x \subset A$ *iff* $\exists N_x$ *such that* $N_x \cap (X \setminus A) = \emptyset$.
$x \notin \mathrm{Int(A)}$ *iff* $\forall N_x, N_x \cap (X \setminus A) \neq \emptyset$ *iff* $x \in \mathrm{Cl(X \setminus A)}$.
Notice that the duality is between Int(A) *and* Cl(A).

Example 2.8.9. *Let* $\mathbb{R}$*, with the standard topology.*

1. $\mathrm{Cl}([0,1[) = [0,1]$.

2. $\mathrm{Cl}(\mathbb{Q}) = \mathbb{R}$.

3. $\mathrm{Cl}(\mathbb{R} \setminus \mathbb{Q}) = \mathbb{R}$.

2.8.2 Dense

Definition 2.8.10. *Let X be a topological space and let $A \subset X$. A is said to be* **(everywhere) dense** *in X if* $\mathrm{Cl}(A) = X$.

Example 2.8.11. *Let $\mathbb{R}$ be the space with the standard topology. $\mathbb{Q}$ is dense in $\mathbb{R}$ because we proved* $\mathrm{Cl}(\mathbb{Q}) = \mathbb{R}$.

At the other extreme,

Definition 2.8.12. *Let X be a topological space and let $A \subset X$. A is said to be* **nowhere dense** *in X if* $\mathrm{Cl}(A)$ *contains no open subset of X.*

Example 2.8.13. *The Cantor set $C \subset [0,1]$ is dense in $[0,1]$. As intersection of closed subsets of $[0,1]$, C is closed, so $\mathrm{Cl}(C) = C$. We need only show that C has an empty interior in $[0,1]$. If C has an interior point in $[0,1]$, it would contain an interval which is not (see 1.3.14). Thus C is nowhere dense in $[0,1]$.*

2.8.3 Frontier

Definition 2.8.14. *The* **Frontier** *of a subset $A \subset X$ is the set* $\mathrm{Cl}(A) \setminus \mathrm{Int}(A)$. It is denoted $\mathrm{Fr}(A)$.

Example 2.8.15. *Let A be some subset of the topological space (X, τ).*

1. *Let $[0,1[\subset \mathbb{R}$ with the standard topology. Then* $\mathrm{Fr}([0,1[) = \{0,1\}$.

2. Show that $\mathrm{Fr}(A) = \mathrm{Fr}(X \setminus A)$.

3. Show that A is closed iff $\mathrm{Fr}(A) \subset A$.

2.8.4 Exercises

1. Let $d_R : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the map defined as follows (cf ex 2.3.1.1d):
For $x = (x_1, x_2)$ and $y = (y_1, y_2)$, $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$.
$d_R(x, y) = \begin{cases} d(x, y) & \text{if } x, y, 0 \text{ are collinear} \\ d(x, 0) + d(0, y) & \text{if not} \end{cases}$

   For the metric d_R on the real plane, $\mathbb{R}^2$, determine (and draw the pictures).
   Let $p_0 = (1, 0)$.

   (a) the circle of radius $\frac{1}{2}$ with center p_0.
   (b) the circle of radius 2 with center p_0.
   (c) the ball $B(p, r) = \{q \in \mathbb{R}^2 \mid d_R(p, q) < r\}$ where $p \in \mathbb{R}^2$ and $r > 0$.
   (d) the ball $B(p_0, 2) = \{q \in \mathbb{R}^2 \mid d_R(p_0, q) < 2\}$. Show that for any $q \in B(p_0, r)$, there exists a ball $B(q, r') \subset B(p_0, r)$.
   (e) the ball $B\left(p_0, \frac{1}{2}\right) = \left\{q \in \mathbb{R}^2 \mid d_R(p_0, q) < \frac{1}{2}\right\}$. Show that for any $q \in B(p_0, r)$, there exists a ball $B(q, r') \subset B(p_0, r)$.
   (f) $\mathrm{Cl}(B(p_0, 2))$.
   (g) $\mathrm{Cl}\big(B\left(p_0, \frac{1}{2}\right)\big)$.
   (h) $\mathrm{Fr}(B(p_0, 2))$.
   (i) $\mathrm{Fr}\big(B\left(p_0, \frac{1}{2}\right)\big)$.

2. Let A, B be two subsets of the topological space (X, τ).

   (a) Is it true that $\mathrm{Int}(A) \cup \mathrm{Int}(B) = \mathrm{Int}(A \cup B)$ and $\mathrm{Cl}(A) \cap \mathrm{Cl}(B) = \mathrm{Cl}(A \cap B)$?
   (b) Show that A is nowhere dense in X if $\mathrm{Int}(\mathrm{Cl}(A)) = \emptyset$.
   (c) Let (X, d) be a metric space. Then $\mathrm{Cl}\big(B(x, r)\big) = \{y \in X \mid d(y, x) \leq r\}$.
   (d) Let A and B be dense in X. Prove that if A and B are open, then $A \cap B$ is also dense in X.

2.8.5 Adherent Points. Accumulation Points

Definition 2.8.16. *A point $x \in X$ is an* **adherent** *point of the subset $A \subset X$ if all the open neighbourhoods of x intersect A.*

$$\forall N_x, \text{ open neighbourhood of } x, \; N_x \cap A \neq \emptyset$$

Definition 2.8.17. *A point $x \in X$ is an* **accumulation** *point of a subset A if each neighbourhood of x contains some points of $A \setminus \{x\}$.*

$$\forall N_x, \text{ open neighbourhood of } x, \; N_x \cap (A \setminus \{x\}) \neq \emptyset$$

In other words every neighbourhood of x contains points of A different from x.

Example 2.8.18. *Consider the following subsets of $\mathbb{R}$ assigned with the standard topology.*

1. *Every point in the closed interval $[0, 1]$ is an accumulation point of the open interval $]0, 1[$ since every deleted neighbourhood of $x \in [0, 1]$ intersects some point in $]0, 1[$.*

2. *The set* $]0,1[\cup\{2\}$ *has accumulation points* $[0,1]$. *The number* 2 *is not an accumulation point of the set since there exists a deleted neighbourhood around* 2 *that does not intersect members of the set.*

3. $\mathbb{Z}$ *have no accumulation point even though the set is infinite.*

Definition 2.8.19. *A point* $x \in X$ *is an* **isolated** *point of a subset* A *if* $x \in A$ *but* x *is not an accumulation point, i.e. there exists a neighbourhood of* x *which does not contain any point of* $A \setminus \{x\}$.

Remark 2.8.20. x *is adherent point of* A *if* x *is either an accumulation point of* A *or an isolated point of* A.

Proposition 2.8.21. *Let* A *be a subset of the topological space* X. *Then*

$$\mathrm{Cl}(A) = \{x \in X \mid x \text{ is adherent of } A\}$$

Proof: Let $x \notin \mathrm{Cl}(A)$, then there exists a closed set F containing A and such that $x \notin F$. Therefore, $x \in X \setminus F = N_x$, open neighbourhood of x and $N_x \cap A \subset N_x \cap F = \emptyset$. So x is not adherent of A.
Conversely, let $x \in \mathrm{Cl}(A)$, then $x \in \bigcap\limits_{A \subset F} F$, where F is closed. But $\bigcap\limits_{A \subset F} F = X \setminus \bigcup\limits_{O \subset X \setminus A} O$ where $O = X \setminus F$. Then it does not exist any open neighbourhood N_x of x such that $N_x \cap A = \emptyset$. Therefore x is adherent of A. ■

Example 2.8.22. *Let* $\mathbb{R}$ *with the standard topology and let* $A = \mathbb{Z} \subset \mathbb{R}$.
Then

$$\begin{aligned} \mathrm{Fr}(\mathbb{Z}) &= \mathrm{Cl}(\mathbb{Z}) \setminus \mathrm{Int}(\mathbb{Z}) \\ &= \mathrm{Cl}(\mathbb{Z}) \cap \mathrm{Cl}(\mathbb{R} \setminus \mathbb{Z}) \end{aligned}$$

because $\mathrm{Cl}(\mathbb{R} \setminus \mathbb{Z}) = \mathbb{R} \setminus \mathrm{Int}(\mathbb{Z})$ *(lemma 2.8.7.2)*
Moreover $\mathbb{R}\setminus\mathbb{Z} = \bigcup\limits_{n\in\mathbb{Z}}]n, n+1[$ *is open as union of open sets thus* $\mathbb{Z}$ *is closed and* $\mathrm{Cl}(\mathbb{Z}) = \mathbb{Z}$.
Any $n \in \mathbb{Z}$ *is adherent of the subset* $\mathbb{R}\setminus\mathbb{Z}$ *because any open neighbourhood of* n *contains an interval* $]n-\varepsilon, n+\varepsilon[$ *for some* $\varepsilon > 0$. *So* $\mathrm{Cl}(\mathbb{R} \setminus \mathbb{Z}) = \mathbb{R}$ *and finally* $\mathrm{Fr}(\mathbb{Z}) = \mathrm{Cl}(\mathbb{Z}) = \mathbb{Z}$.

The subset $A \subset X$ is dense in X iff for any non empty open set O, $O \cap A \neq \emptyset$, i.e. any point of $X \setminus A$ is a adherent point of A.

Here is the "dual" characterization of closed sets.

$$\boxed{\mathbf{F \subset X \text{ is closed iff } \forall N_x \text{ open neighbourhood of } x,\ (N_x \cap F) \neq \emptyset, \text{ then } x \in F}}.$$

2.8.6 Exercises

1. Let (X, τ) be a topological space and let $A, B \subset X$ be two subsets of X.
   (a) Show that $\mathrm{Fr}(A \cup B) \subset \mathrm{Fr}(A) \cup \mathrm{Fr}(B)$.
   (b) Show that this inclusion can be strict (Example: $X = \mathbb{R}$, $A = \mathbb{Q} \cap [0,1]$ and $B = [0,1] \setminus A$).

(c) Suppose $\mathrm{Cl}(A) \cap \mathrm{Cl}(B) = \emptyset$, then show that $\mathrm{Fr}(A \cup B) = \mathrm{Fr}(A) \cup \mathrm{Fr}(B)$.

2. Let (X, τ) be a topological space and let $A \subset X$ be a subset of X. Prove: $\mathrm{Fr}(A) = \emptyset$ iff A is both open and closed.

3. Let $X = \{a, b, c\}$ with the topology $\tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$. Find the adherent points of the set $\{a\}$.

4. Let $X = \{a, b\}$ and let $\tau = \{\emptyset, \{a\}, X\}$. Show that τ is a topology on X. Define the adherent points of each subset of X.

5. Prove that a finite subset A of a Hausdorff space X has no accumulation points. Define the adherent points of A. Conclude that A must be closed.

2.9 Convergence

2.9.1 Filters

The expression "*a sequence (x_n) of real numbers has a limit (or converges to) a real number x_0*" means "*every open interval containing x_0 contains all but a finite number of the x_n*".

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}, \forall n > n_0, |x_n - x_0| < \varepsilon \text{ i.e. } x_n \in]x_0 - \varepsilon, x_0 + \varepsilon[$$

For real-valued functions we also define the expression "*$f(x)$ tends to y_0 when x tends to x_0*" or "*... when x tends to zero on the right* ", etc. Other elementary concepts of limits are also important. For example, the concept of a doubled sequence $(x_{p,q})$ which converges to x_0 when p and q tend to infinity.
If we consider the case of a sequence (x_n) tending to x_0 when n tends to infinity we can make the following observations about the definition:

1. The expression "*all the x_n except for a finite number*" means that we consider the complements (w.r.t. the set of the x_n) of finite subsets. If we denote these complements by $A, A', \dots$ none of them is empty and the intersection $A \cap A'$ of any two is again the complement of a finite subset. Thus the set of complements with respect to the set of x_n, of the finite subsets is a fundamental family which does not contain the empty set.

2. The expression "*every open interval I containing x_0 contains all the x_n except for a finite number* " means that every neighbourhood I of x_0 is contained in some A.

For example, let (x_n) be the sequence where $x_n = (-1)^n + \dfrac{(-1)^n}{n}$. This sequence does not converge. However, the subsequences (x_{2n}) and (x_{2n+1}) converge towards the points -1 and $+1$. The complements of finite subsets of $\{x_n \mid n \in \mathbb{N}\}$ satisfy the first property, but not the second. These examples give rise to the definitions of filters.

Definition 2.9.1. *Let A be a set. We call* **filter**, *a set $\mathcal{F}$ of nonempty subsets of A such that:*

1. *if $B \in \mathcal{F}$ and $B' \in \mathcal{F}$, then $B \cap B' \in \mathcal{F}$.*

2. *if $B \in \mathcal{F}$ and $B \subset B'$, then $B' \in \mathcal{F}$.*

Definition 2.9.2. *We call* **filterbase** *on the set A, a set $\mathcal{B}$ of nonempty subsets of A such that if $B \in \mathcal{B}$ and $B' \in \mathcal{B}$, there exists $C \in \mathcal{B}$, $C \subset B \cap B'$.*

A set $\mathcal{B}$ of subset of X is a filter base if

- $\mathcal{B} \neq \emptyset$.
- $\emptyset \notin \mathcal{B}$.
- $(B, B' \in \mathcal{B}) \Longrightarrow (B \cap B' \in \mathcal{B})$.

A filter is a filterbase but the converse is false. However, the set of subsets of A which contain an element of the filterbase $\mathcal{B}$ is a filter, i.e. $\{B \subset X \mid \exists B' \in \mathcal{B}, B' \subset B\}$ is a filter.
We say that $\mathcal{B}$ is a basis for the filter $\mathcal{F}$, or $\mathcal{F}$ is generated by $\mathcal{B}$.

Example 2.9.3. 1. *Let X be a topological space; the set of all neighbourhoods of a point x is a filter $\mathcal{F}_x$.*

$$\exists N \in \mathcal{F}_x \text{ such that } V \cap N = V' \cap N, \text{ where } V, V' \subset X$$

is an equivalence relation denoted R_x. The quotient set X/R_x is called the set of **germs** *at the point x.*

2. *Let X be a topological space, then the set of all neighbourhoods of a non empty subset of X is a filter.*

3. *Let $A \subset X$, and let $x_0 \in \mathrm{Cl}(A)$. Then the set of the subsets $A \cap N$ where N is a neighbourhood of x_0 in X, is a filter on A.*

4. *Let $X = \mathbb{N}$, For each n let A_n be the set of integers greater than n. Then $\mathcal{B}_{\mathbb{N}} = \{A_n \mid n \in \mathbb{N}\}$ is a filterbase on $\mathbb{N}$. The filter generated by $\mathcal{B}_{\mathbb{N}}$ is called the* **natural filter***.*

5. $\{]x_0 - \varepsilon, x_0 + \varepsilon[, \text{ where } \varepsilon > 0\}$ *is a filterbase on* $\mathbb{R}$.

6. $\{[x_0, x_0 + \varepsilon[, \text{ where } \varepsilon > 0\}$ *is a filterbase on* $\mathbb{R}$.

7. $\{]x_0, x_0 + \varepsilon[, \text{ where } \varepsilon > 0\}$ *is a filterbase on* $\mathbb{R}$.

8. $\{]x_0 - \varepsilon, x_0], \text{ where } \varepsilon > 0\}$ *is a filterbase on* $\mathbb{R}$.

9. $\{]x_0 + \varepsilon, x_0[, \text{ where } \varepsilon > 0\}$ *is a filterbase on* $\mathbb{R}$.

10. $\{]x_0 + \varepsilon, x_0[\cup]x_0, x_0 + \varepsilon[, \text{ where } \varepsilon > 0\}$ *is a filterbase on* $\mathbb{R}$.

2.9.2 Limits and Adherent Points of a Filterbase

Definition 2.9.4. *Let (X, τ) be a topological space and $\mathcal{B}$ a filterbase on X.*

1. *A point $x_0 \in X$ is said to be a* **limit point** *of $\mathcal{B}$ if every neighbourhood of x_0 contains some $B \in \mathcal{B}$ i.e.*

$$\forall N_{x_0} \text{ neighbourhood of } x_0, \exists B \in \mathcal{B} : B \subset N_{x_0}$$

2. *A point $x_0 \in X$ is said to be* **adherent** *to $\mathcal{B}$ if every neighbourhood of x_0 meets every $B \in \mathcal{B}$ i.e.*

$$\forall N_{x_0} \text{ neighbourhood of } x_0, \forall B \in \mathcal{B} : B \cap N_{x_0} \neq \emptyset$$

2.9.3 Images of Limits and Adherent Points

Notice that the point x_0 is adherent to the filterbase $\mathcal{B}$ iff x_0 is adherent to every element of $\mathcal{B}$.

Definition 2.9.5. *Let f be a map from the set A to the topological space X and let $\mathcal{B}$ be a filterbase on A.*

- *The point $x \in X$ is said to be a* **limit** *of f w.r.t. $\mathcal{B}$ if for any neighbourhood N_x of x, there exists $B \in \mathcal{B}$ such that $f(B) \subset N_x$.*
- *The point $x \in X$ is said to be a* **adherent** *of f w.r.t. $\mathcal{B}$ if for any neighbourhood N_x of x, for any $B \in \mathcal{B}$ then $f(B) \cap N_x \neq \emptyset$.*

Case of the Sequences

Recall that any map $\mathbb{N} \longrightarrow X : n \mapsto x_n$ is called a **sequence** of points of X and is denoted $(x_n)_{n\in\mathbb{N}}$. Let $\mathbb{N}$ be the set of natural integers equipped with the filterbase $\mathcal{B}_\mathbb{N}$ defined above.

1. Saying that x is the **limit** of this sequence w.r.t. $\mathcal{B}_\mathbb{N}$ means that:

$$\forall N_x, \text{ neighbourhood of } x, \ \exists N \in \mathbb{N} \ \ \forall n \geq N \Longrightarrow x_n \in N_x$$

   We write $\lim\limits_{n\to\infty} x_n = x$. We also say that the sequence **converges** to x w.r.t. $\mathcal{B}_\mathbb{N}$.

2. Saying that x is **adherent** of this sequence w.r.t. $\mathcal{B}_\mathbb{N}$ means that:

$$\forall N_x, \text{ neighbourhood of } x, \ \forall N \in \mathbb{N} \ \ \exists n \geq N \Longrightarrow x_n \in N_x$$

For example, let $X = \mathbb{R}$ (with the standard topology). The sequence $\frac{1}{4}, 1-\frac{1}{4}, \frac{1}{5}, 1-\frac{1}{5}, \frac{1}{6}, 1-\frac{1}{6}, \cdots$ has two adherent points 0 and 1 w.r.t. to the filterbase $\mathcal{B}_\mathbb{N}$.

Remark 2.9.6. *It is important to distinguish the adherent points of a sequence w.r.t. to the filterbase $\mathcal{B}$ and the adherent points of the set of all points of the sequence. Any adherent points of a sequence w.r.t. to the filterbase $\mathcal{B}$ is an adherent point of the set but the converse is false. A sequence is a set, but it is not only a set.*

Remark 2.9.7.

1. *Let $f : \mathbb{N} \longrightarrow X$, $f(n) = x_n$. When we say that x_n tends to x_0, we mean that every neighbourhood of x_0 contains some $f(A)$, which, here, is the set of points x_k where k belongs to the complement of a finite subset of $\mathbb{N}$. Consequently for every neighbourhood N_0 of x_0 there is an integer $p(N_0)$ such that for every $k \geq p(N_0)$ we have $x_k \in N_0$. Conversely, if for every N_0, there exits an integer $p(N_0)$ such that $x_k \geq N_0$ for $k \geq p(N_0)$ we see that every N_0 contains all the x_k for which k belongs to the complement of a finite subset of $\mathbb{N}$. This in fact amounts to the elementary definition of convergence of x_n to x_0.*
2. *If A is the complement of a finite subset of $\mathbb{N}$, an element, that is, of the natural filter, $f(\mathbb{N}) \setminus f(A)$ is not, in general, the complement of a finite set of points of the sequence. For example, $x_{2k+1} = 0, x_{2k} = \frac{1}{2k}, k \in \mathbb{N}$. Let $A = \mathbb{N} \setminus \{0\}$.*
3. *If the points or values x_n are all distinct, and if X denotes the set of points x_n, the image of the natural filter by the sequence (x_n) consists of the complements of finite subsets of X.*

4. *Let X denotes the set of points x_n. If x_0 is adherent to X, x_0 is not necessarily an adherent point of the filter. But if x_0 is an adherent point of the sequence (x_n), x_0 is adherent to X. For example, on $\mathbb{R}$, the sequence $\left(\frac{1}{n}\right)_{n>0}$ has the single adherent point 0, but every point $\frac{1}{n}$ is adherent to the set of values of the sequence.*

5. *Every subsequence of a sequence converging to x_0 also converges to x_0.*

General Case

Let X and Y be two topological spaces and let f be a map from the set X to the set Y. Let $x_0 \in X$ and $y_0 \in Y$. Let $\mathcal{B}$ be the filterbase of the neighbourhoods of x_0 in X.

1. $y_0 \in Y$ is a **limit** of f w.r.t. $\mathcal{B}$ iff

$$\forall N_{y_0} \text{ neighbourhood of } y_0,\ \exists A \in \mathcal{B} \text{ such that } x \in A \Longrightarrow f(x) \in N_{y_0}$$

   We write $\lim\limits_{x \xrightarrow{\mathcal{B}} x_0} f(x) = y_0$.
   Choosing the other filterbases given in the previous example, define other well known limits.

2. $y_0 \in Y$ is a **adherent** of f w.r.t. $\mathcal{B}$ iff

$$\forall N_{y_0} \text{ neighbourhood of } y_0,\ \forall A \in \mathcal{B}, \exists x \in A \text{ such that } f(x) \in N_{y_0}$$

Proposition 2.9.8. *Let $\mathcal{B}$ be a filterbase on the set X and let f be a map from X to the Hausdorff space Y.*

1. *If f has a limit w.r.t. $\mathcal{B}$, then this limit is unique.*

2. *If f has a limit w.r.t. $\mathcal{B}$, then this limit is the unique adherent point of f w.r.t. $\mathcal{B}$.*

Proof:

1. Let y and y' be two distinct limits of f w.r.t. $\mathcal{B}$. The space Y is Hausdorff, so there exists two disjoints neighbourhoods N_y and $N_{y'}$ of y and y'. There exist $A, A' \in \mathcal{B}$ such that $f(A) \subset N_y$ and $f(A') \subset N_{y'}$. Then there exists $A'' \in \mathcal{B}$ such that $A'' \subset A \cap A'$. Thus $f(A'') \subset N_y \cap N_{y'}$ which is a contradiction because $A'' \neq \emptyset$.

2. Let y be the limit of f w.r.t. $\mathcal{B}$ and let $A \in \mathcal{B}$. There exists $A' \in \mathcal{B}$ such that $f(A') \subset U_y$. Then $A \cap A' \neq \emptyset$, so $f(A \cap A') \neq \emptyset$ and $f(A \cap A') \subset f(A) \subset U_y$. Then $f(A) \subset N_y \neq \emptyset$ and y is adherent of f w.r.t. $\mathcal{B}$.
   Let y' a second distinct adherent point of f w.r.t. $\mathcal{B}$. There exist disjoint neighbourhoods N_y and $N_{y'}$ of y and y'. Then there exists $A \in \mathcal{B}$ such that $f(A) \subset N_y$. So $f(A) \cap N_{y'} = \emptyset$ which is a contradiction.

■

Remark 2.9.9. *If Y is not Hausdorff, f can have several limits w.r.t. $\mathcal{B}$. For example, let τ_Y be the trivial topology on Y. Then every point of Y is a limit of f w.r.t. $\mathcal{B}$.*

Remark 2.9.10. *If f has a limit w.r.t. $\mathcal{B}$, then this limit is adherent. However, what can happen if f has no limit?*

1. *f can have no adherent point. For example, the sequence $(0, 1, 2, 3, \ldots)$ in $\mathbb{R}$ (with the standard topology) has no adherent point.*

2. *f can have a unique adherent point. For example, the sequence $(0, 1, 0, 2, 0, 3, 0, 4, \ldots)$ in $\mathbb{R}$ (with the standard topology) has no limit but 0 is the unique adherent point.*

3. *f can have several adherent points (cf. examples above).*

Metric Spaces

In the following of this section, we only consider metric spaces.

Proposition 2.9.11. *Let X be a metric space and $A \subset X$ and $x \in X$. Then the followings are equivalent:*

1. *$x \in \mathrm{Cl}(A)$.*

2. *There exists a sequence of points of A which converges to x w.r.t. the filter of neighbourhoods of x in X.*

Proof: $\Longleftarrow$) Any neighbourhood of x contains at least one x_n so intersects A; therefore $x \in \mathrm{Cl}(A)$.
$\Longrightarrow$) If $x \in \mathrm{Cl}(A)$, for any $n \in \mathbb{N}^*$, there exists $x_n \in A$ contains in the closed ball of center x and radius $\dfrac{1}{n}$. Then, the sequence (x_n) converges to x. ■

Proposition 2.9.12. *Let X be a metric space and let $(x_1, x_2, \ldots)$ a sequence of X and $x \in X$. Then the followings are equivalent:*

1. *x is adherent to $(x_1, x_2, \ldots)$ w.r.t. the filter of neighbourhoods of x in X..*

2. *There exists a subsequence $(x_{n_1}, x_{n_2}, \ldots)$ with $n_1 < n_2 < \ldots$ which converges to x w.r.t. the filter of neighbourhoods of x in X.*

Proof: $\Longleftarrow$) x is adherent of the sequence $(x_{n_1}, x_{n_2}, \ldots)$ then it is adherent to the sequence (x_n).
$\Longrightarrow$) Suppose x is adherent to $(x_1, x_2, \ldots)$. There exists n_1 such that $d(x_{n_1}, x) \leq 1$. Then, there exists $n_2 > n_1$ such that $d(x_{n_2}, x) \leq \dfrac{1}{2}$ and so on. Then, the sequence $(x_{n_1}, x_{n_2}, \ldots)$ converges to x. ■

2.9.4 Exercises

1. Show that a finite subset of a metric space has no limit and is therefore a closed set.

2. Let f is a mapping of a set X into a set X'. Show that:

   (a) The image by f of a filter on X is a filter on E'.

   (b) The inverse image by f of a filter $\mathcal{F}'$ on X' is a filter on X if every set of $\mathcal{F}'$ meets $f(X)$. In particular, if f is a mapping of X onto X', the direct and inverse images of filters are again filters.

Example: Let (x_n) be a sequence of points in a set X, i.e. a mapping of $\mathbb{N}$ into X. The image in X under this mapping of the natural filter $\mathcal{B}_{\mathbb{N}}$ on $\mathbb{N}$ is a filter, but, in general, it does not consist of the family of complements of finite subsets of the set of values of the sequence. If for example X consist of a single element x, we have $x_n = x$ for all n, and the image of the filter consisting of the complements of finite subsets of X is a filter consisting of a single element X, whilst the complements of a finite subset is $\emptyset$.

3. Show that the sequence $\left((-1)^n\left(1+\dfrac{1}{n}\right)_{n\in\mathbb{N}\setminus\{0\}}\right)$ does not converge w.r.t. the filterbase $\mathcal{B}_{\mathbb{N}}$ but the subset $\left\{(-1)^n\left(1+\dfrac{1}{n}\right)\right\}$ of $\mathbb{R}$, (with the standard topology), has two adherent points.

4. Let $X_i, i = 1,2$ be two sets and let $\mathcal{F}_i$ be a filter on $X_i, i = 1,2$. Let $X = X_1 \times X_2$ be the product of X_1 and X_2. Show that the set $\{A_1 \times A_2 \mid A_i \in \mathcal{F}_i, i = 1,2\}$ is a filterbase on X.

5. Let (X,d) be a metric space and let $(x_1, x_2, \ldots)$ be a sequence of points of X. Show that a point $a \in X$ is adherent point of the sequence w.r.t. the filterbase $\mathcal{B}_{\mathbb{N}}$ if there exists a subsequence which admits a as a limit.

6. Let (x_n) be a sequence of the metric space (X,d). Show that x_0 is adherent to (x_n) iff there exists a strictly increasing function $f : \mathbb{N} \longrightarrow \mathbb{N}$ such that for any $n \in \mathbb{N}, f(n) \geq n$ and $w.r.t.$ $\lim\limits_{n\mapsto\infty} x_{f(n)} = x_0$.

Chapter 3

Continuous Maps

3.1 Continuous Maps between Metric Spaces

3.1.1 Continuity

A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous at x_0 if $f(x)$ tends to $f(x_0)$ when x tends to x_0, both for $x < x_0$ and $x > x_0$.
More generally, let (X, d_X) and (Y, d_Y) be two metric spaces.

Definition 3.1.1. *1. The map $f : X \longrightarrow Y$ is said to be* **continuous at the point** $x_0 \in X$ *if $\forall \varepsilon > 0, \exists \delta > 0, \forall x \in X$ satisfying $d_X(x, x_0) < \delta$, then $d_Y(f(x), f(x_0)) < \varepsilon$.*

2. The map $f : X \longrightarrow Y$ is said to be **continuous** *if it is continuous at x for any $x \in X$.*

Remark 3.1.2. *This definition can be rephrased in terms of balls as follows: The map $f : X \longrightarrow Y$ is said to be* **continuous at the point** $x_0 \in X$ *iff $\forall \varepsilon > 0, \exists \delta > 0$ such that $f\big(B_X(x_0; \delta)\big) \subset B_Y(f(x_0); \varepsilon)$.*
In the definition, the strict inequalities can be replaced by $\leq$.
If we replace the metrics d_X and d_Y by some strongly equivalent metrics, then it does not change the continuity of f.

Such a characterization is not possible for an arbitrary topological space, so we had to find another way to define continuity.

Proposition 3.1.3. *The map $f : X \longrightarrow Y$ is continuous iff $f^{-1}(O)$ is open (resp. closed) for any open (resp. closed) set $O \subset Y$.*

Proof: $\Longrightarrow$) Let $O \subset Y$ be an open set and let $x \in f^{-1}(O)$. Then $f(x) \in O$. There exists $\epsilon > 0$ such that $B_Y(f(x); \varepsilon) \subset O$. The map f is continuous, so there exists $\delta > 0$ such that $f(B_X(x; \delta)) \subset B_Y(f(x); \varepsilon)$, then $f(x') \in O$ for all $x' \in B_X(x; \delta)$ and $B_X(x; \delta) \subset f^{-1}(O)$ which is open.
$\Longleftarrow$) Let $O \subset Y$ be an open set, then $f^{-1}(O)$ is open. Let $x \in X$. Given $\varepsilon > 0$ the ball $B_Y(f(x); \varepsilon)$ is open, hence $f^{-1}(B_Y(f(x); \varepsilon))$ is an open set containing the point x. Then $\exists \delta > 0$ such that $B_X(x; \delta) \subset f^{-1}(B_Y\big(f(x); \varepsilon\big))$. Therefore $\forall \epsilon > 0, \exists \delta > 0$ such that $f\big(B_X(x; \delta)\big) \subset B_Y(f(x); \varepsilon)$. ■

In this proposition, the property characterizing continuity is topological and it will be extended to any topological spaces.

3.1.2 Uniform Continuity

Definition 3.1.4. *Let (X, d_X) and (Y, d_Y) be two metric spaces. A map $f : X \longrightarrow Y$ is* **uniformly continuous** *if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for every $x, x' \in X$ with $d_X(x, x') < \delta$, we have $d_Y(f(x), f(x')) < \varepsilon$.*

For continuity at each point, one takes an arbitrary ε, and then there must exist some δ.
While for uniform continuity, δ must work for all points.
A uniformly continuous map is continuous but the converse is false. For example, consider the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by $f(x) = \dfrac{1}{x}$.

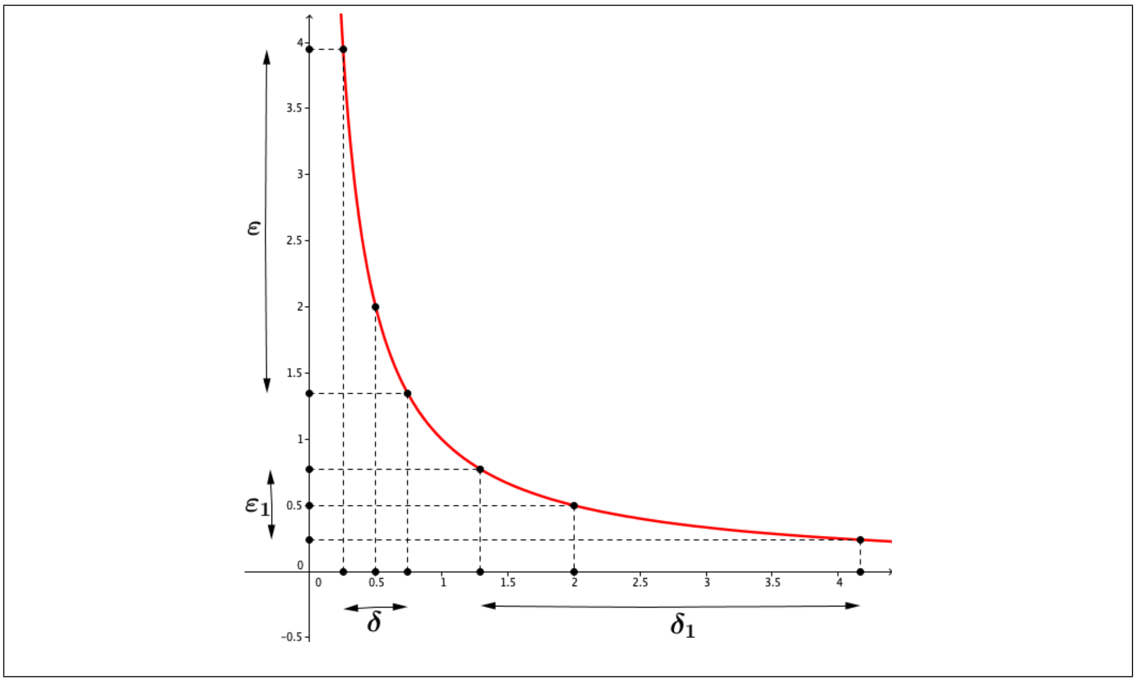


Figure 3.1

3.1.3 Exercises

1. Show that the function $f : \mathbb{R} \longrightarrow \mathbb{R}, x \longmapsto \dfrac{1}{x}$ is not uniformly continuous.

2. Show that the function $f : \mathbb{R} \longrightarrow \mathbb{R}, x \longmapsto x^2$ is not uniformly continuous.
   What about the function $f_{|[a,b]}$ for any $a, b \in \mathbb{R}, a < b$?
   More generally, is every continuous function on any closed bounded interval uniformly continuous?

3.1.4 Isometries

Let us now consider the maps between metric spaces which respect the distances.

Definition 3.1.5. *Let (X, d_X) and (Y, d_Y) be two metric spaces. An* **isometry** *$f : X \longrightarrow Y$ is a bijection such that for any $a, b \in X$, $d_X(a, b) = d_Y\big(f(a), f(b)\big)$.*

Let (X, d_X) be a metric space and let $f : X \longrightarrow Y$ a bijection from X to the set Y. We define a distance d_Y on Y by $d_Y\big(f(x), f(y)\big) = d_X(x, y)$ (exercise, verify that d_Y is a distance). Therefore, f is an isometry from (X, d_X) onto (Y, d_Y).

Example 3.1.6. *Let*

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \ ]-1, +1[\\ x &\longmapsto \frac{x}{1+|x|} \end{aligned}$$

which is a bijection (exercise) such that

$$\begin{aligned} f^{-1} :]-1, +1[&\longrightarrow \mathbb{R} \\ x &\longmapsto \frac{x}{1-|x|} \end{aligned}$$

Let $[-1, +1]$ be the closure of $]-1, +1[$ (for the standard topology) and $\overline{\mathbb{R}}$ denotes $\mathbb{R} \cup \{-\infty, +\infty\}$ where $-\infty$ and $+\infty$ are two new elements. Notice that we dont assign any meaning to the two elements $-\infty$ and $+\infty$. We extend the bijection f to a bijection from $\overline{\mathbb{R}}$ onto $[-1, +1]$ by saying $f(-\infty) = -1$ and $f(+\infty) = +1$.
$[-1, +1]$ is a metric space for the distance $d(x, y) = |x - y|$ (exercise). Let define a metric d' on $\overline{\mathbb{R}}$ by $d'(x, y) = |f(x) - f(y)|$. Then with this distance d', $\overline{\mathbb{R}}$ is a metric space. Notice that for $x \geq 0$, $d'(\infty, x) = \dfrac{1}{1+|x|}$ and for $x \leq 0$, $d'(-\infty, x) = \dfrac{1}{1+|x|}$. Moreover, d' restricts to $\mathbb{R}$ is a metric which is different from the standard metric $|x - y|$, i.e. $|x - y| \neq d'(x, y)$ for some $x, y \in \mathbb{R}$.

3.1.5 Exercises

1. Show that an isometry f is a continuous map and the inverse map f^{-1} is also continuous.

2. Consider the function

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto x - 1 \text{ if } x \leq 3 \\ x &\longmapsto \frac{1}{2}(x + 5) \text{ if } x > 3 \end{aligned}$$

   Find an open set O of $\mathbb{R}$ such that $f^{-1}(O)$ is not open.
   Deduce a property of the function f.
   What about if the function $f : \mathbb{R} \setminus \{3\} \longrightarrow \mathbb{R}$?

3. Let $\mathbb{R}^2$ be the real plane equipped with the two distances d_1, d_2 defined as follows: $d_1(x, y) = |x_1 - y_1| + |x_2 - y_2|$ and $d_2(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$.
   Show that $(\mathbb{R}^2, d_1)$ and $(\mathbb{R}^2, d_2)$ are not isometric.
   (Hint: Suppose there exist some isometry $f : (\mathbb{R}^2, d_1) \longrightarrow (\mathbb{R}^2, d_2)$. Let $a = (1, 1)$, $b = (-1, 1)$, $c = (1, -1)$, $e = (-1, -1)$ be four points. Then $2 = d_1(a, b) = d_1(a, c) = d_1(b, e) = d_1(c, e) = \frac{1}{2} d_1(a, e)$. Thus, for the Euclidean metric d_2, both $f(b)$ and $f(c)$ are the middle of the segment $[f(a), f(e)]$ so $f(b) = f(c)\big)$ (cf exercise 2.3.4.9).

4. Let (X, d_1) and (X, d_2) be two metric spaces on the same set. Then the following assertions are equivalent
   (a) $A \subset X$ is open for d_1 iff A is open for d_2.
   (b) For any $x \in X$ and any $r > 0$, there exist $r' > 0, r'' > 0$ such that $B_{d_1}(x; r') \subseteq B_{d_2}(x; r)$ and $B_{d_2}(x; r'') \subseteq B_{d_1}(x; r)$.
   (c) The identity maps $(X, d_1) \longrightarrow (X, d_2)$ and $(X, d_2) \longrightarrow (X, d_1)$ are continuous.

3.2 Continuous Maps between Topological Spaces

A metric space is a topological space. We defined the continuity of maps between metric spaces using the metric and we showed that continuity can be given using topologies. So, there exists a natural way to extend the definition of continuity to any topological spaces.

Definition 3.2.1. *Let (X, τ_X) and (Y, τ_Y) be two topological spaces. The map $f : X \longrightarrow Y$ is said to be* **continuous** *if $f^{-1}(O)$ is open, i.e. $f^{-1}(O) \in \tau_X$, for any open set $O \in \tau_Y$.*

If there is no ambiguity on the topologies, we don't mention them for simplicity. If not, we denote $f : (X, \tau_X) \longrightarrow (Y, \tau_Y)$.
The map $f : X \longrightarrow Y$ from the set X to the set Y can defined several continuous maps or not, $f : (X, \tau_X) \longrightarrow (Y, \tau_Y)$ according to the choices of the topologies τ_X and τ_Y.
For example, consider the map

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto \begin{cases} -1 & \text{if} \quad x \leq 0 \\ +1 & \text{if} \quad x > 0 \end{cases} \end{aligned}$$

- defines a map $f : (\mathbb{R}, \tau) \longrightarrow (\mathbb{R}, \tau)$ which is not continuous, where τ is the standard topology,
- defines a continuous map $f : (\mathbb{R}, \tau_1) \longrightarrow (\mathbb{R}, \tau_2)$ where either τ_1 is the discrete topology, or if τ_2 is the trivial topology.

Proposition 3.2.2. *$f : X \longrightarrow Y$ is continuous iff $f^{-1}(C)$ is closed for any closed set C in Y.*

Proof: It is enough to notice that $X \setminus f^{-1}(C) = f^{-1}(Y \setminus C)$. ■

Remark 3.2.3. *Notice the "duality" open $\longleftrightarrow$ closed. In other words, we could also define the continuous maps with closed sets in the similar terms as we define with open sets.*
The continuous maps are the morphisms in the category of topological spaces (where the objects are the topological spaces).

Remark 3.2.4. *A morphism in a category respects the structures on the objects. For example, in the category of groups, a morphism sends the composite of two elements to the composite of the images. Another example in the category of vector spaces, a morphism is a linear map which sends any linear combination of vectors to the linear combination of the images.*
In the category of the topological spaces, a morphism is a continuous map and the definition seems strange because the reverse image of an open set has to be open.
We have to recall that a topology is defined by taking some subsets. So, we have to consider not the map from a topological space to a topological space, but the map from the set of subsets of the space to the set of subsets of the other set. In other words, we have to consider not the map $f : X \longrightarrow Y$ but the map $f : \mathcal{P}(X) \longrightarrow \mathcal{P}(Y)$, and we saw that f does not respect the operations on sets, union, intersection, although the map $f^{-1} : \mathcal{P}(Y) \longrightarrow \mathcal{P}(X)$ respects these operations, and consequently, the conditions defining the topologies (cf. 1.3.3).

A map $f : X \longrightarrow Y$ that sends open (resp. closed) sets onto open (resp. closed) sets is called **open** (resp. **closed**) map.

Example 3.2.5. *1. If X and Y are metric spaces, then we recover the definition of continuity given in the previous section.*

2. Let (X, τ) be a topological space and $Id : X \longrightarrow X$ be the identity map ($Id(x) = x$ for any $x \in X$). Then Id is continuous.

3. However let (X, τ_1) and (X, τ_2) be two topological spaces on the same set $X \neq \{x\}$ such that τ_1 is the trivial topology and τ_2 is the discrete topology. Then $Id : X \longrightarrow X$, where the domain is (X, τ_1) and the range is (X, τ_2), is not continuous. (Exercise). Is Id continuous if the domain is (X, τ_2) and the range is (X, τ_1)?

Another characterization of continuity using filters.

Proposition 3.2.6. *Let X, Y be two topological spaces. Then a map $f : X \longrightarrow Y$ is continuous iff f sends converging filters to converging filters.*

Proof: $\Longrightarrow$) Suppose the map f continuous. Let $\mathcal{F}$ be a filter in X converging to the point x, i.e. x is a limit point of $\mathcal{F}$. We have to show that $f(\mathcal{F})$ converges to $f(x)$. Let N be a neighbourhood of $f(x)$. There is an open set $O_{f(x)} \subset N$. So $f^{-1}\left(O_{f(x)}\right)$ is open and contains x, which means that $f^{-1}\left(O_{f(x)}\right) \in \mathcal{F}$ by assumption. Hence, $f\left(f^{-1}\left(O_{f(x)}\right)\right) \in f(\mathcal{F})$. Since $f\left(f^{-1}\left(O_{f(x)}\right)\right) \subset O_{f(x)} \subset N$, we have $N \in f(\mathcal{F})$.
$\Longleftarrow$) Suppose the map f preserves converging filters. Let $O_{f(x)}$ be an open set containing $f(x)$. We have to find an open set $O \subset X$ containing x, such that $f(O) \subset O_{f(x)}$. Let $\mathcal{F}$ be the neighbourhood filter of x. So, $\mathcal{F}$ converges to x. By assumption, $f(\mathcal{F})$ converges to $f(x)$. Since O is an open neighbourhood of $f(x)$, we have $\in f(\mathcal{F})$, or $f(F) \subset O$ for some $F \in \mathcal{F}$. Since F is a neighbourhood of x, it contains an open neighbourhood U of x. Furthermore, $f(U) \subset f(F) \subset O$. Since x is arbitrary, the map f is continuous. ∎

Remark 3.2.7. *Let $f : X \longrightarrow Y$ be a continuous map. It is wrong to say*

- *for any open set $O \subset X, f(O)$ is open.*
- *for any closed set $F \subset X, f(F)$ is closed.*

A map $f : X \longrightarrow Y$ is said to be **open** if for any open set $O \subset X, f(O)$ is open.
A map $f : X \longrightarrow Y$ is said to be **closed** for any closed set $F \subset X, f(F)$ is closed.

Example 3.2.8. *1. Let $\mathbb{R}$ be the topological space, with the standard topology and let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the constant map onto x_0. Then f is continuous but, for any open set $O \subset \mathbb{R}$, $f(O) = \{x_0\}$ which is not open.*

2. Let $\mathbb{R}$ be the topological space with the standard topology and let $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f(x) = \dfrac{1}{1+x^2}$. Then f is continuous, and $f(\mathbb{R}) =]0, 1]$ which is neither closed nor open.

Proposition 3.2.9. *Let X, Y and Z be three topological spaces and let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be two continuous maps. Then $g \circ f : X \longrightarrow Z$ is continuous.*

Proof: (exercise) ∎

Extension by continuity

Let $f : A \longrightarrow \mathbb{R}$ be a function defined and continuous on a subset A of the topological space X. Is it possible to extend f to a continuous function $g : X \longrightarrow \mathbb{R}$?

Example 3.2.10.
- *Let $f(x) = x$ if $x < 0$ and $f(x) = x + 1$ if $x > 0$. Then f cannot be continuously extended to $\mathbb{R}$.*
- *Let $f(x) = x \sin \dfrac{1}{x}$ defined and continuous on $\mathbb{R} \setminus \{0\}$. Then f can be continuously extended to the function $g : \mathbb{R} \longrightarrow \mathbb{R}$ such that $g(0) = 0$.*

The next theorem is given without any proof,

Proposition 3.2.11 (Tietze[1] extension)**.** *Let A be a closed set of the metric space (X, d), and $f : A \longrightarrow \mathbb{R}$ a continuous map. Then there exists a continuous function $g : X \longrightarrow \mathbb{R}$ such that the restriction $g_{|A} = f$.*

3.2.1 Exercises

1. Let $f : X \longrightarrow Y$ be a continuous map. Find out whether f remains continuous w.r.t.
   (a) A finer topology on X and the same topology on Y.
   (b) A coarser topology on X and the same topology on Y.
   (c) The same topology on X and a finer topology on Y.
   (d) The same topology on X and a coarser topology on Y.
2. Show that every function from a discrete topological space is continuous. Analogously, verify that every function to a trivial topological space is continuous.
3. As an application of the previous exercise: Let Y be a topological space and let $\alpha : [0, 1] \longrightarrow Y, \beta : [0, 1] \longrightarrow Y$ be two continuous maps such that $\alpha(1) = \beta(0)$ where $[0, 1]$ is equipped with the topology metric given by $d(x, y) = |x - y|$. Let
$$\begin{array}{rcl} \gamma : [0,1] & \longrightarrow & Y \\ t & \longmapsto & \begin{cases} \alpha(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ \beta(2t-1) & \text{if } \frac{1}{2} < t \le 1 \end{cases} \end{array}$$
   Show that γ is a continuous map.
4. Let $f : X \longrightarrow Y$ be a map between the two topological spaces $(X, \tau_X), (Y, \tau_Y)$. Characterize the fact that f is not continuous.
5. Given the two maps f and g
$$\begin{array}{rcl} f : \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & x + 1 \text{ if } x \ge 0 \\ x & \longmapsto & x \text{ if } x < 0 \end{array}$$

[1] Heinrich Franz Friedrich Tietze (August 31, 1880 - February 17, 1964) was an Austrian mathematician, famous for the Tietze extension theorem on functions from topological spaces to the real numbers. He also developed the Tietze transformations for group presentations, and was the first to pose the group isomorphism problem. Tietze's graph is also named after him; it describes the boundaries of a subdivision of the Möbius strip into six mutually-adjacent regions, found by Tietze as part of an extension of the four color theorem to non-orientable surfaces.

$$\begin{aligned} g : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto x - 1 \text{ if } x \geq 0 \\ x &\longmapsto x \text{ if } x < 0 \end{aligned}$$

Are the maps f and g continuous? What about $g \circ f$?

6. Let $f : X \longrightarrow Y$ be a map.
   Show that f is continuous iff $\mathrm{Cl}(f^{-1}(B)) \subset f^{-1}(\mathrm{Cl}(B))$ for any $B \subset Y$.

7. (a) Let X and Y be two topological spaces and let F, G be closed subsets of X such that $X = F \cup G$. Let $f : F \longrightarrow Y; g : G \longrightarrow Y$ be continuous functions such that $f(x) = g(x)$ for all $x \in F \cap G$.
   Show that there exits a unique function $h : X \longrightarrow Y$ such that $h(x) = f(x)$ if $x \in F$ and $h(x) = g(x)$ if $x \in G$ and show that h is continuous.
   (b) More generally, let X and Y be two topological spaces and let $f : X \longrightarrow Y$ be a map.
   Let $X = \displaystyle\bigcup_{i=1}^{n} A_i$ where A_i is closed for any i.
   Show that if the restriction $f_{|A_i}$ is continuous for any i, then f is continuous.

8. Prove the assertions of Remark 3.2.7.

9. Let X and Y be two topological spaces and let $f : X \longrightarrow Y$ be a map. Show that f is continuous iff for any $x \in X$, $f(x)$ is the limit of the filter of neighbourhoods of x, i.e. $\lim_{x' \to x} f(x') = f(x)$.

10. Let $P = \{\{2k-1, 2k\}\}_{k \in \mathbb{Z}}$ be the partition of $\mathbb{Z}$ which defines a topology $\tau_{P_{\mathbb{Z}}}$ on $\mathbb{Z}$. This topology is called *odd-even* topology. Let $(\mathbb{N}, \tau_{P_{\mathbb{N}}})$ be the topological space with the odd-even topology. Show that the map

$$\begin{aligned} f : \mathbb{N} &\longrightarrow \mathbb{Z} \\ 2k &\longmapsto k \\ 2k-1 &\longmapsto k \end{aligned}$$

   is continuous.

11. Let $\mathcal{A}_1(\mathbb{F})$ be the Zariski affine line and let $f : \mathcal{A}_1(\mathbb{F}) \longrightarrow \mathcal{A}_1(\mathbb{F})$ be a map.
    Show that f is continuous w.r.t. Zariski topologies if the inverse image of any finite set is a finite set.

3.3 Homeomorphisms

Definition 3.3.1. *Let X and Y be two topological spaces. The map $f : X \longrightarrow Y$ is said to be an* **homeomorphism** *if f and f^{-1} are continuous bijections. We say that the topological spaces X and Y are* **homeomorphic** *and we denote $X \cong Y$.*
The map f is said to be an **embedding** *if f is a homeomorphism onto its image.*

Remark 3.3.2. *1. Homeomorphisms are isomorphisms in the category of topological spaces.*

2. Homeomorphisms induce one-to-one correspondences between open sets in X and Y and between closed sets in X and Y.

3. *Let (X, τ_1) and (X, τ_2) be two topological spaces on the same set X, where $\tau_1 > \tau_2$. For example, τ_1 is the discrete topology, and τ_2 is the trivial topology. Then, the identity map $\mathrm{Id}_X : (X, \tau_1) \longrightarrow (X, \tau_2)$ is a continuous bijection, but not a homeomorphism.*

Proposition 3.3.3. *Let $f : X \longrightarrow Y$ be an homeomorphism and let $A \subset X$. Then*

1. $f(\mathrm{Cl}(A)) = \mathrm{Cl}(f(A))$.
2. $f(\mathrm{Int}(A)) = \mathrm{Int}(f(A))$.
3. $f(\mathrm{Fr}(A)) = \mathrm{Fr}(f(A))$.
4. *N is a neighbourhood of $x \in X$ iff $f(N)$ is a neighbourhood of $f(x) \in Y$.*

Proof: ■

Example 3.3.4. *An isometry between two metric spaces is an homeomorphism.*

Remark 3.3.5. *A homeomorphism between two metric spaces is not necessarily an isometry.*

Example 3.3.6. *Let $X = Y = \mathbb{R}$, let $d_X(a, b) = |a - b|$, $d_Y(a, b) = 2|a - b|$ be the distances on X and Y and $f = Id_{\mathbb{R}}$. Then f is an homeomorphism but it is not an isometry (exercise).*

Proposition 3.3.7. *Let $f : X \longrightarrow Y$ be a bijection and let τ_Y be a topology on Y. Then there exists a unique topology τ_X on X such that f is a homeomorphism ($O \in \tau_X \Longleftrightarrow f(O) \in \tau_Y$).*

Proof: The topology τ_X is given by the subsets $f^{-1}(O)$ where $O \in \tau_Y$.
By the properties of the inverse map f^{-1}, τ_X is a topology.
It follows that both f and f^{-1} are continuous, so f is a homeomorphism. ■

Remark 3.3.8. *Let $f : X \longrightarrow Y$ a bijective continuous map. Then f is not necessarily a homeomorphism, i.e. the map f^{-1} can be non continuous. Let give an example.*
The following map

$$\begin{aligned} f : [0, 2\pi[&\longrightarrow \mathbb{S}^1 \\ t &\longmapsto (\cos t, \sin t) \end{aligned}$$

is a continuous bijection. But the reverse map f^{-1} is not continuous. Consider the open subset $[0, \pi[\subset [0, 2\pi[$. Then $f([0, \pi[) = \{(x, y) \in \mathbb{S}^1 \mid y > 0\} \cup \{(1, 0)\}$ which is not open although $[0, \pi[$ is open in $[0, , 2\pi[$.

3.3.1 Exercises

1. Let $f : [\alpha, \beta] \longrightarrow [a, b]$ be a surjection such that if $x < y$ then $f(x) < f(y)$ for all x and y. Show that f is continuous, bijective and f^{-1} is continuous.

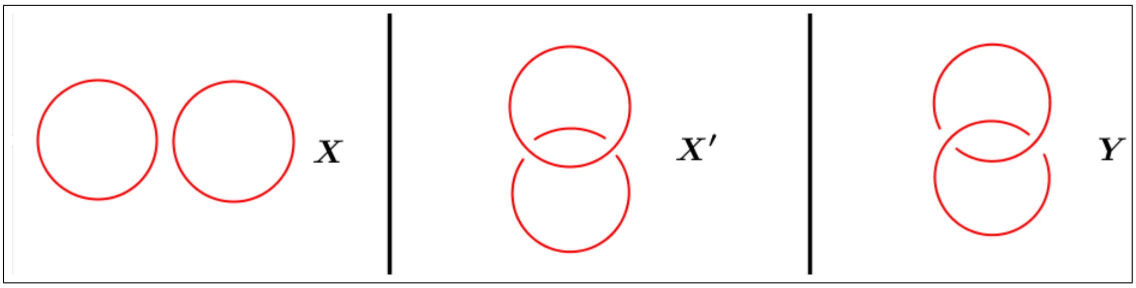
2. Let a, b, c and $d \in \mathbb{R}$. Let define the following map:

$$\begin{array}{rcl} f : \mathbb{R} & \longrightarrow & \mathbb{R} \\ t & \longmapsto & c + \dfrac{(t - a)(d - c)}{(b - a)} \end{array}$$

   is an homeomorphism as a map from $[a, b]$ onto $[c, d]$.

3. Let $\mathbb{R}^2$ be the Euclidean real plane. Show that a circle and a square are homeomorphic subspaces.

4. Let X, X' and Y the three subspaces of $\mathbb{R}^3$, the Euclidean space, where X, X' and Y are made of two circles as in the picture.



Are X and Y homeomorphic? (X' and Y homeomorphic?)
Same question, but X, X' and Y are two subspaces of $\mathbb{R}^2$.

5. Show that the following subspaces of the Euclidean plane are homeomorphic,

   (a) $A = \{(x, y) \mid y > 0\}$.

   (b) $B = \{(x, y) \mid x > y > 0\}$.

   (c) $C = \{(x, y) \mid x^2 + y^2 < 1\} \setminus \{(x, y) \mid x = 2y, x > 0\}$.

6. Let $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ be the sphere and let $N = (0, 0, 1)$ be the North pole. Define the map:

$$\begin{array}{rcl} f : \mathbb{S}^2 \setminus \{N\} & \longrightarrow & \mathbb{R}^2 = \{(x, y, 0) \mid x, y \in \mathbb{R}\} \subset \mathbb{R}^3 \\ (x, y, z) & \longmapsto & \left(\dfrac{x}{1-z}, \dfrac{y}{1-z}, 0\right) \end{array}$$

   Show that f is a homeomorphism and define f^{-1}.

7. The space of real $n \times n$-matrices is a space homeomorphic to $\mathbb{R}^{n^2}$. What about the space of complex $n \times n$-matrices?

8. Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be the map $z \longmapsto z^2$. Let $\mathcal{H} = \{z = x + iy \in \mathbb{C} \mid y > 0\}$.
   Determine $f(\mathcal{H})$. Is the restriction $f_{|\mathcal{H}}$ a homeomorphism onto its image?

9. Consider the topological spaces $(\mathbb{R}, \tau)$ where τ is the standard topology, and the product space $\mathbb{R}^2$.

   (a) Is the function $f : \mathbb{R} \longrightarrow \mathbb{R}, x \longmapsto x^2$ a homeomorphism onto its image $f(\mathbb{R})$?

   (b) Is the function $g : \mathbb{R} \longrightarrow \mathbb{R}^2, x \longmapsto (x, x^2)$ a homeomorphism onto its image $g(\mathbb{R})$?

10. Let $\mathcal{A}_1(\mathbb{F})$ be the Zariski affine line and let $f : \mathcal{A}_1(\mathbb{F}) \longrightarrow \mathcal{A}_1(\mathbb{F})$ be any bijection. Show that f is a homeomorphism w.r.t. Zariski topologies.

11. Let (X, τ_X) and (Y, τ_Y) be two topological spaces and $f : X \longrightarrow Y$ a homeomorphism. Characterize f if the topologies are as follows:

(a) A finer topology on X and the same topology on Y.

(b) A coarser topology on X and the same topology on Y.

(c) The same topology on X and a finer topology on Y.

(d) The same topology on X and a coarser topology on Y.

3.4 Classification of Topological Spaces

To classify some mathematical objects, for example, the sets, or the vector spaces, needs to consider these objects up to isomorphism, for example, bijections for the sets, linear isomorphisms for the vector spaces. In general, the classification has no solution, and we have to weaken conditions.

3.4.1 Some Examples

Sets

Consider the sets $\{a, b\}, \{x, y\}, \{1, 2\}, \{\bigcirc, \square\}, \ldots$. For any two such sets, there exist some bijections between them, i.e. all these sets are "equivalent" and the equivalence class is the cardinality, 2. Notice that $\mathbb{N}$ and $\mathbb{R}$ are infinite sets, but there does not exist a bijection between them, so they have not the same cardinality.
Two sets with the same cardinality have the same properties as sets. But they can have distinct properties if we consider some structures on them, (for example, algebraic structures, topological structures, etc.).
Then, the problem of classification of sets is completely solved.

Vector spaces

The classification of the vector spaces consists to consider the linear isomorphisms between vector spaces.
Any vector space has a basis, and any two bases of a vector space have the same cardinality.
Two vector spaces are isomorphic iff they have isomorphic bases.
Then, the finite-dimensional vector spaces are classified by their dimension. For any vector spaces, it is the cardinality of their bases.

Groups

The classification of groups consists to consider the isomorphisms between groups. It is an extremely difficult question which has only partial answers.
A complete classification is known for finitely generated Abelian groups. This is given by the fundamental theorem of finitely generated Abelian groups: Every finitely generated Abelian group is a direct sum of finitely many non-split cyclic subgroups some of which are finite and primary, while the others are infinite. In particular, finite Abelian groups split into a direct sum of primary cyclic groups. Such splittings are, in general, not unique, but any two splittings of a finitely generated Abelian group into direct sums of non-split cyclic groups are isomorphic, so that the number of infinite cyclic summands and the collection of the orders of the primary cyclic summands do not depend on the splittings chosen. These numbers are called invariants of the finitely generated Abelian group. They constitute a complete system of invariants, in the sense that two (finitely generated) Abelian groups are isomorphic if and only if they have the same invariants. Each subgroup of a finitely generated Abelian group is itself finitely generated.

3.5 Homeomorphism Problem. Topological Invariants

The main problem in topology is to find out whether two topological spaces are homeomorphic or not, it is the ***homeomorphism problem***.
The equivalence relation for topological spaces is the homeomorphism, which places spaces of the same topological types into the same class.
In order to show that two topological spaces X and Y are homeomorphic, one need only construct a homeomorphism between them.
To show they are not homeomorphic is trickier. Since we cannot consider and then reject every possible function in turn, we instead need to find a topological property which one has and the other does not. This problem is far to be solved and we have to weaken to conditions.
In a nutshell, a topological property is something one can say about a space in terms of open sets. Alternatively, it is a property which is preserved by any homeomorphism.

Definition 3.5.1. *A property of topological spaces is called* **topological** *provided that has if a space X has the property, then so does any homeomorphic space Y.*
If X has the property, but Y does not, then X and Y are not homeomorphic.

Example 3.5.2.
- *The Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ for $n \neq m$ are not homeomorphic.*
- *The spheres $\mathbb{S}^n$ and $\mathbb{S}^m$ for $n \neq m$ are not homeomorphic.*
- *In the Euclidean plane $\mathbb{R}^2$, the circles and the squares are homeomorphic.*

Properties that are shared by homeomorphic spaces are called **topological properties** and **invariants**.
A **topological invariant** is a map that assigns the same object to spaces of the same topological type, i.e. which are homeomorphic. An invariant f is only useful through the following:

$$f(X) \neq f(Y) \implies X \ncong Y$$

A *trivial* invariant assigns the same object to all spaces and is therefore useless.
The *complete* invariant assigns different objects to non-homeomorphic spaces, which is the best situation.
Most invariants fall in-between these extremes.
The most powerful an invariant, the harder it is to compute it. As we relax the classification to be coarser, the computation becomes easier.
The invariants of topological spaces given by maps to algebraic structures, for example, some integers, or polynomials, or groups, vector spaces, ..., or by the map to combinatorial structures, are, in general, easier to compute than invariants given by a map to topological structures. This is the subject of algebraic topology.

Example 3.5.3. Fixed-point Brouwer[2] Theorem
It does not exist continuous map $f : \mathbb{D}^2 \longrightarrow \mathbb{S}^1$, where $\mathbb{D}^2$ is the disk, which is an extension $\mathrm{Id} : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$.
Suppose such map f exists, i.e. the diagram is commutative ($f \circ \iota = \mathrm{Id}$).

$$\begin{tikzcd}
\mathbb{S}^1 \arrow[rr, "\mathrm{Id}_{\mathbb{S}^1}"] \arrow[dr, hook, "\iota"'] & & \mathbb{S}^1 \\
& \mathbb{D}^2 \arrow[ur, dotted, "f"'] &
\end{tikzcd}$$

[2]Luitzen Egbertus Jan Brouwer; 27 February 1881 - 2 December 1966), usually cited as L. E. J. Brouwer but known to his friends as Bertus, was a Dutch mathematician and philosopher, who worked in topology, set theory, measure theory and complex analysis. He was the founder of the mathematical philosophy of intuitionism.

We will show in chapter 7, how we associate to any topological space its fundamental group, $\mathbb{Z}$ *for* $\mathbb{S}^1$, 0 *for* $\mathbb{D}^2$ *and to any continuous map, some homomorphism,* $\mathrm{Id}_{\mathbb{Z}}$ *for* $\mathrm{Id}_{\mathbb{S}^1}$*, the constant homomorphism for* ι*, so that the diagram of continuous maps is transformed into the following diagram of homomorphisms of groups*

$$\begin{tikzcd}
\mathbb{Z} \arrow[rr, "\mathrm{Id}_{\mathbb{Z}}"] \arrow[dr] & & \mathbb{Z} \\
& 0 \arrow[ur, dotted, "f^*"'] &
\end{tikzcd}$$

Such a commutative diagram cannot exist, i.e. there does not exist any homomorphism f^* *such that* $f^* \circ 0 = \mathrm{Id}_{\mathbb{Z}}$*, so there does not exist a continuous map* f*.*
The question was initially topological and it becomes algebraic. However, it only gives an answer if the algebraic problem has no solution.

Chapter 4

Topological Constructions

4.1 Introduction

In the chapter one, we showed how to construct some sets from some given sets and maps, the constructions have to satisfy the universal mapping property. Our purpose is to make the similar constructions when the sets are equipped with topologies and to construct some new topological spaces satisfying the universal mapping property, i.e. to consider the category of topological spaces.
Let us give two examples as follows. Let (X, τ) be a topological space. Given a map f from a set Y to the set X, the goal is to define a "suitable" topology τ_Y on Y such that the map f is continuous, i.e. $f^{-1}(O) \in \tau_Y$ for any $O \in \tau$.

Dually, let (X, τ) be a topological space. Given a map f from the set X to a set Y, the goal is to define a "suitable" topology τ_Y on Y such that the map f is continuous, i.e. $f^{-1}(O) \in \tau$ for any $O \in \tau_Y$.

In both cases, there exists an obvious solution. For the first case, take τ_Y as the discrete topology and in the second case , τ_Y the trivial topology. But generally, there exist many more topologies, coarser topologies in the first case and finer in the second case.
The "suitable" topology will be the coarsest in the first case and the finest in the second case.

4.2 Initial Topology

4.2.1 Topological Subspace - Induced Topology

Let us begin with the following example. Let (X, d_X) be a metric space and $Y \subset X$. Then, the metric d_X induces a metric d_Y making Y a metric space (Y, d_Y). Let $y, y' \in Y$ then $d_Y(y, y') := d_X(y, y')$. Let $B_X(x_0; r) = \{x \in X \mid d_X(x_0, x) < r\}$ be the open ball of X of center x_0 and radius r. For $x_0 \in Y$, we have the open ball $B_Y(x_0; r) = \{y \in Y \mid d_Y(x_0, y) < r\}$ of (Y, d_Y). It is clear that $B_Y(x_0; r) = B_X(x_0; r) \cap Y$.

Let (X, τ) be a topological space and let $A \subset X$ be a subset. We want to define a "suitable" topology on A from the topology τ on X.

Remark 4.2.1. *We have two sets X and A with a relation between them. In a first step, we must translate this relation in terms of morphism, i.e. of map. This is a general principle.*

Let us pointed out that the subset $A \subset X$ could be written (in terms of morphism in the category of sets) as the *inclusion* (canonical map) $\iota : A \hookrightarrow X$ where $\iota(a) = a$ for any $a \in A$. The problem is to defined a topology τ_A on A such that ι is a morphism in the category of topological spaces, i.e. ι is a continuous map. For example, the discrete topology on A makes the map ι continuous but it does not depend on the topology τ on X. We can define some other topologies making the map ι continuous.
The "suitable" topology is the coarsest (i.e. weakest or smallest) one making ι continuous. It is defined as follows (exercise):

$$\boxed{\tau_{\mathbf{A}} = \{\mathbf{O} \cap \mathbf{A} \mid \mathbf{O} \in \tau\}.}$$

Then the map $\iota : (A, \tau_A) \longrightarrow (X, \tau)$ is continuous and the topology τ_A is defined from the topology τ. Moreover, τ_A is the smallest set with this property, i.e., it is the coarsest (i.e. weakest or smallest) topology.

Definition 4.2.2. *The topology τ_A is called the* **induced** *topology and (A, τ_A) is called a* **topological subspace** *of (X, τ).*

All the finer topologies than τ_A make the map $\iota : A \hookrightarrow X$ continuous. So, why do we choose the coarsest topology?

Universal Mapping Property: Let (A, τ_A) be a subspace of (X, τ), i.e. τ_A is the coarsest topology making ι continuous.
Let (Z, τ_Z), $f : Z \longrightarrow X$ continuous map such that $f(Z) \subseteq A$. There exists a unique map h such that $f = \iota \circ h$. We have $h(z) = f(z)$ for any $z \in Z$.
The map h is continuous: Let $O_A \in \tau_A$, then $O_A = O \cap A$ where $O \in \tau$.
$h^{-1}(O_A) = h^{-1}(O \cap A) = h^{-1}(\iota^{-1}(O)) = f^{-1}(O) \in \tau_Z$.
If $\tau' \supsetneqq \tau_A$, (ι remains continuous), there exist $O' \in \tau' \setminus \tau_A$ such that $h^{-1}(O') \neq f^{-1}(O)$ for some $O \in \tau$. Then h may be not continuous

$$\begin{tikzcd} Z \arrow[d, dotted, "h"'] \arrow[dr, "f"] & \\ A \arrow[r, hook, "\iota"] & X \end{tikzcd}$$

Example 4.2.3. *Let $A = X, Z = X$, $\tau_X = \tau_Z \neq \tau_{discrete}$ and $f = Id, \iota = Id$. Let $\tau_A = \tau_{discrete}$, then $f = \iota \circ h$ where $h = Id$. The maps f and ι are continuous, but h is not. Then for any $a \in A$, $h^{-1}(\{a\}) = \{a\}$ and there is some $\{a\} \notin \tau_X$, so h is not continuous.*

Injections

More generally, let A be a set and let $f : A \longrightarrow X$ be an injection. Then $f = \iota \circ g$ where $g : A \longrightarrow f(A)$ is a bijection and $\iota : f(A) \hookrightarrow X$ is the inclusion.
We have the following commutative diagram:

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "g"'] & & X \\ & f(A) \arrow[ur, hook, "\iota"'] & \end{tikzcd}$$

Let τ be a topology on X. Let $\tau_{f(A)}$ be the induced topology on $f(A) \subset X$.

Definition 4.2.4. *The induced topology τ_A on A is the topology making the map $g : A \to f(A)$, an homeomorphism $g : (A, \tau_A) \longrightarrow \big(f(A), \tau_{f(A)}\big)$.*

$$\boxed{\mathbf{O} \in \tau_\mathbf{A} \Longleftrightarrow \exists\ \mathbf{O}' \in \tau_\mathbf{X} \text{ such that } \mathbf{O} = \mathbf{f^{-1}}(\mathbf{O}' \cap \mathbf{f}(\mathbf{A})).}$$

The map f is an **embedding** if the topology on A is such that the bijection g is a homeomorphism where $f(A)$ is a subspace of X.

Proposition 4.2.5. *Let A be a set and let $f : A \longrightarrow X$ be an injection and let τ be a topology on X. The subset $F \subset A$ is closed (for the topology τ_A) iff $F = f^{-1}(F' \cap f(A))$ where F' is closed in X.*

Proof: (exercise) ■

Remark 4.2.6. *Let $A \subset X$ where (X, τ) is a topological space. Then (A, τ_A) is a topological space and, as such, A is open and closed for the topology τ_A but, in general, A is neither open nor closed for the topology τ.*
An open set for the induced topology for A is not necessarily open for the topology of the space X. For example, consider the subset $\mathbb{R} \times \{0\} \subset \mathbb{R}^2$ equipped with the Euclidean topology. There is only one open set in $\mathbb{R} \times \{0\}$ for the induced topology which is open in $\mathbb{R}^2$, it is $\emptyset$. It is sufficient to consider the open intervals $]a, b[\times\{0\}$. Let $(x, 0)$ where $a < x < b$, then there is no ball $B(x, r) \subset \mathbb{R}^2$ contained in $]a, b[\times\{0\}$.

Example 4.2.7. *Let $[0, 1[$ be a subset of $\mathbb{R}$ equipped with the standard topology. Then $[0, 1[$ is neither open nor closed. But as a topological subspace (with the induced topology), it is both open and closed.*

Proposition 4.2.8. *Let (A, τ_A) be the subspace of the topological space (X, τ_X). Let (Y, τ_Y) be some topological space and $f : Y \longrightarrow A$ be a map. Then f is continuous iff $\iota \circ f$ is continuous.*

Proof: (exercise) ■

4.2.2 Retractions

Definition 4.2.9. *Let $A \subset X$ be a subspace of the topological space X. The continuous map $\iota : A \longrightarrow X$ has a* **retraction** *$r : X \longrightarrow A$ if r is continuous and the restriction $r_{|A} = Id_A$. The subspace A is called* **retract**.

Example 4.2.10.

- *Let $a \in X$, then the map $r : X \longrightarrow \{a\}$ is a retraction.*
- *Let $a, b \in X$ be two distinct points of X. For example, let $X = [0, 1]$ be the subspace of $\mathbb{R}$ with the usual topology, and $a = 0$, $b = 1$. Then it does not exist a retraction $r : [0, 1] \longrightarrow \{0, 1\}$.*
  Any closed interval of $\mathbb{R}$ is a retract of $\mathbb{R}$. Let $[a, b]$ and $r : \mathbb{R} \longrightarrow [a, b], r(x) = a$ if $x \leq a, r(x) = x$ if $a \leq x \leq b$ and $r(x) = b$ if $x \geq b$. Then r is a retraction.
  But an open interval $]a, b[$ is not a retract. If $r(x) = x$ for any $a < x < b$, then by continuity of r, we have $r(b) = b$, thus, there is no continuous function on $\mathbb{R}$ with image $]a, b[$.

We summarize the characterization of the retraction in the following equivalent statements. Let $A \subset X$ and $r : X \longrightarrow A$ be a continuous map.

1. r is a retraction.
2. $r(a) = a,\ \forall a \in A$.

3. $r \circ \iota = \mathrm{Id}_A$, i.e. r is a left inverse of the inclusion. $$\begin{tikzcd} A \arrow[r, "\iota"] \arrow[rr, bend left, "Id_A"] & X \arrow[r, "r"] & A \end{tikzcd}$$

4. $r : X \longrightarrow A$ is an extension of the identity map $A \longrightarrow A$.

There is a generalization for any injection.
Let A and X be two topological spaces, $f : A \longrightarrow X$ a continuous map.

Definition 4.2.11. *The map $f : A \longrightarrow X$ has a* **retraction** *if there exists a continuous map $r : X \longrightarrow A$ such that $r \circ f = \mathrm{Id}_A$.*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend left, "Id_A"] & X \arrow[r, "r"] & A \end{tikzcd}$$

If f has a retraction, then f is injective and r is surjective. So f defines a bijection from A onto $f(A)$ that we also call f.
The set $f(A)$ inherits of two topologies, the topology τ_1 as subspace of (X, τ_X) and the topology τ_2 induces from the topology τ_A by the bijection $f : A \longrightarrow f(A)$ and $\tau_1 = \tau_2$.

Proposition 4.2.12. *Let $f : Y \longrightarrow X$ be a continuous map and Z be a topological space. There is a retraction $r : X \longrightarrow Y$ iff every continuous map $g : Y \longrightarrow Z$ has an extension to a continuous map $h : X \longrightarrow Z$, i.e. $h \circ f = g$.*

Proof: It follows from the commutative diagram.

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[rr, bend left, "Id_Y"] \arrow[dr, "g"'] & X \arrow[r, "r"] \arrow[d, dotted, "h"] & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

Take $h := g \circ r$ and we have $h \circ f = g \circ r \circ f = g$.
Conversely, Let $Z = Y$ and $g = Id_Y$ then there exists a map that we call $r : X \longrightarrow Y$ such that $r \circ f = Id_Y$. ∎

In particular, let $Z = f(Y)$, then $$\begin{tikzcd} Y \arrow[r, "f"] \arrow[rr, bend left, "Id_Y"] \arrow[dr, "\bar{f}"'] & X \arrow[r, "r"] \arrow[d, dotted, "h"] & Y \arrow[dl, "\bar{f}"] \\ & f(Y) & \end{tikzcd}$$ where $\bar{f} : Y \longrightarrow f(Y), y \longmapsto f(y)$ is a homeomorphism.
We also have the commutative diagram

$$\begin{tikzcd} & f(Y) \arrow[d, "\iota"] & \\ Y \arrow[ur, "\bar{f}"] \arrow[r, "f"] \arrow[rr, bend right, "Id_Y"'] & X \arrow[r, "r"] & Y \arrow[ul, "\bar{f}"'] \end{tikzcd}$$

These two diagrams altogether gives the following $$\begin{tikzcd} f(Y) \arrow[r, "\iota"'] \arrow[rr, bend left, "Id_{f(Y)}"] & X \arrow[r, "h"'] & f(Y) \end{tikzcd}$$ and h is a retraction.

Fixed Point Property

If a space X is such that every continuous map of X into itself has a fixed point, then X is said to have the **fixed point property**.

Proposition 4.2.13. *A retract of a space with the fixed point property also has the fixed point property.*

Proof: Suppose X has the fixed point property and A is a retract of X. let $f : A \longrightarrow A$ be a continuous map. Then f extends to a continuous map $h : X \longrightarrow A$. By assumption, there is an $x_0 \in X$ such that $h(x_0) = x_0$. But since h maps into A, let $x_0 \in A$, so, $f(x_0) = h(x_0) = x_0$ and f has also a fixed point at x_0. ■

4.2.3 Exercises

1. Let (X, d) be a metric space and A a subset of X.
   (a) Show that there exists a metric d_A on A induced by the metric d.
   (b) The metric d defines a topology τ_d on X, then an induced topology τ_A on A. The metric d_A defines a topology τ_{d_A} on A. Compare the two topologies τ_A and τ_{d_A}.
   (c) Let Σ be a basis for the topology τ on X and $A \subset X$. Show that $\Sigma_A = \{V \cap A \mid V \in \Sigma\}$ is a basis for the induced topology on A.
2. A continuous map $f : X \longrightarrow Y$ is an embedding if the map $\tilde{f} : X \longrightarrow f(X)$ is a homeomorphism.
   Given a topological space, show that the inclusion of a subspace into the space is an embedding.
3. Let (X, τ) be a topological space and $B \subset A \subset X$. Let τ_A (resp. τ_B) be the induced topology on $A \subset X$ (resp. $B \subset X$ and τ_{BA} the induced topology on $B \subset A$. Show that $\tau_{AB} = \tau_B$.
4. Show that a subspace of a Hausdorff space is Hausdorff.
5. Show that a retract A of a Hausdorff space X is necessarily closed in X. (Hint: Consider the two maps $\iota \circ r$ and Id_X and $y \notin \{x \in X \mid \iota(r(x)) = Id_X(x)$. Then $\iota(r(y)) \neq Id_X(y)$. Apply Hausdorff property to $\iota(r(y))$ and $Id_x(y)$ and the continuity of the two maps $\iota \circ r$ and Id_X.)
6. Show that a map $r : X \longrightarrow X$ such that $r^2 = r$ is retraction.
7. Show that $\mathbb{R}$ with the standard topology has a countable subspace where the induced topology is discrete.
   Show that there is no uncountable subspace where the induced topology is discrete.
8. Let (X, d) be a metric space and let $A \subset X$. Then A is a metric space. Show that the topology on A defined by the metric coincides with the induced topology.
9. Let Y be a subspace of X. If A is closed in Y, and if Y is closed in X, show that A is closed in X
10. Let A be a subspace of the topological space (X, τ), and $B \subset X$.
    Prove that $\mathrm{Cl(A \cap B)} \subset \mathrm{Cl(B) \cap A}$, where the first closure is taken in τ_A, the second closure is taken in τ.
    Show that the equality does not hold in general.

11. Let $\mathbb{R}^2$ be the Euclidean plane and let a, b be two distinct points of $\mathbb{R}^2$. Let $f : \mathbb{R} \longrightarrow \mathbb{R}^2$ be the map defined by $t \longmapsto tb + (1-t)a$. Show that f is an injection. Define the induced topology on $\mathbb{R}$ via f. Same question where we replace the Euclidean topology on the real plane $\mathbb{R}^2$ by the metric topology defined by $d_1(a, b) = |x_a - x_b| + |y_a - y_b|$.

12. Show that the fixed point property is a topological property, i.e. if X has the fixed point property and if Y is homeomorphic to X, then Y has the fixed point property.

13. Consider the unit-circle $\mathbb{S}^1 = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\}$ as a subspace of $\mathbb{R}^2$ with the usual topology.

    (a) Define $A = \{x \in \mathbb{S}^1 \mid x_1 > 0\}$. Is A an open set in $\mathbb{S}^1$? Is A a open set in $\mathbb{R}^2$?

    (b) Define $B = \{x \in \mathbb{S}^1 \mid x_1 \geq 0\}$. Is B a closed set in $\mathbb{S}^1$? Is B a closed set in $\mathbb{R}^2$?

14. Let (A, τ_A) be a subspace of (X, τ_X), and let (Y, τ_Y) be a Hausdorff space. Let $f : A \longrightarrow Y$ be a continuous map. Show that there exists at most one continuous map $g : \mathrm{Cl(A)} \longrightarrow \mathrm{Y}$ such that $g_{|A} = f$. Show that g exists if for any $a \in \mathrm{Cl(A)} \setminus \mathrm{A}$, $\lim\limits_{\mathrm{x \in A, x \to a}} \mathrm{f(x)}$ exists.

4.2.4 Topological Product - Product Topology

Let (X_1, τ_1) and (X_2, τ_2) be two topological spaces.
We would like to define a "suitable" topology τ on the product $X_1 \times X_2$ from the topologies τ_1 and τ_2. The product $X_1 \times X_2$, as a set, is defined by two canonical surjections p_1 and p_2 such that

$$X_1 \xleftarrow{p_1} X_1 \times X_2 \xrightarrow{p_2} X_2$$

where $p_1(x_1, x_2) = x_1$ and $p_2(x_1, x_2) = x_2$ for $(x_1, x_2) \in X_1 \times X_2$.
The topology τ has to make p_1 and p_2 continuous. Notice that the discrete topology on $X_1 \times X_2$ makes p_1 and p_2 continuous, but it is not defined from the topologies τ_1 and τ_2.
The "suitable" topology is the coarsest (i.e. weakest or smallest) one making p_1 and p_2 continuous. τ is the smallest set with this property.

Definition 4.2.14. *The topology τ on $X_1 \times X_2$ is called the* **product topology** *of τ_1 and τ_2 making p_1 and p_2 continuous.*

$$\boxed{\tau = \left\{ \bigcup_{\mathbf{i}} \mathbf{O_{1_i}} \times \mathbf{O_{2_i}} \mid \mathbf{O_{1_i}} \in \tau_\mathbf{1}, \mathbf{O_{2_i}} \in \tau_\mathbf{2} \right\}}.$$

Exercice 4.2.15.
- *Show that τ as defined above is a topology and p_1 and p_2 are continuous.*
- *Show that if τ' is a topology on $X_1 \times X_2$ and p_1 and p_2 are continuous then $\tau \subseteq \tau'$.*

More generally, we define the product topology on $X_1 \times \cdots \times X_n$ where $(X_i, \tau_i)_{i=1,\ldots,n}$ are some topological spaces.

Exercice 4.2.16. *Let $\mathbb{R}$ with the standard topology. Then $\mathbb{R}^2$ equipped with the product topology is homeomorphic to the Euclidean space.*

Example 4.2.17. *Consider a robot arm made of two bars connected by a revolving joint, the longer bar can rotate around its fixed endpoint, the shorter bar can freely rotate around the joint. The configuration space is the space of all states of the robot arm. It is encoded by the angles* *between each bar with the horizontal axis, i.e.* $\{(\alpha, \beta) \mid \alpha, \beta \in [0, 2\pi]\} = \mathbb{S}^1 \times \mathbb{S}^1$. *It is the torus. The topology on the torus is the product topology. The topology on the torus gives informations on the stability of the robot.*

Proposition 4.2.18. *Let* $(X_1 \times X_2, \tau)$ *be the product space of the two topological spaces* (X_1, τ_1) *and* (X_2, τ_2). *Let* (Y, τ_Y) *be a topological space.*

1. *Let* $f : Y \longrightarrow X_1 \times X_2$. *Then* f *is continuous iff each coordinate map* $p_1 \circ f$ *and* $p_2 \circ f$ *is continuous.*

2. ***Universal Mapping Property**: Let* $f_i : Y \longrightarrow X_i, i = 1, 2$ *be two continuous maps. Then there exists a unique continuous map* $h : Y \longrightarrow X_1 \times X_2$ *such that* $p_i \circ h = f_i, i = 1, 2$.

Proof:

1. $\Longrightarrow$) p_i is continuous, f is continuous, so the composition $p_i \circ f$ is continuous.

$$\begin{tikzcd} & Y \arrow[dl, "p_1 \circ f"'] \arrow[d, "f"] \arrow[dr, "p_2 \circ f"] & \\ X_1 & X_1 \times X_2 \arrow[l, "p_1"] \arrow[r, "p_2"'] & X_2 \end{tikzcd} \qquad \begin{tikzcd} & Y \arrow[dl, "f_1"'] \arrow[d, "h"] \arrow[dr, "f_2"] & \\ X_1 & X_1 \times X_2 \arrow[l, "p_1"] \arrow[r, "p_2"'] & X_2 \end{tikzcd}$$

$\Longleftarrow$) Denote $f_1 = p_1 \circ f$ and $f_2 = p_2 \circ f$. Suppose f_1 and f_2 continuous.
It is enough to consider the open sets $O = O_1 \times O_2$ in $X_1 \times X_2$ where O_1 is an open set of X_1 and O_2 is an open set of X_2.
Then $O_1 \times O_2 = (X_1 \times O_2) \cap (O_1 \times X_2)$ and $f^{-1}(O_1 \times O_2) = f^{-1}(X_1 \times O_2) \cap f^{-1}(O_1 \times X_2)$.
We have $p_2^{-1}(O_2) = X_1 \times O_2$ and $f^{-1}(X_1 \times O_2) = f^{-1}(p_2^{-1}(O_2) = f_2^{-1}(O_2) \in \tau_Z$. Similarly, we have $f^{-1}(O_1 \times X_2) = f_1^{-1}(O_1) \in \tau_Z$. Hence f is continuous.

2. We have $h(z) = (x_1, x_2)$, $p_1(x_1, x_2) = p_1(h(z)) = f_1(z)$ and $p_1(x_1, x_2) = x_1$, and similarly $p_2(x_1, x_2) = f_2(z)$. So $h(z) = (f_1(z), f_2(z))$ and h is unique.
h is continuous because f_1 and f_2 are continuous.

■

Remark 4.2.19. *The map* h *stays continuous if the topology on* $X_1 \times X_2$ *is replaced by a coarser topology. But if it is replaced by a finer topology,* h *may be not continuous and the universal mapping property is not satisfied. Hence, the coarsest topology is the "suitable".*

We define the product of n topological spaces $(X_1, \tau_1), \ldots, (X_n, \tau_n)$ as $\displaystyle\prod_{i=1}^{n}(X_i, \tau)$ where

$$\tau = \left\{ \bigcup_i O_{1i} \times \cdots \times O_{ni} \mid O_{ji} \in \tau_j,\ j = 1, \ldots, n \right\}$$

The product of countable topological spaces $(X_n, \tau_n), n \in \mathbb{N}$, is defined as $\left(\displaystyle\prod_{n \in \mathbb{N}} X_n, \tau \right)$ where

$$\tau = \left\{ \bigcup_i O_{1i} \times \cdots \times O_{ni} \times \prod_{i=n+1}^{\infty} X_i \mid O_{ji} \in \tau_j,\ j = 1, \ldots, n \right\}$$

Retracts & Products

Proposition 4.2.20. *If A is a retract of X and B is a retract of Y, then $A \times B$ is a retract of $X \times Y$.*

Remark 4.2.21. *The space $X \times Y$ has the product topology and $A \times B \subset X \times Y$ has the induced topology. But $A \subset X$ and $B \subset Y$ have the induced topologies and the space $A \times B$ has the product topology. We have to show that these two topologies are equal. (exercise)*

Proof: We have to show that there exists a continuous map $r : X \times Y \longrightarrow A \times B$ such that

$$\begin{tikzcd} A \times B \arrow[r, "\iota"] \arrow[rr, bend left, "Id_{A\times B}"] & X \times Y \arrow[r, "r"] & X \times Y \end{tikzcd}$$

For any space Z and any continuous maps f_X and f_Y, there exists a unique continuous map h such that the following diagram is commutative, where p_X and p_Y are the canonical projection:

$$\begin{tikzcd} X & X \times Y \arrow[l, "p_X"'] \arrow[r, "p_Y"] & Y \\ & Z \arrow[ul, "f_X"] \arrow[u, dotted, "h"] \arrow[ur, "f_Y"'] & \end{tikzcd}$$

and similarly, for any space Z and any continuous maps f_A and f_B, there exists a unique continuous map k such that the following diagram is commutative, where p_A and p_B are the canonical projection:

$$\begin{tikzcd} A & A \times B \arrow[l, "p_A"'] \arrow[r, "p_B"] & B \\ & Z \arrow[ul, "f_A"] \arrow[u, dotted, "k"] \arrow[ur, "f_B"'] & \end{tikzcd}$$

So these two diagrams altogether give the commutative diagram where we replace Z by $A \times B$ in the first one and by $X \times Y$ in the second one, with $f_A = r_A \circ p_X, f_B = r_B \circ p_Y, f_X = \iota_A \circ p_A$ and $f_Y = \iota_B \circ p_B$, as follows

$$\begin{tikzcd} A & A \times B \arrow[l, "p_A"'] \arrow[r, "p_B"] & B \\ X \arrow[u, "r_A"] & X \times Y \arrow[l, "p_X"] \arrow[r, "p_Y"'] \arrow[u, "r"] \arrow[ul, "f_A"] \arrow[ur, "f_B"'] & Y \arrow[u, "r_B"'] \\ A \arrow[u, "\iota_A"] \arrow[uu, bend left=60, "Id_A"] & A \times B \arrow[l, "p_A"] \arrow[r, "p_B"'] \arrow[u, "\iota"] \arrow[ul, "f_X"] \arrow[ur, "f_Y"'] & B \arrow[u, "\iota_B"'] \arrow[uu, bend right=60, "Id_B"'] \end{tikzcd}$$

We have $r(x, y) = (r_A(x), r_B(y))$ and $\iota(a, b) = (\iota_A(a), \iota_B(b))$. It remains to verify that $r \circ \iota = Id_{A\times B}$ which is straightforward. ■

Properties of the Product

Proposition 4.2.22. *Let (X_1, τ_1) and (X_2, τ_2) be Hausdorff spaces. Then the product space $(X_1 \times X_2, \tau)$ is Hausdorff.*

Proof: Let $(x_1, x_2) \neq (y_1, y_2) \in X_1 \times X_2$, then $x_1 \neq y_1$ or $x_2 \neq y_2$. Suppose $x_1 \neq y_1$. X_1 is Hausdorff so there exists open disjoint neighbourhoods O_{x_1} of x_1 and O_{y_1} of y_1. Then $O_{x_1} \times X_2$ and $O_{y_1} \times X_2$ are open disjoint neighbourhoods of (x_1, x_2) and (y_1, y_2), then $(X_1 \times X_2, \tau)$ is Hausdorff. ■

Proposition 4.2.23. *Let (X_1, τ_1) and (X_2, τ_2) be topological spaces and let $A_1 \subset X_1, A_2 \subset X_2$. Then*

1. $\mathrm{Int}(A_1 \times A_2) = \mathrm{Int}(A_1) \times \mathrm{Int}(A_1)$

2. $\mathrm{Cl}(A_1 \times A_2) = \mathrm{Cl}(A_1) \times \mathrm{Cl}(A_1)$

Proof:

$\Longleftarrow$) $\mathrm{Int}(A_1) \times \mathrm{Int}(A_1) \subset \mathrm{Int}(A_1 \times A_2)$
Let $a_1 \in \mathrm{Int}(A_1)$ and $a_2 \in \mathrm{Int}(A_2)$. Then there exists open neighbourhoods $O_1 \subset A_1$ of a_1 and $O_2 \subset A_2$ of a_2, so $O_1 \times O_2 \subset \mathrm{Int}(A_1 \times A_2)$ is an open neighbourhood of (a_1, a_2), hence $(a_1, a_2) \in \mathrm{Int}(A_1 \times A_2)$.

$\Longrightarrow$) $\mathrm{Int}(A_1 \times A_2) \subset \mathrm{Int}(A_1) \times \mathrm{Int}(A_1)$
Let $(a_1, a_2) \in \mathrm{Int}(A_1 \times A_2)$, then there exists an open neighbourhood $O \subset A_1 \times A_2$ of (a_1, a_2). By definition of open set in a product, $O = \bigcup_i O_{1_i} \times O_{2_i}$ where $O_{1_i} \in \tau_1,\ O_{2_i} \in \tau_2$ and $O_{1_i} \subset A_1$ (resp.$O_{2_i} \subset A_2$) is an open neighbourhood of a_1 (resp. a_2), hence $a_1 \in \mathrm{Int}(A_1)$ and $a_2 \in \mathrm{Int}(A_2)$. Therefore $\mathrm{Int}(A_1 \times A_2) \subset \mathrm{Int}(A_1) \times \mathrm{Int}(A_1)$.

The case of closures is similar (exercise). ■

Proposition 4.2.24. *Let $(X_1 \times X_2, \tau)$ be the product space of the two topological spaces (X_1, τ_1) and (X_2, τ_2). Then p_1 (resp. p_2) sends open sets onto open sets.*

Proof: Let O be an open set of $X_1 \times X_2$, so that O is a union of products of open sets. For any $x_1 \in p_1(O)$, take $x_2 \in X_2$ such that $(x_1, x_2) \in O$. Hence, there exists $O_1 \in \tau_1, O_2 \in \tau_2$ such that $(x_1, x_2) \in O_1 \times O_2 \subset O$. Then $O_1 \subset p_1(O)$ is an open neighbourhood of x_1 in O_1, so $O_1 \in \tau_1$. ■

However it is not true that p_1 (resp. p_2) sends closed sets onto closed sets.

Example 4.2.25. *Let $X = Y = \mathbb{R}$ with the standard topology and let $X \times Y = \mathbb{R}^2$ be the product space. Let $A = \{(x, y) \in \mathbb{R}^2 \mid xy = 1\}$. The hyperbola A is closed and $p_1(A) = p_2(A) = \mathbb{R} \setminus \{0\}$ which is not closed. (Hint: Let $A^c = \{(x, y) \in \mathbb{R}^2 \mid xy \neq 1\}$. We have to show that A^c is open. Let $(x, y) \in A^c$ and let $\mathcal{L}$ be a line through (x, y) which meets A in two points. For example choose $\mathcal{L} = \{(x + t, y + t) \mid t \in \mathbb{R}\}$. Then find an open rectangle or an open quadrant containing (x, y) and contained in A^c.)*

Remark 4.2.26. *The duality does not work. So, this example is important. It shows that the duality has to be considered with caution.*

4.2.5 Initial Topology

The induced topology and the product topology are some particular cases of a more general setting.
Let X be a set, $(Y_i, \tau_i)_{i \in I}$ a family of topological spaces and let $f_i : X \longrightarrow Y_i$ be a family of maps.

Definition 4.2.27. *The* **initial topology** τ *on* X *w.r.t. the family* $(f_i)_{i\in I}$ *is the coarsest topology which makes all the maps* f_i *continuous.*

Then τ is the topology generated by $\left\{f_i^{-1}(O_i) \mid O_i \in \tau_i, i \in I\right\}$.
If $\mathcal{B}_i$ is a basis of open sets of Y_i, for any $i \in I$, then the set of all finite intersections of elements of $\left\{f_i^{-1}(O_i) \mid O_i \in \mathcal{B}_i, i \in I\right\}$ is a basis of τ.

Example 4.2.28. *The case of induced or subspace topology consists to consider* X *as a subset of the topological space* Y*, i.e. the family* $(Y_i)_{i\in I}$ *is reduced to one element,* Y*. It is the initial topology w.r.t. the inclusion* ι*.*
The product topology is the initial topology w.r.t. the family of projections $(p_j)_{j\in I}$ *where* $p_j : \prod\limits_i X_i \longrightarrow X_j$ *such that* $p_j((x_i)) = x_j$*.*

The initial topology has the following property

Proposition 4.2.29. *Let* τ *be the initial topology on* X *w.r.t. the family* $(f_i)_{i\in I}$ *where* $f_i : X \longrightarrow Y_i$*. For any topological space* (Z, τ_Z) *and any map* $g : Z \longrightarrow X$*, we have* g *is continuous iff* $f_i \circ g$ *is continuous for any* $i \in I$*.*

Proof: If g is continuous, then $f_i \circ g$ is continuous since each f_i is continuous.
Conversely, $\{f_i^{-1}(O_i) \mid O_i \in \tau_i, i \in I\}$ is a subbase of (X, τ).
Each $g^{-1}(f_i^{-1}(O_i)) = (f_i \circ g)^{-1}(O_i) \in \tau_Z$. Hence g is continuous. ■

Remark 4.2.30. *The property given in the proposition is not the universal mapping property. However, some authors call it universal.*

4.2.6 Exercises

1. Let $(X_1, \tau_1), (X_2, \tau_2)$ be two topological spaces with the basis Σ_1, Σ_2. Then, show that $\Sigma = \Sigma_1 \times \Sigma_2$ is a basis for the product topology on $X_1 \times X_2$.

2. Show that the topological subspace can be characterized by a universal property.

3. Let (X, d_X) and (Y, d_Y) be two metric spaces. There are several ways of making the product $X \times Y$ a metric space:
$$d_1\left((x_1, y_1), (x_2, y_2)\right) = d_X(x_1, x_2) + d_Y(y_1, y_2)$$
$$d_2\left((x_1, y_1), (x_2, y_2)\right) = \sup\left(d_X(x_1, x_2), d_Y(y_1, y_2)\right)$$
Show that $(X \times Y, d_1)$ and $(X \times Y, d_2)$ are metric spaces.
Show that the sequence $\big((x_n, y_n)\big)$ converges to (x_0, y_0) iff we have coordinate convergences.

4. Let (X, d) be a metric space. Then the map $d : X \times X \longrightarrow \mathbb{R}$ is continuous.

5. The standard topology on $\mathbb{R}^n$ is metric, either d_1, d_2 or d_∞. Show that this metric topology is the n times product of the standard topology on $\mathbb{R}$.

6. Prove that $\mathbb{R}^2 \setminus \{0\}$ as subspace of the Euclidean plane, is homeomorphic to $\mathbb{S}^1 \times]0, +\infty[$ with the product topology where S^1 is the subspace of the Euclidean plane and $]0, +\infty[$ is the subspace of $\mathbb{R}$ with the standard topology.

7. Let $f : X \longrightarrow Y$ be a homeomorphism from the topological space (X, τ_X) onto the topological space (Y, τ_Y). Let $A \subset X$ be a subset of X.
Show that the map $f_{|X\setminus A}$, restriction of f to the subspace $X \setminus A$ defines a homeomorphism $g : X \setminus A \longrightarrow Y \setminus f(A)$.

8. Show that $(\mathbb{R}, \tau_s)$ where τ_s is the standard topology, and $(\mathbb{R}_{>0}, \tau)$ where τ is the induced topology of τ_s, are two homeomorphic topological spaces and give a homeomorphism.

9. Show that $(\mathbb{R}^2 \setminus \{(0,0)\}, \tau_1)$, where τ_1 is the induced topology of the Euclidean topology, and $(\mathbb{S}^1 \times \mathbb{R}, \tau_2)$, where τ_2 is the product topology of the subspace $\mathbb{S}^1 \subset \mathbb{R}^2$ with τ_s, are homeomorphic.

10. Let X, Y, Z and T be the four following subspaces of $\mathbb{R}^2$:

$$\begin{aligned}
X &= \{(x,y) \in \mathbb{R}^2 \mid x \geq y \geq 0\} \\
Y &= \{(x,y) \in \mathbb{R}^2 \mid y \in [0,1[\} \\
Z &= \{(x,y) \in \mathbb{R}^2 \mid x \in [0,+\infty[, y \in [0,\pi/4]\} \\
T &= \{(x,y) \in \mathbb{R}^2 \mid y \geq 0\}
\end{aligned}$$

   (a) Draw the pictures of X, Y, Z and T.
   (b) Find a homeomorphism $f : X \longrightarrow Z$.
   (c) Find a homeomorphism $g : Z \longrightarrow T$.
   (d) Find a homeomorphism $h : T \longrightarrow Y$.
   (e) Show that $X \cong Y$.

11. Let $\mathbb{D}^n = \{x \in \mathbb{R}^n \mid d(O,x) = \|x\| \leq 1\}$ be the closed unit ball of $\mathbb{R}^n$ where $O = (0,\ldots,0)$ and $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$. Show that the following map f is a homeomorphism:

$$\begin{array}{rcl}
f : \mathbb{D}^1 \times \mathbb{D}^2 & \longrightarrow & \mathbb{D}^3 \\
(x,y) & \longmapsto & \begin{cases} \dfrac{\|y\|}{\sqrt{\|x\|^2 + \|y\|^2}}(x,y) & \text{if } 0 < \|x\| < \|y\| \\ \dfrac{\|x\|}{\sqrt{\|x\|^2 + \|y\|^2}}(x,y) & \text{if } 0 < \|y\| < \|x\| \end{cases}
\end{array}$$

12. Let $\mathbf{P}_1(\mathbb{R})$ be the projective line with the Zariski topology. Let $s : \mathbf{P}_1(\mathbb{R}) \times \mathbf{P}_1(\mathbb{R}) \longrightarrow \mathbf{P}_3(\mathbb{R})$ be the map defined by $s((a_0 : a_1), (b_0 : b_1)) = (a_0 b_0 : a_0 b_1 : a_1 b_0 : a_1 b_1)$.

   (a) Show that the image of s is a closed set w.r.t. Zariski topology.
   (b) Show that $\mathbf{P}_1(\mathbb{R}) \times \mathbf{P}_1(\mathbb{R})$ w.r.t. Zariski topology is not the product in $\mathbf{P}_3(\mathbb{R})$ w.r.t. Zariski topologies.

13. Let C be the Cantor set. Let $\{0,1\}$ be the set equipped with the discrete topology and let $\{0,1\}^{\mathbb{N}}$ be the product space. Show that the map

$$\begin{array}{rcl}
f : \{0,1\}^{\mathbb{N}} & \longrightarrow & C \\
(x_i)_{i \in \mathbb{N}} & \longmapsto & \displaystyle\sum_{i \geq 0} \frac{2x_i}{3^{i+1}}
\end{array}$$

   is a homeomorphism. Show that $C^{\mathbb{N}}$ and C are homeomorphic.

14. Show that the Zariski topology on the product $X \times Y$ is not the product of the topological spaces X and Y equipped with the Zariski topology.

15. Let (X, τ_X) and (Y, τ_Y) be two topological spaces and $A \subseteq X, B \subseteq Y$. The product $A \times B$ can be viewed as a subspace of the product space $X \times Y$ and as the product of the two subspaces (A, τ_{X_A}) and (B, τ_{Y_B}).
Do we have the same topology on $A \times B$?

16. Let $\{X_\alpha\}_{\alpha \in A}$ be a family of topological spaces. Consider the family of the sets of the form $\prod_{\alpha \in A} O_\alpha$ where O_α is an open set in X_α for all $\alpha \in A$.
Show that it is a basis for a topology on the product space $\prod_{\alpha \in A} X_\alpha$. This topology is called the **box topology**.
Show that the box topology is, in general, finer than the product topology.
If the set A is finite, then the two topologies are identical.
There are many important theorems about finite products that also hold for arbitrary products if we use product topology, but not if we use bow topology. So, the box topology is not so important.

4.3 Final Topology

4.3.1 Topological Quotient - Quotient Topology

Let X be a set and let $\mathcal{R}$ be an equivalence relation on X. The quotient set denoted $X/\mathcal{R}$ is the set of all the equivalence classes modulo $\mathcal{R}$. The equivalence classes define a partition of X. Then there is a canonical surjection:

$$\begin{array}{rcl} p : X & \longrightarrow & X/\mathcal{R} \\ x & \longmapsto & p(x) = \{y \in X \mid y\mathcal{R}x\} := [x] \end{array}$$

Let τ be a topology on X. We have to define a "suitable" topology on the quotient set $X/\mathcal{R}$ which makes the map p continuous. For example, the trivial topology on $X/\mathcal{R}$ makes p continuous and this topology does not depend on τ but there exists many others.
We will choose the finest (i.e. strongest or largest) topology $\tau_\mathcal{R}$ on $X/\mathcal{R}$ which makes the map p continuous.

Definition 4.3.1. *Let (X, τ) be a topological space, let $X/\mathcal{R}$ be the quotient set and let define*

$$\boxed{\tau_\mathcal{R} = \left\{\mathbf{O} \subset \mathbf{X}/\mathcal{R} \mid \mathbf{p^{-1}(O)} \in \tau\right\}}.$$

Then $\tau_\mathcal{R}$ is called the **quotient topology** *of the topology τ mod $\mathcal{R}$.*

Notice that $\tau_\mathcal{R}$ is the finest (i.e. strongest or largest) topology making p continuous.
For any coarser topology τ than $\tau_\mathcal{R}$, the map p is continuous, so why do we choose this topology $\tau_\mathcal{R}$?
The quotient topology satisfies the following ***universal mapping property***, i.e. for any (Z, τ_Z) and any continuous map $f : X \longrightarrow Z$ such that $x\mathcal{R}x' \Longrightarrow f(x) = f(x')$, there exists a unique continuous map $h : X/\mathcal{R} \longrightarrow Z$ such that $f = h \circ p$.
For a coarser topology than $\tau_\mathcal{R}$ on the quotient $X/\mathcal{R}$, the map h may be not continuous, so the universal mapping property may be not satisfied.

Example 4.3.2. *Let $(Z, \tau_Z) = (X/\sim, \tau_\mathcal{R})$, then $f = p$ and $h = Id$. Let $\tau \subsetneqq \tau_\mathcal{R}$, and let $O \in \tau_\mathcal{R} \setminus \tau$, then $h^{-1}(O)$ is not open, so h is not continuous.*

Surjections

More generally, let $f : X \longrightarrow B$ be a surjection, where (X, τ) is a topological space and B a set. Let $\mathcal{R}_f$ be the equivalence relation on X where $x\mathcal{R}_f x'$ if $f(x) = f(x')$.
Let denote $[x] = \{x' \in X \mid f(x') = f(x)\}$. Then $\bigcup_{x\in X} [x] = X$ and $[x] \cap [x'] = \emptyset$ if $f(x) \neq f(x')$, $[x] \cap [x'] = [x] = [x']$ if $f(x) = f(x')$. Hence, we get a partition of X. We obtain the following commutative diagram:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & B \\ & X/\mathcal{R}_f \arrow[ur, "g", "\simeq"'] & \end{tikzcd}$$

where p is the canonical surjection, and $g([x]) = f(x)$, so that g is a bijection.
Let $\tau_{\mathcal{R}_f}$ be the quotient topology on $X/\mathcal{R}_f$.

Definition 4.3.3. *The* **identification topology** τ_B *on* B *is the topology making the bijection* $g : X/\mathcal{R}_f \xrightarrow{\simeq} B$ *an homeomorphism* $g : (X/\mathcal{R}_f, \tau_{\mathcal{R}_f}) \xrightarrow{\cong} (B, \tau_B)$.

Recall that, given a bijection from one topological space onto a set, there is a topology on the set such that the bijection is a homeomorphism.

Definition 4.3.4. *Let* (X, τ_X) *be a topological space. A surjective continuous map* $f : X \longrightarrow B$ *is called a* **quotient map** *if the topology on* B *is the identification topology.*

It means that a subset O of B is open iff the set $f^{-1}(O)$ is open in X.
An equivalent definition of quotient map is given by replacing open by closed, i.e. a subset C of B is closed iff the set $f^{-1}(C)$ is closed in X.
Notice that the continuity follows from the definition.
A continuous surjection that is either open or closed is a quotient map. But there exist quotient maps that are neither open nor closed.
Notice that the quotient maps and the embeddings are dual notions.
It is possible to extend the construction of quotient map to any map $f : X \longrightarrow Y$. If f is continuous, f can be factored as $f = g \circ \pi_f : X \longrightarrow X/\mathcal{R}_f \longrightarrow Y$ where π_f and g are continuous, π_f surjective and g injective.

Remark 4.3.5. *We focused on the duality between initial and final topologies. However, duality has its own limits. For example, a subspace of Hausdorff space is Hausdorff as well any product of Hausdorff spaces is Hausdorff. But the quotient space of a Hausdorff space is not necessarily Hausdorff. Let us give an example. Let* $X = [0, 1]$ *and* $\tau = \{\{0\}, (0, 1), \{1\}\}$ *which is a partition of* X*, is not Hausdorff. So, duality must be used with care.*

Remark 4.3.6. *Notice that there is a well known similar process for group homomorphisms; more precisely, the Fundamental Homomorphism Theorem is as follows: Let* $f : G \longrightarrow H$ *be a homomorphism of groups. Then* $f(G)$ *is a group and there is a canonical isomorphism of* $f(G)$ *with* $G/\ker(f)$*. In other words,* f *can be factored as* $f = g \circ \pi_f : G \longrightarrow G/\ker(f) \longrightarrow H$ *where* $\ker(f)$ *and* g *are group homomorphisms,* π_f *surjective and* g *injective.*

Proposition 4.3.7. *Let* (X, τ_X) *be a topological space and let* $(X/\mathcal{R}, \tau_{\mathcal{R}})$ *be the quotient topological space. Let* (Z, τ_Z) *be a topological space and let* $g : X/\mathcal{R} \longrightarrow Z$ *be a map. Then* g *is continuous iff* $g \circ \pi$ *is continuous.*

Proof: (exercise) ■

4.3.2 Sections

Let X and Y be two topological spaces.

Definition 4.3.8. *The continuous map $s : Y \longrightarrow X$ is said to be a* **section** *if there exists $f : X \longrightarrow Y$ such that $f \circ s = \mathrm{Id}_Y$, i.e. s is a right inverse of f.*

If f has a section, then f is surjective and s is injective. Moreover, s is a right inverse of f, so, f is a left inverse of s and we can say that s is a section of f iff f is a retraction of s.

Proposition 4.3.9. *Let $f : X \longrightarrow Y$ be a surjection and Z a topological space. There is a section $s : Y \longrightarrow X$ iff for every continuous map $g : Z \longrightarrow Y$, there is a continuous map $h : Z \longrightarrow X$ such that $f \circ h = g$.*

Proof: (exercise)

$$\begin{tikzcd}
Y \arrow[r, "s"] \arrow[rr, bend left, "Id_Y"] & X \arrow[r, "f"] & Y \\
& Z \arrow[ul, "g"] \arrow[u, dotted, "h"] \arrow[ur, "g"'] &
\end{tikzcd}$$

■

In particular, let $Z = X/\mathcal{R}_f$. We have the two following commutative diagrams

$$\begin{tikzcd}
Y \arrow[r, "s"] \arrow[rr, bend left, "Id_Y"] & X \arrow[r, "f"] \arrow[d, "p"] & Y \\
& X/\mathcal{R}_f \arrow[ul, "g"] \arrow[ur, "g"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& X/\mathcal{R}_f \arrow[dl, "g"'] \arrow[d, dotted, "h"] \arrow[dr, "g"] & \\
Y \arrow[r, "s"] \arrow[rr, bend right, "Id_Y"'] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

where the map g is a homeomorphism such that $h = s \circ g$ and $p = g^{-1} \circ f$ so

$$p \circ h = (g^{-1} \circ f) \circ (s \circ g) = g^{-1} \circ (f \circ s) \circ g = g^{-1} \circ Id_Y \circ g = Id_{X/\mathcal{R}_f}$$

These two diagrams altogether show that the map h is a section of the map p,

$$\begin{tikzcd}
X/\mathcal{R}_f \arrow[r, "h"] \arrow[rr, bend left, "Id_{X/\mathcal{R}_f}"] & X \arrow[r, "p"] & X/\mathcal{R}_f
\end{tikzcd}$$

4.3.3 Topology Sum - Coproduct Topology

Let (X_1, τ_1) and (X_2, τ_2) be two topological spaces. We defined the disjoint sum $X_1 \coprod X_2$ of the two sets X_1 and X_2. There are two canonical maps $\iota_1 : X_1 \longrightarrow X_1 \coprod X_2,\ x_1 \mapsto (1, x_1)$ and $\iota_2 : X_2 \longrightarrow X_1 \coprod X_2,\ x_2 \mapsto (2, x_2)$.

$$X_1 \xleftarrow{\ \iota_1\ } X_1 \coprod X_2 \xrightarrow{\ \iota_2\ } X_2$$

We have to define a topology τ on $X_1 \coprod X_2$ which makes the maps ι_1 and ι_2 continuous. Notice that the trivial topology on $X_1 \coprod X_2$ makes the maps ι_1 and ι_2 continuous. The "suitable" topology is the finest (i.e. strongest or largest) one making the maps ι_1 and ι_2 continuous.
Recall that we identified X_1, (resp. X_2) with the subset $(\{1\} \times X_1)$, (resp. $(\{2\} \times X_2)$).

Definition 4.3.10. *$(X_1 \coprod X_2, \tau)$ is called the* **sum** *or* **coproduct** *of the topological spaces (X_1, τ_1) and (X_2, τ_2) if*

$$\boxed{\tau = \left\{\mathbf{O} \subset \mathbf{X_1} \coprod \mathbf{X_2} \mid \mathbf{O} \cap \mathbf{X_1} \in \tau_1 \text{ and } \mathbf{O} \cap \mathbf{X_2} \in \tau_2\right\}}.$$

Remark 4.3.11. *This definition extends to any family $(X_i)_{i\in I}$ of topological spaces to define the sum $\coprod_{i\in I} X_i = \bigcup_{i\in I} \{i\} \times X_i$.*

Proposition 4.3.12. *Let $X_1 \coprod X_2$ be the sum of the two topological spaces (X_1, τ_1) and (X_2, τ_2). Let (Z, τ_Z) be a topological space.*

1. *Let $f : X_1 \coprod X_2 \longrightarrow Y$. Then f is continuous iff each map $f \circ \iota_1$ and $f \circ \iota_2$ is continuous.*

2. ***Universal Mapping Property**: Let $f_1 : X_1 \coprod X_2 \longrightarrow Y, f_2 : X_1 \coprod X_2 \longrightarrow Y$ be two continuous maps. Then there exists a unique continuous map $h : X_1 \coprod X_2 \longrightarrow Y$ such that $h \circ \iota_1 = f_1$ and $h \circ \iota_2 = f_2$.*

Proof: (exercise) ■

For a coarser topology than the topology τ on the sum $X_1 \coprod X_2$, the map h may be not continuous, so the universal mapping property should not satisfied.

4.3.4 Final Topology

The sum topology, the coproduct topology are some particular cases of a more general setting. Let X be a set, $(Y_i, \tau_i)_{i\in I}$ a family of topological spaces and let $f_i : Y_i \longrightarrow X$ be family of maps.

Definition 4.3.13. *The* **final topology** *τ on X defined by the family $(f_i)_{i_inI}$ is the finest topology which makes all the maps f_i continuous.*

Then $\tau = \left\{O \mid f_i^{-1}(O) \in \tau_i, i \in I\right\}$.
The case of sum topology consists to consider X as $\coprod_{i\in I} X_i$ and f_i as the canonical maps ι_i.

Remark 4.3.14. *Notice the duality between initial and final topologies.*

The final topology has the following property

Proposition 4.3.15. *Let τ be the final topology on X w.r.t. the family $(f_i)_{i\in I}$ where $f_i : Y_i \longrightarrow X$. For any topological space (Z, τ_Z) and any map $g : X \longrightarrow Z$, we have g is continuous iff $g \circ f_i$ is continuous for any $i \in I$.*

Proof: If g is continuous, then $g \circ f_i$ is continuous since each f_i is continuous.
Conversely, let $O \in \tau_Z$ and $g^{-1}(O) = U \subset X$. The maps f_i are continuous, so $f_i^{-1}(U) = f_i^{-1}(g^{-1}(O)) = (g \circ f_i)^{-1}(O) \in \tau_i$ for any $i \in I$. It follows that $U \in \tau$ and g is continuous. ■

Remark 4.3.16. *The property given in the proposition is not the universal mapping property. However, some authors call it universal.*

4.3.5 Initial & Final Topologies

Subspaces and Product of Spaces

Let (X, τ_X) and (Y, τ_Y) be two topological spaces and $A \subset X, B \subset Y$ two subsets. Consider the product $A \times B$ of the two sets. Then $A \times B$ is assigned with the product topology of the two subspaces (A, τ_{A_X}) and (B, τ_{B_Y}). But $A \times B$ is also assigned with the induced topology of the product topology on $X \times Y$.
The question is: are these two topologies on $A \times B$ the same?
Notice that the induced topology and the product topology are two initial topologies.

Quotient Spaces and Sum of Spaces

Let (X, τ_X) and (Y, τ_Y) be two topological spaces and $\mathcal{R}_X$ (resp. $\mathcal{R}_Y$) be an equivalence relation on X (resp. Y). Consider the sum $(X \coprod Y, \tau)$ and the equivalence relation $\mathcal{R}$ on the sum given by the two equivalence relations on X and Y. Another way to see it, is to consider the partitions of X and Y defined by the equivalence relations on X and Y and the partition of the disjoint union. The two sets $(X \coprod Y)/\mathcal{R}$ and $(X/\mathcal{R}_X) \coprod (Y/\mathcal{R}_Y)$ are the same up to a bijection.
The question is: Are the quotient topology on the sum $X \coprod Y$ and the topology sum of the quotient spaces $(X/\mathcal{R}_X) \coprod (Y/\mathcal{R}_Y)$, the same?
Notice that the quotient topology and the sum topology are two final topologies.

Quotient Spaces and Product of Spaces

Let (X, τ_X) and (Y, τ_Y) be two topological spaces and $\mathcal{R}_X$ (resp. $\mathcal{R}_Y$) be an equivalence relation on X (resp. Y), i.e. there exist a partition of X and a partition of Y. Then we get a partition of the product $X \times Y$, i.e. an equivalence relation $\mathcal{R}$ on $X \times Y$. Are the two sets $X/\mathcal{R}_X \times Y/\mathcal{R}_y$ and $X \times Y/\mathcal{R}$ the same? Moreover, $X/\mathcal{R}_X \times Y/\mathcal{R}_y$ is assigned with the product of the quotient topologies and $X \times Y/\mathcal{R}$ with the quotient topology of the product space. Are these two topologies same?
Notice that the product topology is initial topology and quotient topology is final topology.

Subspaces and Sum of Spaces

Let (X, τ_X) and (Y, τ_Y) be two topological spaces and $A \subset X, B \subset Y$ two subsets. Consider the sum $A \coprod B$ of the two sets. Then $A \coprod B$ is assigned with the sum topology of the two subspaces (A, τ_{A_X}) and $(B, \tau_{B?Y})$. But $A \coprod B$ is also assigned with the induced topology of the sum topology of $X \coprod Y$. Are the two topologies the same?
Notice that the induced topology is initial topology and sum topology is final topology.

Subspaces and Quotient Spaces

Let (X, τ_X) be a topological space and $A \subset X$. Let $p : X \longrightarrow X/\sim$ be a quotient map. Is the restriction $p_{|A}$ a quotient map?
Notice that the induced topology is initial topology and quotient topology is final topology.

Product and Sums of Spaces

Let $(X_1, \tau_{X_1}), (X_2, \tau_{X_2}), (Y_1, \tau_{Y_1}), (Y_2, \tau_{Y_2})$ be four topological spaces. Compare the topological spaces $(X_1 \coprod X_2) \times (Y_1 \coprod Y_2)$ and $(X_1 \times X_2) \coprod (Y_1 \times Y_2)$.

4.3.6 Exercises

1. Define the topology sum and the topology quotient as final topologies.

2. Answer to the two last questions in the previous subsection.

3. Let $s_1 : X_1 \longrightarrow Y_1$ and $s_2 : X_2 \longrightarrow Y_2$ be two sections, then show that there is a section $X_1 \coprod X_2 \longrightarrow Y_1 \coprod Y_2$ associated to s_i.

4. Consider the following relation $\mathcal{R}$ on $\mathbb{R}^2$
$$(x, y)\mathcal{R}(x', y') \text{ if } xy = x'y'$$
Show that $\mathcal{R}$ is an equivalence relation and $\mathbb{R}^2/\mathcal{R}$ is homeomorphic to $\mathbb{R}$ where the topologies on $\mathbb{R}^2$ and $\mathbb{R}$ are the Euclidean and standard ones.

5. Show that the following map is continuous
$$\begin{array}{rcl} f : [0,1] \times [0,1] & \longrightarrow & B \\ (s,t) & \longmapsto & f(s,t) = se^{2i\pi t} \end{array}$$
where $B = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ and $[0, 1] \times [0, 1]$ are subspaces of $\mathbb{R}^2$ with the Euclidean topology.
Determine an equivalence relation on $[0, 1] \times [0, 1]$ such that the quotient space $[0, 1] \times [0, 1]/\sim$ is homeomorphic to B.
Determine the open sets O in B such that $f^{-1}(O)$ is $(a, b) \times (c, d)$, for $0 \leq a < b \leq 1$ and $0 \leq c < d \leq 1$.

6. Let (X, τ_X) and (Y, τ_Y) be two topological spaces. Show that the map $p_X : X \times Y \longrightarrow X$ is a quotient map. Are the two topologies τ_X and the quotient topology on X are the same?

7. Consider the following map
$$\begin{array}{rcll} f : [0,1] \times [0,1] & \longrightarrow & \mathbb{R}^2 & \\ (s,t) & \longmapsto & (-1+2s, 2s(2t-1)) & \text{for } 0 \leq s \leq \dfrac{1}{2} \\ (s,t) & \longmapsto & (-1+2s, 2(1-s)(2t-1)) & \text{for } \dfrac{1}{2} \leq s \leq 1 \end{array}$$
   (a) Show that f is a continuous map.
   (b) Determine $f([0, 1] \times [0, 1])$.
   (c) Show that $f([0, 1] \times [0, 1])$ is homeomorphic to the quotient space $[0, 1] \times [0, 1]/\sim$ and define the equivalence relation.
   (d) Determine the open sets of $f([0, 1] \times [0, 1])$.

8. Given two topological spaces $(X, \tau_X), (Y, \tau_Y)$ and $\mathcal{R}_X, \mathcal{R}_Y$ some equivalence relations on X and Y respectively.
   (a) Let $\mathcal{R}$ be the equivalence relation on the sum $X \coprod Y$ generated by $(0, x)\mathcal{R}(0, x')$ if $x\mathcal{R}_X x'$ and $(1, y)\mathcal{R}(1, y')$ if $y\mathcal{R}_Y y'$.
   Let $(X/\mathcal{R}_X, \tau_{\mathcal{R}_X})$ and $(Y/\mathcal{R}_Y, \tau_{\mathcal{R}_Y})$ be the two quotient spaces. Compare the two topological spaces $(X/\mathcal{R}_X \coprod Y/\mathcal{R}_Y, \tau)$ and $(X \coprod Y)/\mathcal{R}, \tau')$ where τ is the sum of the quotient topologies and τ' is quotient topology of the sum topology.

(b) Let $\sim$ be the equivalence relation on the product $X \times Y$ defined as follows: $(x, y) \sim (x', y')$ if $x\mathcal{R}x'$ and $y\mathcal{R}y'$.
Show that the map $X \times Y \longrightarrow X/\mathcal{R}_X \times Y/\mathcal{R}_Y : (x, y) \longmapsto (p_X(x), p_Y(y))$ induces a continuous bijection $\varphi : (X \times Y)/\sim \longrightarrow (X/\mathcal{R}_X) \times (Y/\mathcal{R}_Y)$.
Show that if the canonical surjections p_X and p_Y are open, then the map φ is a homeomorphism.

9. Let $\mathbf{U} = \{z \in \mathbb{C} \mid |z| = 1\}$ as a subspace of $\mathbb{R}^2$.
Show that $\mathbf{U}$ is Hausdorff (Find two disjoint neighbourhoods of two distinct points).

10. Let $X \coprod Y$ be the topological sum of the two topological spaces (X, τ_X) and (Y, τ_Y). Show that, given a topological space (Z, τ_Z) and two continuous maps $f_X : X \longrightarrow Z$, $f_Y : Y \longrightarrow Z$, there exists a unique continuous map $h : Z \longrightarrow X \coprod Y$ such that $h \circ i_X = f_X$ and $h \circ i_Y = f_Y$.

11. Show that if the quotient topology is Hausdorff, then the equivalence classes are closed.

12. For any topological spaces X and Y, and continuous functions $f : X \longrightarrow \mathbb{R}$, $g : Y \longrightarrow \mathbb{R}$, prove that the function $h : X \times Y \longrightarrow \mathbb{R}$ such that $h((x, y)) = f(x).g(y)$ is continuous.

13. Consider the equivalence relation on $\mathbb{R}$: $x \sim y$ if $x - y \in \mathbb{Q}$. Show the the quotient topology on $\mathbb{R}/\sim$ is the trivial topology. (Hint: Show that $p^{-1}(A)$, where $p : \mathbb{R} \longrightarrow \mathbb{R}/\sim$, is dense.)

4.4 Applications

4.4.1 Contraction

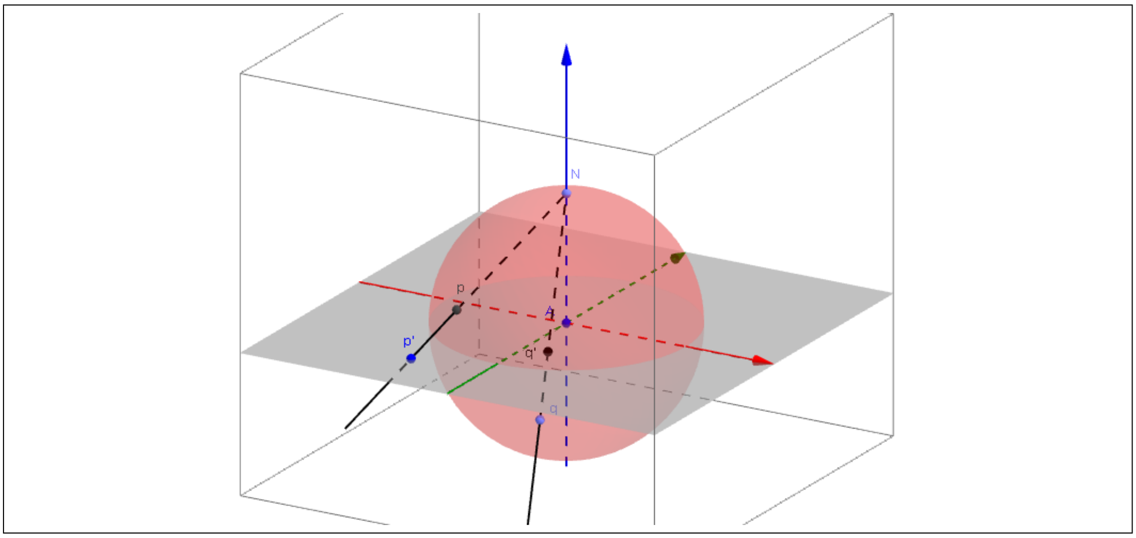
Let (X, τ) be a topological space and let $A \subset X$. Then we get the partition of X as the subsets A and all the one-point sets $\{x\}$ where $x \notin A$. Denote $(X/A, \tau_A)$ the quotient space called the **contraction** of A in X. For example,

1. Let $X = [0, 1]$ be the interval equipped with the induced topology of the standard topology on $\mathbb{R}$. Let $A = \{0, 1\}$. Then $X/A \cong \mathbb{S}^1$ (homeomorphism) (exercise).

2. Let $\mathbb{R}^n$ be the Euclidean space where $\| (x_1, \ldots, x_n) \| = \sqrt{x_1^2 + \cdots + x_n^2}$.
Let $\mathbb{D}^n = \{x \in \mathbb{R}^n \bigm| \| x \| \leq 1\}$ and $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n \bigm| \| x \| = 1\}$ as subspaces of the Euclidean space. Then $\mathbb{D}^n/\mathbb{S}^{n-1} \cong \mathbb{S}^n$ (homeomorphism).
Proof:: Let $N = (0, \ldots, 0, 1) \in \mathbb{S}^{n-1}$ be the north pole. $\mathbb{D}^n = B^n \cup \mathbb{S}^{n-1}$ where $B^n = \{x \in \mathbb{R}^n \bigm| \| x \| < 1\}$ is the open unit ball and $B^n \cap \mathbb{S}^{n-1} = \emptyset$.
Define the map

$$\begin{array}{rcl} f : \mathbb{S}^n \setminus \{N\} & \longrightarrow & \mathbb{R}^n \\ (x_1, \ldots, x_{n+1}) & \longmapsto & \left(\dfrac{x_1}{1 - x_{n+1}}, \cdots, \dfrac{x_n}{1 - x_{n+1}} \right) \end{array}$$

For $n = 1$, we get $(x_1, x_2) \longmapsto \dfrac{x_1}{1 - x_2}$, so $p_i \circ f(x_1, \ldots, x_n) = \dfrac{x_i}{1 - x_n}$ where p_i is the projection on the i^{th} factor.
This map f is called the stereographic projection.



Define the map

$$\begin{array}{rcl} g:\mathbb{R}^n & \longrightarrow & \mathbb{S}^n\setminus\{N\} \\ (x_1,\ldots,x_n) & \longmapsto & \left(\dfrac{2x_1}{1+\parallel x\parallel^2},\cdots,\dfrac{2x_n}{1+\parallel x\parallel^2},\dfrac{\parallel x\parallel^2-1}{\parallel x\parallel^2+1}\right) \end{array}$$

where $\parallel x\parallel^2=x_1^2+\cdots+x_n^2$.
The two maps f and g are continuous and inverses of each other, i.e. $\mathbb{S}^n\setminus\{N\}$ and $\mathbb{R}^n$ are homeomorphic.
We show it for $n=2$.
$g(x_1,x_2)=\left(\dfrac{2x_1}{1+\parallel x\parallel^2},\dfrac{2x_2}{1+\parallel x\parallel^2},\dfrac{\parallel x\parallel^2-1}{\parallel x\parallel^2+1}\right)$.

$$\begin{aligned} \parallel g(x_1,x_2)\parallel^2 &= \frac{(2x_1)^2+(2x_2)^2+(x_1^2+x_2^2-1)^2}{(x_1^2+x_2^2+1)^2} \\ &= \frac{4x_1^2+4x_2^2+x_1^4+x_2^4+1+2x_1^2x_2^2-2x_1^2-2x_2^2}{x_1^4+x_2^4+1+2x_1^2x_2^2+2x_1^2+2x_2^2} \\ &= 1 \end{aligned}$$

so g is a map from $\mathbb{R}^2$ to $\mathbb{S}^2$.
For all (x_1,x_2) we have $x_1^2+x_2^2-1<x_1^2+x_2^2+1$, so $g(x_1,x_2)\neq N$ and g is a map from $\mathbb{R}^2$ to $\mathbb{S}^2\setminus\{N\}$.
If $g(x_1,x_2)=(x,y,z)$, then $x=\dfrac{2x_1}{x_1^2+x_2^2+1}$ and

$$1-z=\frac{x_1^2+x_2^2-1}{x_1^2+x_2^2+1}=\frac{(x_1^2+x_2^2+1)-(x_1^2+x_2^2-1)}{x_1^2+x_2^2+1}=\frac{2}{x_1^2+x_2^2+1}$$

so $\dfrac{x}{1-z}=x_1$ and similarly $\dfrac{y}{1-z}=x_2$. Hence $f(g(x_1,x_2))=(x_1,x_2)$.
Let us show that $g(f(x,y,z))=(x,y,z)$ where $(x,y,z)\in\mathbb{S}^2$ and $f(x,y,z)=(x_1,x_2)$.

We have $x^2+y^2+z^2=1$ and $x_1=\dfrac{x}{1-z}, x_2=\dfrac{y}{1-z}$. So,

$$x_1^2+x_2^2+1=\frac{x^2+y^2+(1-z)^2}{(1-z)^2}=\frac{x^2+y^2+z^2+1-2z}{(1-z)^2}=\frac{2-2z}{(1-z)^2}=\frac{2}{1-z}$$

Thus

$$\frac{2x_1}{x_1^2+x_2^2+1}=\frac{2x}{1-z}\Big/\frac{2}{1-z}=x$$

Similarly, we have $\dfrac{2x_2}{x_1^2+x_2^2+1}=y$. We also have $x^2+y^2=1-z^2$, so

$$\begin{aligned}x_1^2+x_2^2-1=&\frac{x^2+y^2-(1-z)^2}{(1-z)^2}\\ =&\frac{x^2+y^2-z^2-1+2z}{(1-z)^2}\\ =&\frac{1-z^2-z^2-1+2z}{(1-z)^2}\\ =&\frac{2z(1-z)}{1-z}=\frac{2z}{1-z}\end{aligned}$$

We see that $\dfrac{x_1^2+x_2^2-1}{x_1^2+x_2^2+1}=z$ and it follows $g(f(x,y,z))=(x,y,z)$.
Define the map

$$\begin{array}{rcl} h: B^n & \longrightarrow & \mathbb{R}^n \\ (x_1,\dots,x_n) & \longmapsto & \left(\dfrac{x_1}{1-\parallel x\parallel},\dots,\dfrac{x_n}{1-\parallel x\parallel}\right)\end{array}$$

The map h is a homeomorphism. (exercise)
We have

$$B^n \xrightarrow{h} \mathbb{R}^n \xrightarrow{g} \mathbb{S}^n\setminus\{N\} \qquad \text{and} \qquad \mathbb{S}^{n-1}\longrightarrow\{N\}$$

So, we can define the map

$$\begin{array}{rcl} k:\mathbb{D}^n=B^n\cup\mathbb{S}^{n-1} & \longrightarrow & \mathbb{S}^n \\ (x_1,\dots,x_n) & \longmapsto & g(h(x_1,\dots,x_n)) \text{ if } \parallel \mathrm{x}\parallel<1 \text{ i.e. } \mathrm{x}\in \mathrm{B}^{\mathrm{n}} \\ (x_1,\dots,x_n) & \longmapsto & N \text{ if } \parallel \mathrm{x}\parallel=1 \text{ i.e. } \mathrm{x}\in\mathbb{S}^{\mathrm{n}-1}\end{array}$$

The map k is a quotient map which implies that $\mathbb{D}^n/\mathbb{S}^{n-1}\cong\mathbb{S}^n$ and $\mathbb{S}^n$ is homeomorphic to the contraction of $\mathbb{S}^{n-1}$ in the space $\mathbb{D}^n$. ■

4.4.2 Gluing

Let $(A,\tau_A),(B,\tau_B)$ be two homeomorphic topological subspaces of the topological space (X,τ) and let $h:A\longrightarrow B$ be a homeomorphism. Define the partition of X given by the one-point sets $\{x\}$ where $x\notin A\cup B$ and the sets $\{x,h(x)\}$ where $x\in A$. Denote $\mathcal{R}_f$ or $\mathcal{R}_{AB}$ the equivalence relation defined by this partition. Then the quotient space $X/\mathcal{R}_f$ is obtained by **identifying** (or **gluing**) the sets A and B via the homeomorphism h.

Example 4.4.1. *Let $I=[0,1]$ and I^2 the subspaces of $\mathbb{R}$ and $\mathbb{R}^2$ equipped with the Euclidean topologies.*

1. **Circle**
Consider the following commutative diagram:

$$\begin{tikzcd} I \arrow[rr, "f"] \arrow[dr, "p"'] & & \mathbb{S}^1 \\ & I/\mathcal{R}_f \arrow[ur, "g", "\cong"'] & \end{tikzcd}$$

where $f(t) = e^{2i\pi t}, t \in I$ *and* $\mathcal{R}_f$ *is the equivalence relation such that* $0\mathcal{R}_f 1$ *and the other equivalence classes are the one-point sets.*
This construction consists to identify the points 0 *and* 1 *of* I *into one point to obtain the circle.*
Given I *with the induced topology as subspace of* $\mathbb{R}$ *with standard topology, we define the continuous map* f *onto the circle (subspace of* $\mathbb{R}^2$ *with the Euclidean topology). Then we get the homeomorphism* g*. (exercise: give explicitely* g *and prove it is a homeomorphism).*
Notice that the quotient space represents the "abstract" topological space and the homeomorphism gives a "concrete" space if it is suitably defined. These two spaces are defined as a quotient space for one and as a subspace for the other one.
The "dual" construction consists to delete one point of the circle. Let $f :]0,1[\longrightarrow \mathbb{S}^1$ *be the injection defined by* $f(t) = e^{2i\pi(t-t_0)}$ *and denote* $* = e^{2i\pi t_0}$*. Then*

$$\begin{tikzcd}]0,1[\arrow[rr, "f"] \arrow[dr, "\cong", "g"'] & & \mathbb{S}^1 \\ & \mathbb{S}^1 \setminus \{*\} \arrow[ur, hook, "\iota"] & \end{tikzcd}$$

Given the circle with the induced topology as subspace of $\mathbb{R}^2$ *with the Euclidean topology, the injection defines the induced topology on* $]0,1[$*. Then the homeomorphism* g *identifies* $]0,1[$ *and* $\mathbb{S}^1 \setminus \{*\}$*.*

2. **Cylinder**
Let $A = \{(0,t), t \in I\}$ *and* $B = \{(1,t), t \in I\}$ *be two subsets of* I^2 *and let* $h : A \longrightarrow B$ *be the homeomorphism defined by* $h((0,t)) = (1,t), t \in I$*.*
Let $p : I \longrightarrow \mathbb{S}^1$ *be the continuous surjection defined by* $p(s) = e^{2i\pi s}$*. Let* $f : I^2 \longrightarrow \mathbb{S}^1 \times I$ *be the map defined by* $f((s,t)) = (p(s), t)$*. If* $(s,t) \in A$*, i.e.* $(s,t) = (0,t)$*, then* $f((0,t)) = (1,t) = f(1,t) = f(h(0,t),t)$*. Then* f *is a continuous surjection so that* f *is a quotient map. The equivalence relation* $\mathcal{R}_f = \mathcal{R}_{AB}$ *is in fact defined by the homeomorphism* h *and we have the following commutative diagram.*

$$\begin{tikzcd} I^2 \arrow[rr, "f"] \arrow[dr, "p"'] & & \mathbb{S}^1 \times I \\ & I^2/\mathcal{R}_{AB} \arrow[ur, "g", "\cong"'] & \end{tikzcd}$$

Notice that the square represents the **cylinder** $\mathbb{S}^1 \times I$*, the two vertical sides with the two upwards arrows, represent the homeomorphism* h*, i.e. the gluing. The two horizontal sides without any arrow, represent the boundary of the cylinder.*
Define the "dual" construction (exercise).

More generally, let X be a topological space. Then, $X \times I$ is called the **cylinder** *on X. As exercise, determine the homeomorphism g.*

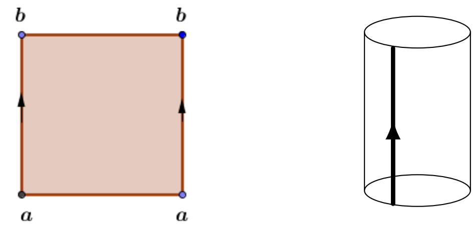


Figure 4.1 – Cylinder

3. **Möbius strip**
Let $A = \{(0,t), t \in I\}$ and $B = \{(1,t), t \in I\}$ be two subsets of I^2 and let $h : A \longrightarrow B$ be the homeomorphism defined by $h((0,t)) = (1, 1-t), t \in I$. Let $\mathcal{R}_{AB}$ be the equivalence relation defined by the homeomorphism h.
$I^2/\mathcal{R}_{AB}$ is the **Möbius**[1] **strip**.
Notice that Möbius strip is not a product of two spaces. It is represented by the square with some arrows on the vertical sides upwards for one and downwards for the other one. The two horizontal sides are the boundary of the Möbius strip, i.e. a circle.

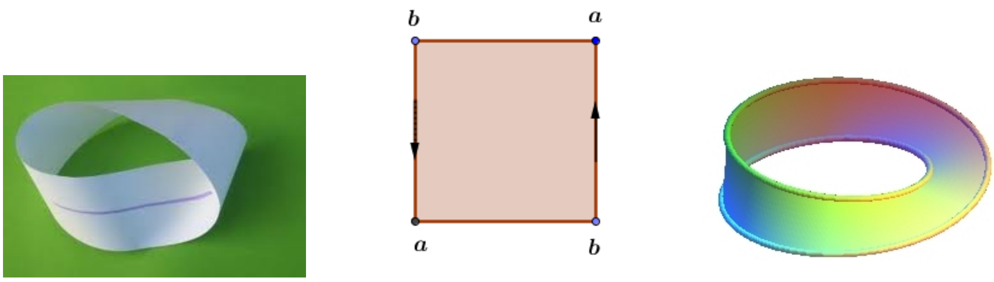


Figure 4.2 – Möbius strip

4. **Torus**
Let $A = \{(0,t), t \in I\} \cup \{(s,0), s \in I\}$ and $B = \{(1,t), t \in I\} \cup \{(s,1), s \in I\}$ be two subsets of I^2 and let $h : A \longrightarrow B$ be the homeomorphism defined by $h((s,0)) = (s,1), s \in I$ and $h((0,t)) = (1,t), t \in I$.
Let $f : I^2 \longrightarrow \mathbb{S}^1 \times \mathbb{S}^1$ be the continuous surjection defined by $f(s,t) = (e^{2i\pi s}, e^{2i\pi t})$. Then f is a continuous surjection so that f is a quotient map. The equivalence relation $\mathcal{R}_f = \mathcal{R}_{AB}$

[1] August Ferdinand Möbius (November 17, 1790 - September 26, 1868) was a German mathematician and theoretical astronomer.

is defined by the homeomorphism h and we have the following commutative diagram..

$$\begin{tikzcd} I^2 \arrow[rr, "f"] \arrow[dr, "p"'] & & \mathbb{S}^1 \times \mathbb{S}^1 \\ & I^2/\mathcal{R}_{AB} \arrow[ur, "g", "\cong"'] & \end{tikzcd}$$

$\mathbf{T} = \mathbb{S}^1 \times \mathbb{S}^1$ *is called the* **Torus**.
As exercise, determine the homeomorphism g.

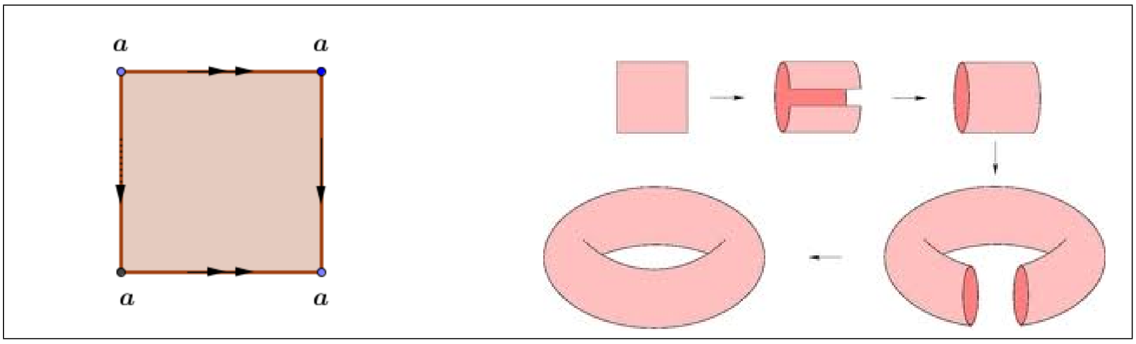
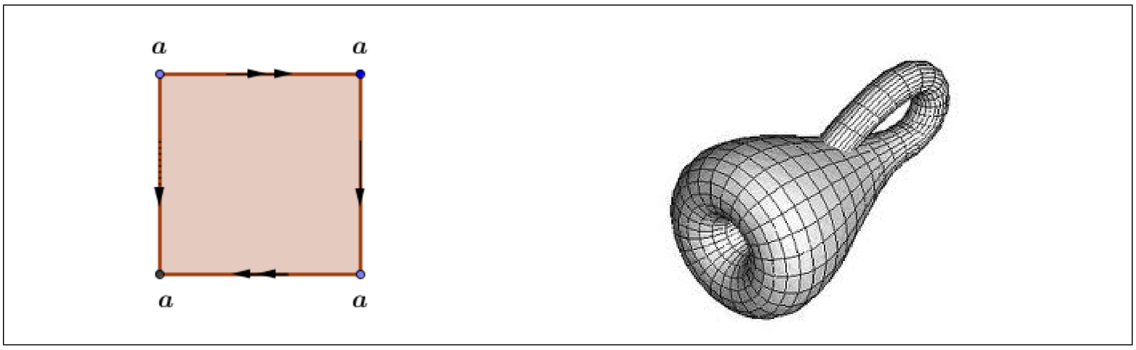


Figure 4.3 – Torus

5. **Klein bottle**
Let $A = \{(0,t), t \in I\} \cup \{(s,0), s \in I\}$ and $B = \{(1,t), t \in I\} \cup \{(s,1), s \in I\}$ and let $h : A \longrightarrow B$ be the homeomorphism defined by $h((s,0)) = (1-s,1), s \in I$ and $h((0,t)) = (1,t), t \in I$.
$I^2/\mathcal{R}_{AB}$ is the **Klein[2] bottle**.
Notice that the Klein bottle is not a product of spaces. It is represented by the square with the arrows on two vertical sides are upwards, and with the double arrows on the horizontal sides are to the right for one and to the left for the other one. There is no boundary.



[2]Christian Felix Klein (25 April 1849 - 22 June 1925) was a German mathematician, known for his work in group theory, function theory, non-Euclidean geometry, and on the connections between geometry and group theory.

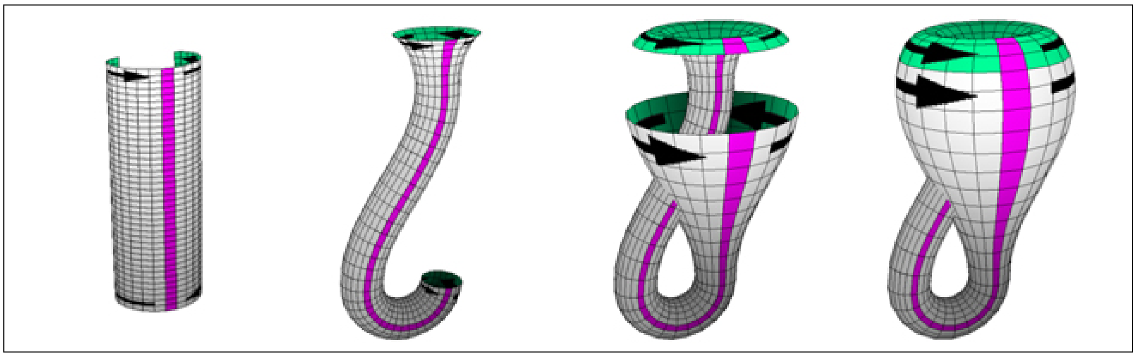
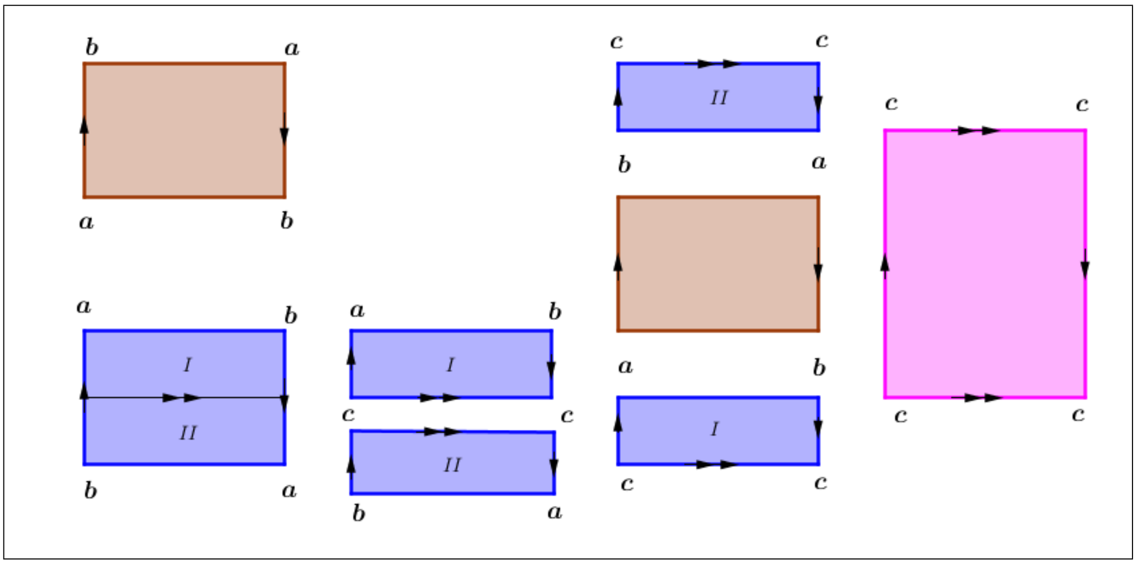


Figure 4.4 – Klein bottle

The Klein bottle was discovered in 1882 by Felix Klein. The bottle is a one-sided surface - like the well known Möbius strip - but is even more fascinating, since it is closed and has no border and neither an enclosed interior nor exterior.
The Möbius strip has a boundary homeomorphic to the circle $\mathbb{S}^1$*. Moreover, the Möbius strip can be embedded in the Euclidean space* $\mathbb{R}^3$*, but the Klein bottle cannot.*

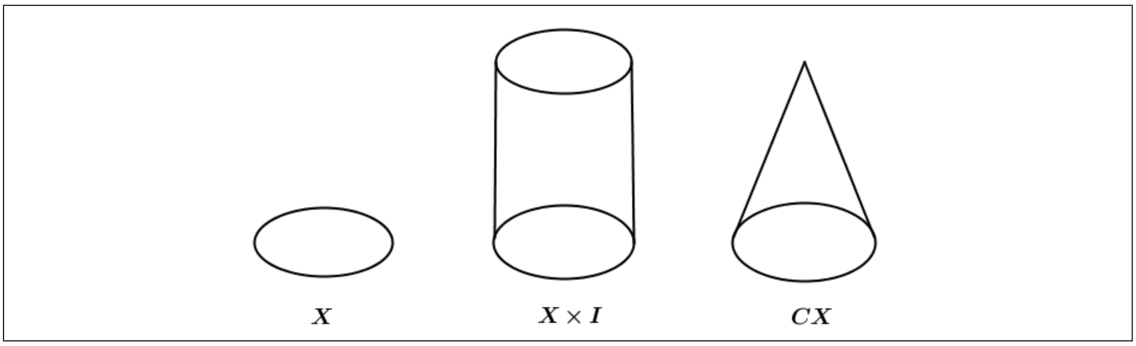
Figure 4.5 – Felix Klein

Remark 4.4.2. *The Klein bottle can be obtained by gluing two Möbius strips along their boundaries, as it is shown in the following pictures.*



Remark 4.4.3. *The picture of the Klein bottle in the Euclidean space $\mathbb{R}^3$ shows that the bottle has self intersection set, which is not. It is not possible to embed the Klein bottle in $\mathbb{R}^3$. It can be shown that there exists some embedding into $\mathbb{R}^4$.*

6. **Cone**
For any space X, the **cone** *CX of X is the quotient space $(X \times I)/\mathcal{R}$ where $\mathcal{R}$ is the equivalence relation $(x, 1)\mathcal{R}(x', 1)$ for all $x, x' \in X$. A continuous map $f : X \longrightarrow Y$ induces a continuous map $Cf : CX \longrightarrow CY$.*



7. **Suspension**
Let J be the interval $[-1, +1]$. For any space X, the **suspension** *SX of X is the quotient space $(X \times J)/\mathcal{R}$ where $\mathcal{R}$ is the equivalence relation where $(x, 1)\mathcal{R}(x', 1)$, $(x, -1)\mathcal{R}(x', -1)$, $(x, t)\mathcal{R}(x, t)$ for all $x, x' \in X, t \neq -1, 1$.*
It is easy to verify $S\mathbb{S}^0$ is homeomorphic to $\mathbb{S}^1$.
SX is homeomorphic to CX/X.
A continuous map $f : X \longrightarrow Y$ induces a continuous map $Sf : SX \longrightarrow SY$.

8. **Real projective space**
Let $\mathbb{R}^n \setminus \{0\}$ the subspace of $\mathbb{R}^n$. Identify two points of $\mathbb{R}^n \setminus \{0\}$ iff they lie on the same straight line through the origin. The space $(\mathbb{R}^n \setminus \{0\})/\mathcal{R}$ is called the **real projective space** *and it is denoted $\mathbf{P}_{n-1}(\mathbb{R})$. The real projective space can also be defined as the quotient space $\mathbb{S}^n/\sim$ where $x \sim -x$.*

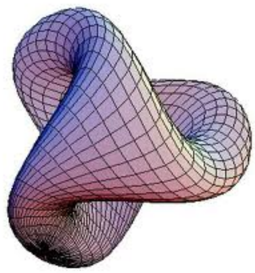


Figure 4.6 – The real projective plane $\mathbf{P}_2(\mathbb{R})$

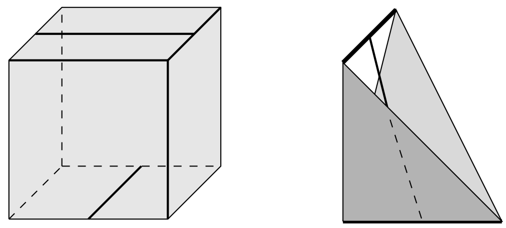
9. **Join**
Let X and Y be two topological spaces. Let $X \times Y \times I$ where $I = [0, 1]$ is the subspace of

$\mathbb{R}$ *with the standard topology. Define the equivalence relation* $\sim$ *as follows:*

$$(x, y_1, 0) \sim (x, y_2, 0) \text{ for any } y_1, y_2 \in Y$$

$$(x_1, y, 1) \sim (x_2, y, 1) \text{ for any } x_1, x_2 \in X$$

Definition 4.4.4. *The quotient space* $(X \times Y \times I)/\sim$ *is called the* **join** *of* X *and* Y *and it is denoted* $X * Y$.



$I \times I \times I$ $\qquad$ $I \vee I$

Figure 4.7

Example 4.4.5.

- *Let* $Y = \{a\}$ *be a one-point set. Then* $X * \{a\} \cong C(X)$.
- *Let* $Y = \mathbb{S}^0 = \{a, b\}$. *Then* $X * \mathbb{S}^0 \cong S(X)$.
- *More generally,* $X * \mathbb{S}^n \cong S(S(\cdots SX)\cdots)$, *the iterated* $n+1$*-suspension of* X.
- $\mathbb{S}^n * \mathbb{S}^m \cong \mathbb{S}^{n+m+1}$.
- $C(X) * C(Y) \cong C(X * Y)$.
- $\mathbb{D}^n \times \mathbb{D}^m \cong C(\mathbb{S}^{n-1}) \times C(\mathbb{S}^{m-1}) \cong C(\mathbb{S}^{n-1} * \mathbb{S}^{m-1}) \cong \mathbb{D}^{n+m}$, *where* $\mathbb{D}^n$ *is the disc of* $\mathbb{R}^n$.
- $\mathbb{D}^n * \mathbb{D}^m \cong \mathbb{D}^{m+n+1}$.

4.4.3 Spaces with Base Points

Let X be a topological space and $x_0 \in X$. then (X, x_0) is called space with base point x_0. These spaces are very important in homotopy theory. Let (X, x_0) and (Y, y_0) be two spaces with base points. We will considered the base-point-preserving continuous maps $f : X \longrightarrow Y$ such that $f(x_0) = y_0$.
Given the spaces with base points (X, x_0) and (Y, y_0); then the product $X \times Y$ is always given the base point (x_0, y_0).
The quotient space of a space with base point x_0 has a natural base point $p(x_0)$ where p is the canonical surjection.
However, for some other constructions, sum, suspension, join, there is no such natural base point and we need to proceed to some modifications.

- **Wedge of two topological spaces with base points**
  We leave as an exercise to extend all these results to the wedge of a family of topological spaces.
  The wedge has the role of the sum for the spaces with base points.

Definition 4.4.6. *Let X and Y be two topological spaces, with base points x_0 and y_0. The* **wedge** *of (X, x_0) and (Y, y_0), denoted $X \vee Y$ is defined as the quotient space $(X \coprod Y)/Z$ where $Z = \{(x_0, 0), (y_0, 1)\}$. Recall that we defined $X \coprod Y$ as $(X \times \{0\}) \cup (Y \times \{1\})$.*

In the notation $X \vee Y$, the base points are suppressed. Strictly speaking, we should write $(X, x_0) \vee (Y, y_0)$, but for simplicity's sake, we take the base points x_0 and y_0 to be understood.
Notice that we have the natural maps $\iota_X : X \longrightarrow X \vee Y$ and $\iota_Y : Y \longrightarrow X \vee Y$. Moreover, there exist two other maps $p_X : X \vee Y \longrightarrow X$ (and $p_Y : X \vee Y \longrightarrow Y$) given by:

- $p_X \circ \iota_X = \mathrm{Id}_X$ and $p_X \circ \iota_Y = x_0$
- $p_Y \circ \iota_Y = \mathrm{Id}_Y$ and $p_Y \circ \iota_X = y_0$

Suppose that $X = Y$, then, we define the **folding** map $\nabla : X \vee X \longrightarrow X$ induced by the map of the disjoint union to X that sends the point $(x, i), i = 0, 1$, to the point $x \in X$. The map ∇ is continuous (exercise).

Proposition 4.4.7. *Let $X_i, Y_i,\ i = 1, 2$ be some topological spaces with base points and let $f_i : X_i \longrightarrow Y_i, i = 1, 2$ be some base-point-preserving continuous maps. Then, there exists a base-point-preserving continuous map $f = f_1 \vee f_2 : X_1 \vee X_2 \longrightarrow Y_1 \vee Y_2$ such that:*

1. *if $g_i : Y_i \longrightarrow Z_i, i = 1, 2$ are two base-point-preserving continuous maps, then $(g_1 \vee g_2) \circ (f_2 \vee f_2) = (g_1 \circ f_1) \vee (g_2 \vee f_2)$.*
2. *if each $f_i, i = 1, 2$ is a copy of $f : X \longrightarrow Y$, then $f \circ \nabla_X = \nabla_Y \circ (f \vee f)$.*

Proof:(exercise). ■

It is possible to regard $X \vee Y$ as a subspace of $X \times Y$ as follows:

Proposition 4.4.8. *Let (X, x_0) and (Y, y_0) be two base point topological spaces. Let Z be the subspace of $X \times Y$ consisting of all points with "at most" one co-ordinate different from the base point, i.e. $Z = \{(x, y) \in X \times Y \mid x = x_0 \text{ or } y = y_0\}$. Then there is a homeomorphism from $X \vee Y$ onto Z.*

Proof: There is an obvious map f from $X \amalg Y$ to Z that sends $(x, 0)$ (resp. $(y, 1)) \in X \amalg Y$ to the point of (x, y_0) (resp. $(x_0, y)) \in Z$. In fact, f is an identification map: it is certainly onto.
Let $U \subset Z$, then $U = (\{x_0\} \times U_Y) \cup (U_X \times \{y_0\})$ where $U_X \subset X$ and $U_Y \subset Y$. Then $f^{-1}(U) = (U_X \times \{0\}) \cup (U_Y \times \{1\}) \bmod \{(x_0, 0), (y_0, 1)\}$.
Suppose $f^{-1}(U)$ open , then U is open in Z. Finally, f identifies together the base points of X and Y, so f induces a bijection from $X \vee Y$ to Z that is a homeomorphism. ■

- **Smash Product (Tensor Product)**

Definition 4.4.9. *Let (X, x_0) and (Y, y_0) be two base point topological spaces. Then the* **smash product** *(also called* **tensor product***) of the two spaces X and Y is denoted $X \wedge Y = (X \times Y)/(X \vee Y)$.*

In the notation $X \wedge Y$, the base points are suppressed. Strictly speaking, we should write $(X, x_0) \wedge (Y, y_0)$, but for simplicity's sake, we take the base points x_0 and y_0 to be understood.
Let $\pi : X \vee Y \longrightarrow X \wedge Y$ be the canonical projection. The point $\pi(x_0 \wedge y_0)$ will be taken as the base point of $X \wedge Y$.

Exercice 4.4.10. *Prove there exists a canonical homeomorphism $X \wedge Y \longrightarrow Y \wedge X$ (commutativity of the smash product).*

Remark 4.4.11. *The smash product is not, in general, associative, i.e. $X \wedge (Y \wedge Z) \ncong (X \wedge Y) \wedge Z)$. However, we will see later (section on compact spaces), that under some specific conditions, the smash product is associative.*

Proposition 4.4.12. *Let X_1, X_2, Y_1, Y_2 be four topological spaces with base points and let $f_i : X_i \longrightarrow Y_i,\ i = 1, 2$ be two continuous maps. Then, there exists a continuous map $f_1 \wedge f_2 : X_1 \wedge X_2 \longrightarrow Y_1 \wedge Y_2$ such that if $h_i : Y_i \longrightarrow Z_i,\ i = 1, 2$ are two continuous maps, then*

$$(h_1 \wedge h_2) \circ (f_1 \wedge f_2) = (h_1 \circ f_1) \wedge (h_2 \wedge f_2)$$

Proof: (exercise) ■

- **Reduced Cone**
  Let (X, x_0) be a base point space. Define the base point of the cone CX as $\pi(\{x_0\} \times I)$ where $\pi : X \times I \longrightarrow CX$ is the canonical projection. We denote it as $C(x, x_0)$. It can be view as $(X \times I)/(X \times \{1\} \cup (\{x_0\} \times I)$, i.e. as $(X, x_0) \wedge (I, 1)$.

- **Reduced Suspension**
  Let (X, x_0) be a base point space. Define the **reduced suspension** $s(X, x_0)$ or $s(X)$ as the quotient space $SX/\pi(\{x_0\} \times J)$. Let $\pi : SX \longrightarrow s(X, x_0)$ be the canonical projection. Then $\pi(\{x_0\} \times J)$ is chosen as the base point of $s(X, x_0)$.
  Show that $s(X, x_0)$ is homeomorphic to the quotient space of the cylinder $X \times J$ by $X \times (\{-1, +1\} \cup \{x_0\} \times J)$.

- **Join**
  Let (X, x_0) and (Y, y_0) be two base points spaces. Then the join $(X, x_0) * (Y, y_0)$ is defined as the quotient space $(X * Y)/\pi(\{x_0\} \times \{y_0\} \times I)$. The base point of $(X, x_0) * (Y, y_0)$ is chosen as $p(\{x_0\} \times \{y_0\} \times I)$ where p is the quotient map.

4.4.4 Fibered products

Let $f : X \longrightarrow Z$ and $g : Y \longrightarrow Z$ be two continuous maps. and let $p_X : X \times Y \longrightarrow X$, $p_Y : X \times Y \longrightarrow Y$ be the two canonical surjections.
We define the **fibered product** of (f, g)

$$X \prod_{f,g} Y = \{(x, y) \in X \times Y \mid f(x) = g(y)\} \subset X \times Y$$

The fibered product is also denoted $X \prod\limits_{Z} Y$ (cf.1.3.7).
For any space W and $h : W \longrightarrow X, g : W \longrightarrow Y$, continuous maps, there exists one continuous map $l : W \longrightarrow X \prod\limits_{f,g} Y$ making commutative the following diagram.

We also call p_X for $P_X \circ \iota$ where ι is the canonical injection (and similarly for p_Y).

$$\begin{tikzcd}
W \arrow[rrd, bend left, "k"] \arrow[rdd, bend right, "h"'] \arrow[rd, dotted, "l"] & & \\
& X \prod\limits_{f,g} Y \arrow[r, "p_Y"'] \arrow[d, "p_X"] & Y \arrow[d, "g"'] \\
& X \arrow[r, "f"] & Z
\end{tikzcd}$$

Example 4.4.13. *Let X be a topological space, A, B two subspaces of X. Then $A \prod\limits_{\iota_A,\iota_B} B = A \cap B$. (exercise)*

Example 4.4.14. *Let X, Y be two topological spaces, $f : X \longrightarrow \{*\}, g : Y \longrightarrow \{*\}$. Then $X \prod\limits_{f,g} Y = X \times Y$. (exercise)*

4.4.5 Fibered sums

Let $f : Z \longrightarrow X$ and $g : Z \longrightarrow Y$ be two continuous maps. and let $\iota_X : X \longrightarrow X \coprod Y$, $\iota_Y : Y \longrightarrow X \coprod Y$ be the two canonical injections. Define the equivalence relation $\sim$ on $X \coprod Y$ as follows, let $u, v \in X \coprod Y$, then $u \sim v$ if there exists $z \in Z$ such that $\iota_X \circ f(z) = \iota_Y \circ g(z)$. The quotient space $X \coprod Y/\sim$ is called the **fibered sum** of (f, g) along Z and denoted $X \coprod\limits_{f,g} Y$ or $X \coprod\limits_{Z} Y$ (cf.1.3.7).
We also call ι_X for $\iota_X \circ p$ where $p : X \times Y \longrightarrow X \times Y/\sim$ is the canonical surjection (and similarly for ι_Y).

$$\begin{tikzcd}
Z \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "\iota_Y"'] \arrow[rdd, bend left, "h"] & \\
X \arrow[r, "\iota_X"] \arrow[rrd, bend right, "k"'] & X \coprod\limits_{f,g} Y \arrow[rd, dotted, "l"] & \\
& & W
\end{tikzcd}$$

Let A be a closed subset of X and $f : A \longrightarrow Y$ be a continuous map. We denote

$$X \cup_f Y = X \coprod_{\iota_A, f} Y$$

the attaching of Y to X along f.

Remark 4.4.15. *If $Z = \emptyset$, then $X \coprod_{f,g} Y = X \coprod Y$.*

Cylinder of a Map. Cone of a Map

- **Cylinder of a map**, also called **Mapping Cylinder**

  Let $f : X \longrightarrow Y$ be a continuous map. Let $X \overset{\iota_X}{\underset{f_1}{\rightrightarrows}} X \coprod Y$ be the maps $x \longmapsto \iota_X(x) = (0, x)$ and $x \longmapsto f_1(x) = (1, f(x))$. Let $M_f = (I \times X) \coprod_{\iota_X, f} Y$ be the fibered sum called **cylinder of the map** f. We have the following diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook, "\iota_X"'] & Y \arrow[d, "j"] \\ I \times X \arrow[r, "i"] & M_f \end{tikzcd}$$

  where i and j are the canonical maps and M_f is the quotient space $((I \times X) \coprod Y)/\sim$ where $\sim$ is the equivalence relation $u \sim v \Longleftrightarrow \begin{cases} u = v & \text{or} \\ u = (0, x), v = f_1(x) & \text{or} \\ u = (0, x), v = (0, x'), f(x) = f(x') \end{cases}$

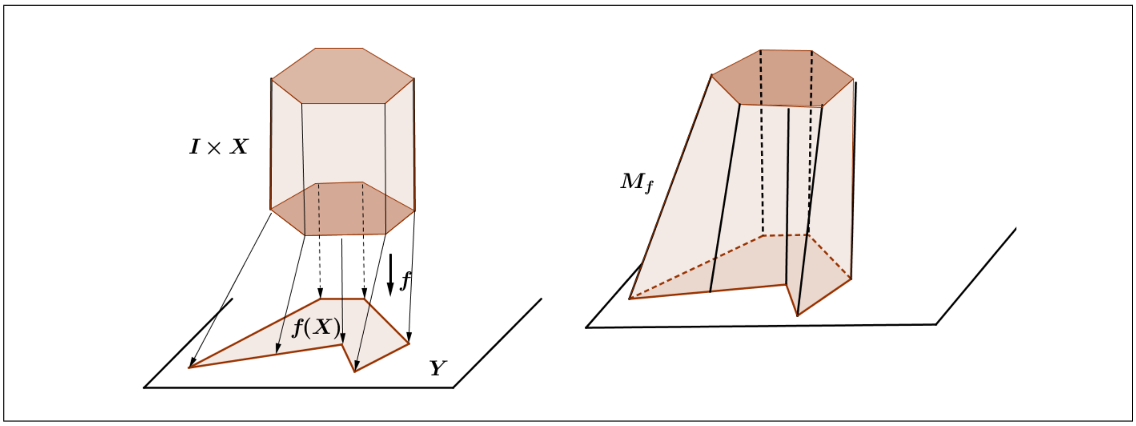


Figure 4.8

  The map $\widetilde{f} = i \circ \iota_X$ is injective, so $r : M_f \longrightarrow Y$ is a retraction. The following diagram is commutative

$$\begin{tikzcd} X \arrow[rr, "\widetilde{f}"] \arrow[dr, "f"'] & & M_f \arrow[dl, "r"] \\ & Y & \end{tikzcd}$$

- **Cone of a map**, also called **Mapping Cone**

  Let $f : X \longrightarrow Y$ be a continuous map and let $\iota : X \hookrightarrow CX$ be the inclusion. We define $C_f = CX \coprod_{\iota, f} Y$ as the fibered sum which is called the **cone of the map** f.

  Example 4.4.16. – *Let $f : \mathbb{S}^{n-1} \hookrightarrow B_n$ be the inclusion. Then $C_f \cong \mathbb{S}^n$.*

  – *Let $f : \mathbb{S}^{n-1} \longrightarrow \mathbf{P}_{n-1}(\mathbb{R})$ be the canonical surjection. Then $C_f \cong P_n(\mathbb{R})$.*

4.4.6 Exercises

1. Let Y be a topological space and X a subspace. Let $c : X \longrightarrow \{*\}$ be the unique map. Show that $\{*\} \coprod_{c,\iota} Y \cong Y/X$ where $\iota : X \longrightarrow Y$ is the inclusion.

2. Let $\mathbb{S}^{n-1}$ be the $n-1$-sphere, and let B_n be the closed unit ball of the Euclidean space $\mathbb{R}^n$. Consider the two maps $B_n \underset{j}{\overset{i}{\rightrightarrows}} \mathbb{S}^n$ where $i(x) = (x_1, \dots, x_n, \sqrt{1- \parallel x \parallel^2})$ and $j(x) = (x_1, \dots, x_n, -\sqrt{1- \parallel x \parallel^2})$. Show that the following diagram defines $\mathbb{S}^n$ as a fibered sum, i.e. $\mathbb{S}^n \cong B_n \coprod_{c,c} B_n$, and $c : \mathbb{S}^{n-1} \longrightarrow \{*\}$:

$$\begin{tikzcd} \mathbb{S}^{n-1} \arrow[r, "c"] \arrow[d, "c"'] & B_n \arrow[d, "i"] \\ B_n \arrow[r, "j"] & \mathbb{S}^n \end{tikzcd}$$

3. Let $\iota : \mathbb{S}^{n-1} \hookrightarrow B_n$ be the inclusion, let $p : B_n \longrightarrow \mathbb{S}^n$ be the map given by $p(x) = \big(2\sqrt{1- \parallel x \parallel^2} x_1, \dots, 2\sqrt{1- \parallel x \parallel^2} x_n, 2 \parallel x \parallel^2 -1\big)$ and let $k : \{*\} \longrightarrow \mathbb{S}^n$ that sends the point $*$ onto the North pole $(0, \dots, 0, 1)$. Show that the following diagram defines $\mathbb{S}^n$ as a fibered sum, i.e. $\mathbb{S}^n \cong \{*\} \coprod_{h,\iota} B_n$:

$$\begin{tikzcd} \mathbb{S}^{n-1} \arrow[r, hook, "\iota"] \arrow[d, "h"'] & B_n \arrow[d, "p"] \\ \{*\} \arrow[r, "k"] & \mathbb{S}^n \end{tikzcd}$$

We can show that $B_n/\mathbb{S}^{n-1} \cong \{*\} \coprod_{h,\iota} B_n$.

4. Imagine you are a little two-dimensional bug living inside the square diagram for the Mobius strip (resp. Klein bottle). You decide to go for a walk. Trace your path.

5. Let X be a topological space and let $I = [0, 1]$. Define the equivalence relation on $X \times I$ whose the equivalence classes are the one-point set $\{(x, t)\}$ where $x \in X, t \in I, t \neq 1$ and the set $X \times \{1\}$. The quotient space, denoted $C(X)$ is also called the cone of X. For all $x \in X$, denote $f(x$ the canonical image of $(x, 0)$ in $C(X)$.
   (a) Show that f is a homeomorphism from X onto $f(X)$.
   (b) Show that X is Hausdorff iff $C(X)$ is Hausdorff.

6. The circle $\mathbb{S}^1$ is the quotient space of $[0, 1] \subset \mathbb{R}$. On the other hand, it is a subspace of $\mathbb{R}^2$. Are the quotient and the induced topologies the same?

7. The cylinder, the Möbius strip and the torus are obtained as quotient spaces of the subspace $[0, 1]^2 \subset \mathbb{R}^2$ and they have the quotient topology. On the other hand, they are subspaces of $\mathbb{R}^3$, and as such, they have the induced topology.
   Are these two topologies are the same or not?
   Notice that the Klein bottle is not a subspace of $\mathbb{R}^3$; it is shown that it is a subspace of $\mathbb{R}^4$.

8. Give an example of a Hausdorff space which has a quotient space which is not Hausdorff.

9. Give an explicit description of the quotient space of the segment $[0,3]$ by the equivalence relation whose the partition consists of $[0,1],]1,2],]2,3]$.

10. Let X and Y be two topological spaces, $A \subset Y$, and let $f : A \longrightarrow X$ be a continuous map. Let $X \coprod Y$ be the disjoint union. Define the partition of $X \coprod Y$ given by the one-point sets $\{y\}$ where $y \in Y \setminus A$ and the sets $\{a, f(a)\}$ where $a \in A$. Denote $\mathcal{R}_f$ the equivalence relation defined by this partition and denote $X \cup_f Y$ the quotient space. We also say that $X \cup_f Y$ is obtained by *gluing* the space Y to the space X via f. Prove that if X is a one-point set, then $X \cup_f Y$ is Y/A.

11. Obtain the sphere $\mathbb{S}^3$ by gluing two copies of the solid torus $\mathbb{S}^1 \times \mathbb{D}^2$ via the map
$$\begin{array}{ccc} \mathbb{S}^1 \times \mathbb{S}^1 & \longrightarrow & \mathbb{S}^1 \times \mathbb{S}^1 \\ (x,y) & \longmapsto & (y,x) \end{array}$$

12. Let X be a set and X_1, X_2 two subsets whose union is X. Assume that $(X_i, \tau_i), i = 1,2$ such that the two topologies coincide on $X_1 \cap X_2$ and $X_1 \cap X_2$ is open both for τ_1 and τ_2. Show there is a unique topology τ on X inducing upon each X_i the topology τ_i.
Let $X = \mathbb{R} \setminus \{0\} \cup \{0_1, 0_2\}, X_i = \mathbb{R} \setminus \{0\} \cup \{0_i\}, i = 1,2$. Give on X_i the standard topology on $\mathbb{R}$.
Show that the topology τ on X is not Hausdorff, so is not metric.

13. Obtain the Klein bottle by gluing two copies of the cylinder $\mathbb{S}^1 \times I$ to each other.

14. Let $f_i : X_i \longrightarrow Y_i,\ i = 1,2$ be two continuous onto maps such that Y_i has the quotient topology determined by $f_i,\ i = 1,2$. Show that the two maps f_i define the continuous map $f : X_1 \times X_2 \longrightarrow Y_1 \times Y_2$.
Show that f is onto and $Y_1 \times Y_2$ as product space is the same as the quotient space determined by f, i.e. the product topology and the quotient topology on $Y_1 \times Y_2$ are the same.

15. Is a quotient space of a subspace the same as a subspace of a quotient space?
Let $f : X \longrightarrow Y$ be a map which is onto, Y has the quotient topology determined by f, A is a subspace of X, and $B = f(A) \subset Y$. Then, we can either topologize B as a subspace of Y, or give it the quotient topology determined by the map $f_{|A} : A \longrightarrow B$. Are the two topologies on B the same?
Let X be a rectangle and Y is a cylinder.
$$X = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq 2\pi,\ 0 \leq y \leq 1\}$$
$$Y = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1,\ 0 \leq z \leq 1\}$$
$$\begin{array}{rcl} f : X & \longrightarrow & Y \\ (x,y) & \longmapsto & (\cos x, \sin x, y) \end{array}$$
Let $A = \{(x,y) \in X \mid 0 \leq x < 2\pi,\ y = 0\}$. Describe $B = f(A)$ and show that B is homeomorphic to A. What about the two topologies on B?

16. Under the same hypotheses as in the previous exercise, if A is closed in X, and $f(C)$ is closed for any closed set C in X, or if A is open and $f(O)$ is open for any open set O in X, show that the subspace and the quotient topologies on B are the same.

17. Show that $X \wedge \mathbb{S}^1 \cong SX$.

18. Show that $CX \cong X \wedge I$.

19. Show that $(X \vee Y) \wedge Z \cong (X \wedge Z) \vee (Y \wedge Z)$.

20. Show that for each $m, n \geq 0, \mathbb{S}^{m+n}$ is homeomorphic to $\mathbb{S}^m \wedge \mathbb{S}^n$. Find two such distinct homeomorphisms. (Hint: 1. Consider the discs $\mathbb{D}^m$ and $\mathbb{D}^n$. 2. Consider $\mathbb{S}^{m+n}$ as a subspace of the disc $\mathbb{D}^{m+n+1}$).

21. Let $\iota : \mathbb{S}^{n-1} \hookrightarrow B_n$ be the inclusion and let $f : \mathbb{S}^{n-1} \longrightarrow X$ be a continuous map. Show that X can be identified with a closed subset of $B_n \coprod_{\iota,f} X$ and $(B_n \coprod_{\iota,f} X) \setminus X$ is homeomorphic to any open ball of dimension n.

22. Let A and B be two closed subsets of the topological space X. Let $\iota_A : A \cap B \hookrightarrow A$ and $\iota_B : A \cap B \hookrightarrow B$ be the inclusions. Show that $A \coprod_{\iota_A, \iota_B} B \cong A \prod_{\iota_A, \iota_B} B$.

23. The n-dimensional torus $\mathbf{T}^n$ is defined by induction as $\mathbf{T}^1 = \mathbb{S}^1$, $\mathbf{T}^n = \mathbf{T}^{n-1} \times \mathbf{T}^1$. Let X (resp. Y) be the subspace of $\mathbb{R})^3$ obtained from the circle of equation $(y-2)^2 + z^2 = 1$ (resp. $(y-2)^2 + z^2 \leq 1$) making a rotation of angle 2π around the axis z.

    (a) Show that there exists a homeomorphism $f : Y \longrightarrow B_2 \times \mathbb{S}^1$ such that the restriction $f_{|X}$ is a homeomorphism from X onto $\mathbb{S}^1 \times \mathbb{S}^1$.

    (b) Show that the fibered sums $Y \coprod_{\iota,\iota} Y$ and $(B_2 \times \mathbb{S}^1) \coprod_{g,g} (B_2 \times \mathbb{S}^1)$ are homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$, where $g = \iota' \times \mathrm{Id}_{\mathbb{S}^1}$, ι' being the inclusion of $\mathbb{S}^1$ into B_2.

    (c) Let X' (resp. Y') be the subspace $\left\{(u,v) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid \ \| u \|^2 + \| v \|^2 = 1, \| u \|^2 = \dfrac{1}{2}\right\}$, $\left(\text{resp. } \left\{(u,v) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid \ \| u \|^2 + \| v \|^2 = 1, \| u \|^2 \leq \dfrac{1}{2}\right\}\right)$.
    Show that there exists a homeomorphism from X' onto $\mathbb{S}^1 \times \mathbb{S}^1$.
    Deduce that the fibered sum $\left(B_2 \times \mathbb{S}^1\right) \coprod_{h,h} (\mathbb{S}^1 \times B_2)$ where $h = \iota' \times \mathrm{Id}_{\mathbb{S}^1}$, is homeomorphic to $\mathbb{S}^3$.
    Notice that according to the way we glue two tori along their boundaries, we obtain either $\mathbb{S}^2 \times \mathbb{S}^1$ or $\mathbb{S}^3$. These two spaces are not homeomorphic.

Chapter 5

Topological Properties

In this chapter, we will study some **topological properties** or **topological invariants**, i.e. properties of topological spaces which are invariant under homeomorphisms. A property of a topological space is topological if every homeomorphic spaces possess this property.
An important consequence is the following. Let X and Y be two topological spaces such that X possesses the topological property P and Y does not, then X and Y are not homeomorphic.

5.1 Path-Connectedness

Let (X, τ) be a topological space and let $x_0, x_1 \in X$ be two points (not necessarily distinct). Consider $[0, 1]$ as a subspace of $\mathbb{R}$ with the standard topology.

Definition 5.1.1. *A* **path** *in X from x_0 to x_1 is a continuous map $p : [0, 1] \longrightarrow X$ such that $p(0) = x_0$ and $p(1) = x_1$. If $x_0 = x_1$ then the path is also called a* **loop** *based at the point x_0.*

Notice that if there exists a path p from x_0 to x_1, then there exists a path q from x_1 to x_0, given by $q(t) = p(1 - t)$.
The existence of a path from x_0 to x_1 depends on the two points and also on the topology on X (see exercises).
In algebraic topology, the set of loops in a topological space gives some important informations on the topology of the space.

Definition 5.1.2. *The topological space X is said to be* **path-connected** *if for any two points $x_0, x_1 \in X$, there exists a path in X from x_0 to x_1.*

Such spaces are also called **pathwise-connected**, or **arcwise-connected**.
Notice that such a path is not necessarily unique. Moreover, don't confuse the set $\{p(t) \mid t \in [0, 1]\}$ with the set $\{(t, p(t) \mid t \in [0, 1]\}$ which is the graph of the path p. Different paths could defined the same set of points. For example, let p be a path from x_0 to x_1. Then the paths p and $q : [0, 1] \longrightarrow X$ such that $q(t) = p(2t)$ for $t \in \left[0, \frac{1}{2}\right]$ and $q(t) = x_1$ for any $t \in \left[\frac{1}{2}, 1\right]$ define the same set of points but they are distinct paths.

Example 5.1.3.

1. *$\mathbb{R}$ with the standard topology is path-connected.*
2. *More generally, the Euclidean spaces $\mathbb{R}^n$ are path-connected.*

3. *The graph of a continuous map $f : \mathbb{R} \longrightarrow \mathbb{R}$ is path-connected.*

4. *The spheres $\mathbb{S}^n, n > 0$ are path-connected.*

5. *However, $\mathbb{S}^0$ and $\mathbb{R} \setminus \{0\}$ are not path-connected.*

6. *A convex subset of $\mathbb{R}^n$ is path-connected. Any two points a, b can be joined inside the convex subset by the path p defined by $p(t) = (1-t)a + tb, t \in [0,1]$. Notice that such path does not exist in an arbitrary topological space.*

Definition 5.1.4. *Let (X, τ) be a topological space. The points $a, b \in X$ are said to be* **path-equivalent** *if there exists a path p in X such that $p(0) = a$ and $p(1) = b$.*
The path-equivalence classes are called **path-connected components** *of X.*

Notice that "path-equivalence" is an equivalence relation (exercise). So there exists a partition of X into path-connected components. In particular, each point belongs to one path-connected component where a path-connected component is a maximal subset (w.r.t. inclusion) such that any pairs of points are path-connected.

Proposition 5.1.5. *A continuous image of a path-connected space is path-connected.*

Proof: Let $f : X \longrightarrow Y$ be a continuous map and suppose X is a path-connected space. We have to show that $f(X)$ is path-connected.
Let $y_0, y_1 \in f(X)$. Then there exist $x_0, x_1 \in X$ such that $f(x_0) = y_0, f(x_1) = y_1$. By assumption, there exists a path $p : I \longrightarrow X$ such that $p(0) = x_0$ and $p(1) = x_1$, so that the map $f \circ p : I \longrightarrow Y$ is a path such that $f \circ p(0) = y_0$ and $f \circ p(1) = y_1$. ■

Corollary 5.1.6. *Let X be path-connected space and let $\mathcal{R}$ be an equivalence relation on X. Then the quotient space $X/\mathcal{R}$ is path-connected.*

Proof: The proof is straightforward. ■

Corollary 5.1.7. *Path-connectedness is a topological property, i.e. let $f : X \longrightarrow Y$ be a homeomorphism, then X is path-connected iff Y is path-connected.*

Proof: The proof follows from the proposition. ■

Corollary 5.1.8. *The number of path-connected components is a topological invariant, i.e. then there exists a bijection between the path-connected components of X and Y.*

Proof: Let $f : X \longrightarrow Y$ be some homeomorphism. For any $x, x' \in X$, there is path from x to x' iff there is a path from $f(x)$ to $f(x')$. Let $\bar{x}$ be the path-connected component of x, and $X/\sim, Y/\sim$ the set of path-connected components of X and Y. Then there exists a bijection $\bar{f}$ such that the diagram is commutative

$$\begin{tikzcd} X \arrow[r, "f", "\cong"'] \arrow[d, "p_X"'] & Y \arrow[d, "p_Y"] \\ X/\sim \arrow[r, "\bar{f}", "\simeq"'] & Y/\sim \end{tikzcd}$$

■

A topological space has only one path-connected component iff it is path-connected.
Let X and Y be two topological spaces. Suppose that the numbers of path-connected components

are distinct, then X and Y are not homeomorphic. However, two topological spaces having the same number of path-connected components are not necessarily homeomorphic.
For example, $\mathbb{R}$ and $\mathbb{S}^1$ are path-connected (one path-connected component), but they are not homeomorphic. Suppose there exist a homeomorphism $h : \mathbb{R} \longrightarrow \mathbb{S}^1$, choose an arbitrary point $a \in \mathbb{R}$, then $\mathbb{R} \setminus \{a\} \cong \mathbb{S}^1 \setminus \{h(a)\}$. But $\mathbb{R} \setminus \{a\}$ has two connected components and $\mathbb{S}^1 \setminus \{h(a)\}$ is path-connected, so it does not exist some homeomorphism $h : \mathbb{R} \longrightarrow \mathbb{S}^1$.

5.1.1 Exercises

1. Prove that path equivalence is an equivalence relation.

2. Explain the examples 5.1.3.

3. Let (X, τ) be a topological space. Is it a path-connected space,

   (a) If τ is the discrete topology?

   (b) If τ is the trivial topology?

4. Determine the subspaces of $\mathbb{R}$ that are path-connected.

5. Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous map.
   Show that for any y between $f(a)$ and $f(b)$, there exists $x \in [a, b]$ such that $f(x) = y$. (*Intermediate value theorem*).
   Show that this result remains valid for any continuous map $f : X \longrightarrow \mathbb{R}$ which takes any value between $f(a)$ and $f(b)$, and where X is path-connected.
   Assume that the Earth is the sphere $\mathbb{S}^2$, and the temperature is a continuous function $f : \mathbb{S}^2 \longrightarrow \mathbb{R}$.
   Show that there is a pair of diametrically opposite places that have the same temperature.

6. Let X be a metric space and let $f : X \longrightarrow \mathbb{R}$ be a continuous function such that $f(x) \neq 0$ for all $x \in X$. Suppose that $y, z \in X$ such that $f(y) < 0 < f(z)$. Show that there is no path from y to z.

7. Let L be a line of the Euclidean plane $\mathbb{R}^2$.
   Show that $\mathbb{R}^2 \setminus L$ is not path-connected but $\mathbb{R}^2 \setminus \{0\}$ is path-connected.
   More generally, let A be a p-dimensional affine subspace of $\mathbb{R}^n$, where $p < n$, i.e. A is defined by $n - p$ linearly independent equations.
   Is $\mathbb{R}^n \setminus A$ path-connected?

8. Show that the product of two path-connected spaces is path-connected.

9. Let A and B two subspaces of the topological space X. Suppose that A and B are path-connected and $A \cap B \neq \emptyset$, prove that $A \cup B$ is path-connected.

10. Let A and B be two path-connected subsets of the topological space X such that $A \cap B \neq \emptyset$. Is $A \cap B$ path-connected?

11. Let X be a path-connected space and let $\mathcal{R}$ be an equivalence relation on X. Recall that the quotient space $X/\mathcal{R}$ is path-connected. Is $\mathbb{S}^1 \times \mathbb{S}^1$ a path-connected space?

12. Let V be a real vector space of finite dimension n.

    (a) Show that there exists a topology on V such that V is homeomorphic to $\mathbb{R}^p$ and determine p.

(b) Show that the map $V \times V \longrightarrow V$ such that $(x, y) \longmapsto x + y$ is continuous.

(c) Let A and B be two path-connected subsets of V.
Show that $A + B = \{x + y \mid x \in A, y \in B\}$ is path-connected.

13. (a) Let $X = \{(x, y) \in \mathbb{R}^2 \mid x < y\}$.
Is X path-connected?

(b) Let $F : X \longrightarrow \mathbb{R}$ be a continuous map such that $F((x, y)) \neq 0$ for all $(x, y) \in X$.
Determine $F(X)$.

(c) Let $f : [0, 1] \longrightarrow \mathbb{R}$ be an injective continuous map.
Deduce that f is either strictly increasing or strictly decreasing .

14. Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function, and y_0 between $f(a)$ and $f(b)$.
Prove there exists $x_0 \in [a, b]$ such that $f(x_0) = y_0$. Is x_0 unique?
Does this result remain valid for any interval, i.e. not necessary closed?

15. Let $f : [a, b] \longrightarrow [a, b]$ be a continuous function.
Prove there exists $c \in [a, b]$ such that $f(c) = c$.
Does this result remain valid for the interval $]a, b[$?
Is $]a, b[$ a retract of $[a, b]$?
Is this result valid for $\mathbb{S}^n, n > 0$?

16. Let $U = \{(x, \sin \frac{\pi}{x}) \in \mathbb{R}^2 \mid x \in]0, 1[\}$. Then $X = U \cup \{(0, 0)\} \subset \mathbb{R}^2$ is not path-connected.
(Hint: Suppose there exists a path p such that $p(0) = (0, 0)$, $p(1) = (1, 0)$. Let $s = \sup\{t \mid p(t) = (0, 0)\}$. Thus prove that $p(s) = (0, 0)$. By continuity of p, there exists $\delta > 0$ such that if $|s - t| < \delta$, then $d\big(p(s), p(t)\big) < \frac{1}{2}$. Hence, get a contradiction.)

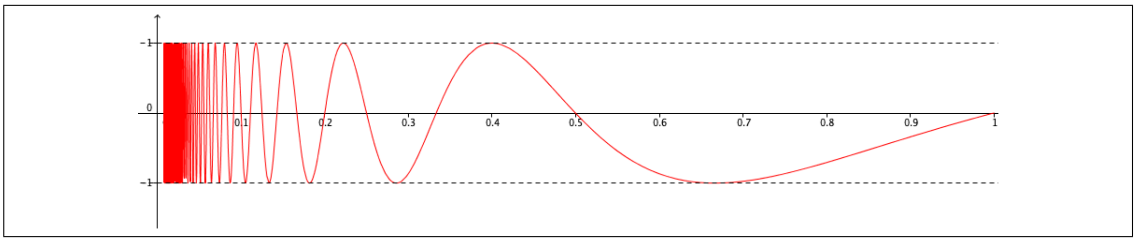


Figure 5.1

17. The space $\mathcal{M}_{n \times n}$ of all real $n \times n$-matrices is a space homeomorphic to $\mathbb{R}^{n^2}$. Find the path-connected components of the subspaces:

(a) $GL(n; \mathbb{R}) = \{A \in \mathcal{M}_{n \times n} \mid \det A \neq 0\}$

(b) Same question with $\mathbb{R}$ replaced by $\mathbb{C}$.

(c) $\{A \mid A^2 = I\}$

18. Let $f : X \longrightarrow Y$ be a homeomorphism.

(a) For any $x \in X$, prove the map $\tilde{f} : X \setminus \{x\} \longrightarrow Y \setminus \{f(x)\}$ such that $\tilde{f}(x') = f(x')$ for any $x' \neq x$, is a homeomorphism.

(b) Prove that the circle $\mathbb{S}^1$ is not homeomorphic to $\mathbb{R}$.

(c) Prove that $\mathbb{R} \setminus \{0\}$ and $\mathbb{R} \setminus \{0,1\}$ are not homeomorphic.

(d) The real line $\mathbb{R}$ (with the standard topology) is not homeomorphic to the Euclidean space $\mathbb{R}^2$.

19. For a path-connected topological space X, the point $a \in X$ is a *cut* point of order k if the complement $X \setminus \{a\}$ consists of k path-connected components.
Show that any homeomorphism $f : X \longrightarrow Y$ between two path-connected topological spaces establishes a bijection between cut points of fixed order k. Hence the number of cut points of order k is a topological invariant.
Deduce that $\mathbb{R}$ and $\mathbb{S}^1$ are not homeomorphic.

5.2 Connectedness

There exists a weaker notion of connectedness than path-connectdness. We mentioned the special role plays by subsets that are both open and closed (Remark 2.5.6.). Here is the explanation of this assertion.

Definition 5.2.1. *A topological space (X, τ) is said to be* **connected** *if X has only two subsets $\emptyset$ and X are both open and closed. Otherwise, X is said to be* **disconnected**.

Proposition 5.2.2. *The topological space X is connected iff X has no partition into two non-empty open sets iff X has no partition into two non-empty closed sets.*

Proof: (exercise)

Example 5.2.3. *1. (X, τ) where τ is the trivial topology is connected.*

2. (X, τ) where τ is the discrete topology and $\operatorname{card}(X) \geq 2$ is disconnected.

3. The subspace $\mathbb{Q} \subset \mathbb{R}$ is disconnected. (Hint: $\mathbb{Q} = \Big(\mathbb{Q} \cap \Big]-\infty, \sqrt{2}\Big[\Big) \cup \Big(\mathbb{Q} \cap \Big]\sqrt{2}, +\infty\Big[\Big)$).

4. $\mathbb{R}$ with the usual topology is connected, while $\mathbb{R}$ with the half-open topology is disconnected.

Lemma 5.2.4. *The topological space X is connected iff any continuous map $f : X \longrightarrow \{0,1\}$ is constant ($\{0,1\}$ with the discrete topology).*

Proof: $\Longrightarrow$) Suppose there exists a non constant continuous map $f : X \longrightarrow \{0,1\}$, so f is surjective. Let $U = f^{-1}(\{0\})$ and $V = f^{-1}(\{1\})$. $\{0\}$ and $\{1\}$ are open sets so U and V are open in X where $X = U \cup V$, and $U \cap V = \emptyset$. Thus $U = X \setminus V$ is also closed and there exists a subset U of X which is both open and closed and X is disconnected.
$\Longleftarrow$) Suppose X disconnected, i.e. $X = O_1 \cup O_2$ where $O_1, O_2 \in \tau$ and $O_1 \cap O_2 = \emptyset$. Let $f : X \longrightarrow \{0,1\}$ be the map such that $f(O_1) = \{0\}, f(O_2) = \{1\}$. Then f is continuous and non constant. ■

Notice that in the following, $\{0,1\}$ is considered as subspace of $\mathbb{R}$ (standard topology), so it is a discrete space.

Proposition 5.2.5. *The closure of a connected set is connected.*

Proof: Let $A \subset X$ be a connected set in the topological space X and let $f : \operatorname{Cl}(A) \longrightarrow \{0,1\}$ be a continuous map. The restriction $f\big|_A$ of f to A is not surjective so, for example $f(A) = \{0\}$. But $f(\operatorname{Cl}(A)) \subset \operatorname{Cl}(f(A)) = \operatorname{Cl}(\{0\}) = \{0\}$ then f is not surjective and $\operatorname{Cl}(A)$ is connected. ■

Proposition 5.2.6. $[0,1]$ *is connected.*

Proof: Suppose $[0,1]$ disconnected, then $[0,1] = U \cup V$ where U and V are non-empty disjoint closed subsets. Take $a \in U$ and $b \in V$ and assume $0 \leq a < b \leq 1$. Then the sets $X = U \cap [a,b]$ and $Y = V \cap [a,b]$ are disjoint and closed in $[0,1]$. Then $\sup X = c$ is a limit point of X, hence $c \in X$ and $c < b$ because $b \in Y$. Thus $X \cap]c,b] = \emptyset$ so $]c,b] \subset Y$ and $c \in Y$ is a limit point of $]c,b]$, then $X \cap Y \neq \emptyset$ which is a contradiction. ■

But $X = [0,1] \setminus \left\{\frac{1}{2}\right\}$ is disconnected (as subspace of $\mathbb{R}$) since $\left[0, \frac{1}{2}\right[$ and $\left]\frac{1}{2}, 1\right]$ are open in X (not in $\mathbb{R}$).

Lemma 5.2.7. *Let $(U_\lambda)_{\lambda \in \Lambda}$ be a family of connected sets in the topological space X. If there exists $\lambda_0 \in \Lambda$ such that for all $\lambda \in \Lambda, U_{\lambda_0} \cap U_\lambda \neq \emptyset$, then $\bigcup_{\lambda \in \Lambda} U_\lambda$ is connected.*

Proof: Any continuous map $f : \bigcup_{\lambda \in \Lambda} U_\lambda \longrightarrow \{0,1\}$ is constant on U_{λ_0} and takes the same value on each U_λ, hence f is constant. ■

Definition 5.2.8. *A* **connected component** *of a topological space X is a maximal* (*w.r.t. inclusion*) *connected subset of X.*

Proposition 5.2.9. *A connected component is closed.*

Proof: Let A be a connected component of X. Then $\mathrm{Cl}(A)$ is connected (Prop.5.2.5) and $A \subset \mathrm{Cl}(A)$ so from definition of the connected component, $A = \mathrm{Cl}(A)$ and A is closed. ■

But a connected component need not to be open. Consider the subspace $\mathbb{Q} \subset \mathbb{R}$. The connected components are the one-point sets. Let $A \subset \mathbb{Q}$ containing two points $a < b$. Let $x \notin \mathbb{Q}$ such that $a < x < b$. Then $A = (A \cap \{y \mid y < x\}) \cup (A \cap \{y \mid y > x\})$. Then A is the disjoint union of two open sets, hence any subset of $\mathbb{Q}$ containing more than one rational number is not connected. Any non-empty set in $\mathbb{R}$ contains an open interval and infinitely many rational numbers.Then, every non-empty open set in $\mathbb{Q}$ is infinite.

Each point of a topological space X belongs to one connected component. So there is a partition of X into closed sets, the connected components.
If for any $x \in X$, the connected component containing x is $\{x\}$, then X is said to be **totally disconnected**.

Proposition 5.2.10. *Let X and Y be two topological spaces, X connected and let $f : X \longrightarrow Y$ be a continuous map. Then $f(X)$ is connected.*

Proof: If $f(X)$ is disconnected, then $f(X) = U \cup V$ where U, V are non-empty open subsets. Thus $X = f^{-1}(U) \cup f^{-1}(V)$ is a non-empty open subset which is a contradiction. ■

Corollary 5.2.11. *Let X be connected space and let $\mathcal{R}$ be an equivalence relation on X. Then the quotient space $X/\mathcal{R}$ is connected.*

The proof is straightforward. ■

Corollary 5.2.12.

1. *Connectedness is a topological property.*
2. *The number of connected components is a topological invariant.*

Proof: (exercise) ■

Up to now, we introduced two concepts, path-connectedness an connectedness which seem without any relation.

Proposition 5.2.13. *Any path-connected space is connected.*

Proof: Let X be a path-connected space. Suppose X disconnected, then $X = U \cup V$ disjoint union of two non-empty open sets. Choose $a \in U, b \in V$ and let $p : [0,1] \longrightarrow X$ be a path such that $p(0) = a$ and $p(1) = b$. Then $p^{-1}(U) \cap p^{-1}(V) = \emptyset$ and $p^{-1}(U), p^{-1}(V)$ are non-empty open subsets. Thus $[0,1] = p^{-1}(U) \cup p^{-1}(V)$ so $[0,1]$ is disconnected which is a contradiction (prop. 5.2.6). ■

Example 5.2.14. $\mathbb{R}$ *is connected.* (*We already know that* $\mathbb{R}$ *is path-connected*).

Remark 5.2.15. *A connected space is not necessarily path-connected.*

Example 5.2.16. $\Big\{(x,y) \in \mathbb{R}^2 \mid y = \sin\dfrac{\pi}{x}, x \in]0,1]\Big\} \cup \big\{(0,0) \in \mathbb{R}^2\big\}$ *is connected but it is not path-connected* (*exercise 5.1.1.15*)(*Notice that* $(0,0) \in \mathrm{Cl}\left(\Big\{(\mathrm{x},\mathrm{y}) \in \mathbb{R}^2 \mid \mathrm{y} = \sin\dfrac{\pi}{\mathrm{x}}, \mathrm{x} \in]0,1]\Big\}\right)$.

Lemma 5.2.17. *Let* $\{X_i, i \in I\}$ *be a family of connected subspaces of the topological space* X. *Suppose there exists* X_{i_0} *such that for any* $i \in I$, $X_{i_0} \cap X_i \neq \emptyset$. *Then* $\cup_{i\in I} X_i$ *is connected.*

Proof: Any continuous map $f : \displaystyle\bigcup_{i\in I} X_i \longrightarrow \{0,1\}$ is constant on X_{i_0} and takes the same value on each X_i, hence f is constant. ■

Proposition 5.2.18. *If the two spaces* X *and* Y *are connected, then* $X \times Y$ *is connected.*

Proof: Let $x \in X$, then $\{x\} \times Y \subset X \times Y$ is connected since it is homeomorphic to Y. Similarly, $X \times \{y\}$ is connected for any $y \in Y$. Then $(\{x\} \times Y) \cup (X \times \{y\})$ is connected because (x,y) is a point of the overlap. Finally, from the previous lemma, $X \times Y = \displaystyle\bigcup_{y\in Y} \left((\{x\} \times Y) \cup (X \times \{y\})\right)$ is connected. ■

The converse is true. (exercise).

Corollary 5.2.19. ■

1. $\mathbb{R}^n$ *is connected.*
2. *The torus* $\mathbb{S}^1 \times \mathbb{S}^1$ *is connected.*

Proof: The proof is straightforward. ■

Notice that $\mathbb{S}^n$, for $n > 0$, is connected because $\mathbb{S}^n \setminus \{a\} \cong \mathbb{R}^n$ for $a \in \mathbb{S}^n$ and $\mathrm{Cl}(\mathbb{S}^n \setminus \{a\}) = \mathbb{S}^n$.

5.2.1 Exercises

1. Let $f : [a,b] \longrightarrow \mathbb{R}$ be a continuous real-valued map. Let y_0 be a real number between $f(a)$ and $f(b)$. Show there is a number $c \in [a,b]$ for which $f(c) = y_0$.
2. Show that the topological subspace $X = \{(x,y) \in \mathbb{R}^2 \mid x \neq 0\} \subset \mathbb{R}^2$ is not connected. Define the connected components.

3. For any $n \in \mathbb{Z}$, define $X = \left\{t \in \mathbb{R} \mid |t-n| < \frac{1}{2}\right\}$. Show that the sets $J_n = \left]n - \frac{1}{2}, n + \frac{1}{2}\right[$, $n \in \mathbb{Z}$, are the connected components of X.

4. Let (X, τ_1) and (X, τ_2) be two topological spaces where τ_2 is finer than τ_1. If (X, τ_1) is connected, is (X, τ_2) connected? If (X, τ_2) is connected, is (X, τ_1) connected?

5. Let $A, B \subset \mathbb{R}^2$. Give some examples such that

   - A, B connected and $A \cap B$ disconnected.
   - A, B disconnected and $A \cup B$ connected.
   - A, B connected, $\mathrm{Cl}(A) \cap \mathrm{Cl}(B) \neq \emptyset$ and $A \cup B$ disconnected.

6. Let $f : [a, b] \longrightarrow [a, b]$ be a continuous map on the closed subspace $[a, b] \subset \mathbb{R}$. Then, there exists $c \in [a, b]$ such that $f(c) = c$. Show that this property does not remain true if $[a, b]$ is replaced by the real space $\mathbb{R}$, by the interval $[a, b[$.

7. Show that the cylinder, the Möbius strip, the torus and the Klein bottle are not homeomorphic.

8. Let $\mathcal{C}$ be a circle in the affine plane $\mathcal{A}_2(\mathbb{R})$.
   Show that $\mathcal{A}_2(\mathbb{R}) \setminus \mathcal{C}$ is disconnected w.r.t. standard topology and connected w.r.t. Zariski topology.

9. Is $\mathbb{R}$ with the half-infinite topology a connected space?

10. Show that the Cantor set is totally disconnected.

5.3 Compactness

Compactness is a sort of topological counterpart for the finiteness in the context of set theory. It is a generalization of the subset of the Euclidean spaces which are both closed and bounded. The term compact was introduced into mathematics by Maurice Fréche[1]t in 1906. Compactness plays an extremely important role in mathematical analysis, because many classical and important theorems of 19th century analysis, such as the extreme value theorem, are easily generalized to this situation.

5.3.1 Compact Spaces

Let (X, τ) be a topological space and let $A \subset X$.

Definition 5.3.1. *A collection of subsets of X is said to* **cover** *A if every point of A belongs to at least one subset of the collection. A* **subcover** *is a subfamily of a cover which is a cover. If all the subsets are open, we say* **open cover** *of A.*

Definition 5.3.2. *X is said to be* **compact** *if every open cover of X has a finite subcover.*

[1]Maurice René Fréchet; 2 September 1878 - 4 June 1973) was a French mathematician. He made major contributions to the topology of point sets and introduced the entire concept of metric spaces. He also made several important contributions to the field of statistics and probability, as well as calculus. His dissertation opened the entire field of functionals on metric spaces and introduced the notion of compactness. Independently of Riesz, he discovered the representation theorem in the space of Lebesgue square integrable functions.

Remark 5.3.3. *This definition does not mean that a space X is compact if there exists a finite open cover of X, (for example, simply take the open finite cover $\{X\}$). However, any topological space (X, τ) where τ is finite is compact.*

Remark 5.3.4. *Given an open cover $(O_i)_{i\in I}$. All the points of the same O_i are "near" w.r.t. the cover, and two points in different O_i are "far". But if they belong to two O_i of the cover whose the intersection is not empty, they are not too far. They are "near" up to one step.*
So, in a compact space, two points are "near" up to finitely many steps, and all the points are not "too far" from each others w.r.t. the given cover.
If the space is not compact, there is some cover with no finite subcover, hence some points are "infinitely far" from some others w.r.t. the cover.

Remark 5.3.5. *Roughly speaking, compact topological spaces have "few" open sets, while Hausdorff ones have "many" open sets, two distinct points have disjoint open neighbourhoods. A trivial topological space is always compact (but never Hausdorff unless X is empty or a point), a discrete topological space is always Hausdorff, but only compact, if X has finitely many elements. In this sense, compact Hausdorff spaces represent a happy middle ground.*

Example 5.3.6. *In the following examples, the real space $\mathbb{R}$ is equipped with the standard topology, if there is no other mention*

1. *$\mathbb{R}$, is not compact (Consider the open cover $(U_n)_{n\in\mathbb{N}^*}$ where $U_n =]-n, +n[$).*
2. *$]0,1[,]0,1]]$ as subspaces of $\mathbb{R}$ are not compact.*
3. *The closed intervals $[a,b] \subset \mathbb{R}$ are compact (Borel-Lebesgue theorem 5.3.26).*
4. *Any finite topological space is compact.*
5. *Let $(\mathbb{R}, \tau)$ where $O \in \tau$ if either $O = \emptyset$ or $\mathbb{R} \setminus O$ is a finite set. Then $(\mathbb{R}, \tau)$ is compact.*

Following the examples 1 and 5, notice that the same set can be compact for some topology and non compact for another one.
We can also define compactness in terms of closed sets.

Definition 5.3.7. *A family of subsets of X is said to have the* **finite intersection property** *if the intersection of any finite subfamily is non-empty.*

Proposition 5.3.8. *X is compact iff for any family of closed subsets of X with the finite intersection property, the intersection of the family is not empty.*

Proof: $\Longrightarrow$) Let $\mathcal{C} = \{V_\lambda\}_{\lambda\in\Lambda}$ be a family of closed sets with the finite intersection property. Suppose $\bigcap\limits_\lambda V_\lambda = \emptyset$, then $\bigcup\limits_\lambda (X \setminus V_\lambda)$ is an open cover of X. Therefore, there exists a finite subcover of X, such that $X = \bigcup\limits_{i=1}^k (X \setminus V_{\lambda_i}) = X \setminus \bigcap\limits_{i=1}^k V_{\lambda_i}$ then $\bigcap\limits_{i=1}^k V_{\lambda_i} = \emptyset$ which is a contradiction, so $\bigcap\limits_\lambda V_\lambda \neq \emptyset$.
$\Longleftarrow$) Let $\mathcal{V} = \{O_\lambda\}_{\lambda\in\Lambda}$ be an open cover of X so $\bigcup\limits_\lambda O_\lambda = X$ and $\bigcap\limits_\lambda X \setminus O_\lambda = \emptyset$. Then the subsets $X \setminus O_\lambda$ are closed. If any finite intersection $\bigcap\limits_i (X \setminus O_{\lambda_i}) \neq \emptyset$, then $\bigcap\limits_\lambda (X \setminus O_\lambda) \neq \emptyset$ which

is a contradiction. Hence, there exists some finite intersection, $\bigcap_i (X \setminus O_{\lambda_i}) = \emptyset$ and $\{O_{\lambda_i}\}$ is a finite subcover of X. ■

Example 5.3.9. *The subspace $\mathbb{Q}$ of $\mathbb{R}$ is not compact. (Hint: Suppose $\mathbb{Q}$ compact. Then for any $a > 0, \mathbb{Q} \cap [-a, a]$ is closed then compact (lemma 5.3.11). Let $x \in [-a, a] \cap (\mathbb{R} \setminus \mathbb{Q})$; then the family of closed subsets $\left(\mathbb{Q} \cap [-a, a] \cap \left[x - \frac{1}{n}, x + \frac{1}{n}\right]\right)_{n \in \mathbb{N}^*}$ has an empty intersection although any finite intersection is non empty; which is a contradiction.)*

Open cover	Finite intersection
$\forall (O_i)_{i \in I}, O_i$ open such that $\bigcup_i O_i = X$, $\exists (O_{i_j}), j = 1, \ldots, n$ such that $\bigcup_{j=1}^{n} O_{i_j} = X$.	$\forall (V_i)_{i \in I}, V_i$ closed such that $\forall (V_{i_j}), j = 1, \ldots, n$, such that $\bigcap_{j=1}^{n} V_{i_j} \neq \emptyset$, then $\bigcap_i V_i \neq \emptyset$. .

5.3.2 Properties of Compact Spaces

Proposition 5.3.10. *The topological space X is compact iff any filter on X has at least an adherent point.*

Proof: $\Longleftarrow$): Let $\mathcal{V} = \{V_\lambda\}_{\lambda \in \Lambda}$ be a family of closed sets such that the intersection is empty. If the intersection of an arbitrary finite subfamily is not empty, then $\mathcal{V}$ defines a filter which would have an adherent point. This adherent point should belong to any element of $\mathcal{V}$, these elements being closed sets; thus it is a contradiction.
$\Longrightarrow$): Let $\mathcal{F}$ be a filter without adherent point. Then the closure of the elements of the filter defines a family of closed sets which contradicts the assumption. ■

Lemma 5.3.11. *Let A be a closed subset of a compact space X. Then A is compact.*

Proof: Let $\mathcal{V}$ be an open cover of A. Let adjoin the subset $X \setminus A$ to get an open cover $\mathcal{V}'$ of X. X is compact so the cover $\mathcal{V}'$ has a finite subcover, then $\mathcal{V}$ has a finite subcover of A. ■

Remark 5.3.12. *This result does not remain valid if the space X is not compact. For example, let $X = \mathbb{R}$ with the standard topology. Then $]-\infty, 0]$ is closed but not compact.*

Lemma 5.3.13. *Let $f : X \longrightarrow Y$ be a continuous map and let A be a compact subset of X. Then $f(A)$ is a compact subset of Y.*

Proof: Let $\mathcal{V}$ be an open cover of $f(A)$. Then A is covered by the open sets $f^{-1}(O)$ for $O \in \mathcal{V}$. But A is compact then there exists a finite subfamily $\{f^{-1}(O_i)\}_{i=1,\cdots,k}$ such that $A \subset f^{-1}(O_1) \cup \cdots \cup f^{-1}(O_k)$ and $f(A) \subset O_1 \cup \cdots \cup O_k$. ■

Remark 5.3.14. *Notice that $f^{-1}(B)$ is not necessarily compact whenever B is a compact subset of Y. For example, let $f : \mathbb{R} \longrightarrow \mathbb{R} : x \mapsto f(x) = y_0$ be the constant map which is continuous, where $\mathbb{R}$ is equipped with the standard topology. Then $\{y_0\}$ is compact and $f^{-1}(\{y_0\}) = \mathbb{R}$ which is not compact.*

Corollary 5.3.15. *Let X be compact space and let $\mathcal{R}$ be an equivalence relation on X. Then the quotient space $X/\mathcal{R}$ is compact.*

The proof is straightforward. ■

Compact subspaces of Hausdorff spaces behave to some extent like points.

Proposition 5.3.16. *Let X be a Hausdorff space and let K be a compact subset of X. Let $x \in X \setminus K$. Then there exists some open sets $V, W \subset X$ such that $x \in V, K \subset W$ and $V \cap W = \emptyset$.*

Proof: X is Hausdorff, so, for all $y \in K$, there exists some open sets $V_{x,y}, W_{x,y}$ such that $x \in V_{x,y}$, $y \in W_{x,y}$, $V_{x,y} \cap W_{x,y} = \emptyset$. The set K is compact so there exists a finite set $\{y_1, \ldots, y_r\}$ such that $K \subset W_{x,y_1} \cup \ldots \cup W_{x,y_r}$. Define $V = \bigcap_{i=1}^{r} V_{x,y_i}$ and $W = \bigcup_{i=1}^{r} W_{x,y_i}$. Then V and W are open, $x \in V$, $K \subset W$ and $V \cap W = \emptyset$. ■

Corollary 5.3.17. *A compact subset of a Hausdorff space is closed.*

Proof: (exercise) ■

Corollary 5.3.18. *Two disjoint compact subsets of a Hausdorff space have two disjoint open neighbourhoods.*

Proof: (exercise) ■

Remark 5.3.19.

- *A compact subset of a metric space is closed.*
- *If the space X is not Hausdorff, then a compact subset is not necessarily closed. For example, let $X = \{a, b\}$ and $\tau = \{\emptyset, \{a\}, X\}$. Notice that (X, τ) is not Hausdorff. Then $\{a\}$ is compact, open and not closed.*

Theorem 5.3.20. *A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.*

Proof: Let $f : X \longrightarrow Y$ be a continuous bijection where X is compact and Y is Hausdorff. Let F be a closed subset of X. Then F is compact. Therefore $f(F)$ is compact and closed. So f takes closed sets onto closed sets which prove that f^{-1} is continuous. ■

Remark 5.3.21. *If the continuous map is only surjective, then it is a closed quotient map.*
If the continuous map is only injective, then it is an embedding.

A continuous bijection from a compact space to a metric space is a homeomorphism.

Example 5.3.22. *Let $\sim$ be the equivalence relation on $\mathbb{R}$ defineds as follows: $x \sim y$ iff $x - y \in \mathbb{Z}$. Let $\mathbf{T} = \mathbb{R}/\sim$ be the quotient space. Then we define the continuous surjection $\pi : \mathbb{R} \longrightarrow \mathbf{T}$.*
Let $\mathbf{U} = \{z \in \mathbb{C} | |z| = 1\}$ be the unit circle of $\mathbb{R}^2$ (after identification with $\mathbb{C}$). Define the map $f : \mathbb{R} \longrightarrow \mathbf{U}$; $x \mapsto f(x) = e^{2\pi i x}$, then f is a surjection and $f(x) = f(y)$ iff $x - y \in \mathbb{Z}$ so f is compatible with the equivalence relation $\sim$ and f is continuous. Hence, there exists a map $g : \mathbf{T} \longrightarrow \mathbf{U}$ such that $f = g \circ \pi$ and f is continuous iff g is continuous.
$\mathbf{T}$ is compact because $\mathbf{T} = \pi([0, 1]), \pi$ is continuous and $[0, 1]$ is compact.
Finally, the map $g : \mathbf{T} \longrightarrow \mathbf{U}$ is a continuous bijection, $\mathbf{T}$ is compact, $\mathbf{U}$ is Hausdorff, then g is a homeomorphism.

Lemma 5.3.23. *Let X and Y be two topological spaces. Let K be a compact subset of Y and let O be an open set of $X \times Y$. Let $V = \{x \in X \mid \{x\} \times K \subset O\}$, then V is open in X.*

Proof: If $O = \emptyset$, then O is open.
If not, let $x \in V$. For each $y \in K$, there exist some open subsets $O_y^X \subset X$ and $O_y^Y \subset Y$ such that $(x, y) \in O_y^X \times O_y^Y \subset O$. The subset K is compact so there exists a finite set $\{y_1, \ldots, y_r\}$ such that $K \subset O_{y_1}^Y \cup \ldots \cup O_{y_r}^Y$. Set $N_x = O_{y_1}^X \cap \ldots \cap O_{y_r}^X$ then N_x is open in X and $N_x \times K \subset \bigcup_{i=1}^{r} N_x \times O_{y_i}^Y \subset \bigcup_{i=1}^{r} O_{y_i}^X \times O_{y_i}^Y \subset O$ so that $N_x \subset V$. It follows that $V = \bigcup_{x \in V} N_x$ and finally V is open. ■

Theorem 5.3.24. *A finite product of topological spaces $X_1 \times \cdots \times X_n$ is compact iff each X_i, $i = 1, \ldots, n$, is compact.*

Proof: It is enough to prove this result for the product of two spaces X and Y.
Suppose X and Y are compact.
Let $\mathcal{V}$ be an open cover of $X \times Y$.
For any $y \in Y$ and any $x \in X$, there exists some open neighbourhoods O_{xy}^X of x, O_{xy}^Y of y, there exists $V_{xy} \in \mathcal{V}$ such that $(x, y) \in O_{xy}^X \times O_{xy}^Y \subset V_{xy}$.
$\{O_{xy}^X \times O_{xy}^Y \mid y \in Y\}$ is an open cover of $\{x\} \times Y$
Let $x \in X$, then $\{x\} \times Y$ is a compact subset of $X \times Y$, as the image of a compact space under the continuous map

$$\begin{array}{ccc} Y & \longrightarrow & X \times Y \\ y & \longmapsto & (x, y) \end{array}$$

$\{x\} \times Y$ is compact, so there exists a finite subcover $\left\{O_{xy_k}^X \times O_{xy_k}^Y \mid k = 1, \ldots, p\right\}$ of $\{x\} \times Y$.
Let $O_x = \bigcap_{k=1}^{p} O_{xy_k}^X$, then $O_x \times Y \subset \bigcup_{k=1}^{p} V_{xy_k}$.
$\{O_x \mid x \in X\}$ is an open cover of the compact space X, so there exists a finite subcover $(O_{x_j})_{j=1,\ldots,r}$ of X, then $X \times Y \subset \bigcup_{j=1}^{r} (O_{x_j} \times Y) \subset \bigcup_{k=1}^{p} \bigcup_{j=1}^{r} V_{x_j y_k}$. ■

Another Proof: Let $\mathcal{V}$ be an open cover of $X \times Y$.
Let $x \in X$, then $\{x\} \times Y$ is a compact subset of $X \times Y$ as proved above. There exists a finite subfamily $O_1, \ldots, O_r$ of $\mathcal{V}$ such that $\bigcup_{i=1}^{r} O_i \supset \{x\} \times Y$. Denote $V_x = \{x' \in X \mid \{x'\} \times Y \subset \bigcup_{i=1}^{r} O_i\}$.
From lemma 5.3.23, $x \in V_x$ and V_x is open. Notice that $V_x \times Y$ is covered by finitely many open sets of $\mathcal{V}$. Moreover, $\{V_x \mid x \in X\}$ is an open cover of X then there exists a finite set $\{x_1, \ldots, x_s\} \subset X$ such that $X = V_{x_1} \cup \cdots \cup V_{x_s}$, $\bigcup_{i=1}^{s} V_{x_i} \times Y = X \times Y$ and each $V_{x_i} \times Y$ is covered by a finite subfamily of $\mathcal{V}$, so $X \times Y$ is compact.
Conversely, if $X \times Y$ is compact, then $X \cong X \times \{y\} \subset X \times Y$ and $X \times \{y\}$ is compact (exercise). ■

Notice that if X is compact, then $X \times \{y\}$ is compact. Suppose Y is infinite, then it does not imply that $X \times Y$ is compact (as $\cup_y X \times \{y\}$).

Example 5.3.25.

- *The cylinder, the Möbius strip, the torus and the Klein bottle are compact as quotient spaces of the compact space* $[0, 1] \times [0, 1]$.
- $[0, 1] \times [0, 1[$ *is not compact.*

- *The cone $C\mathbb{S}^n$ is homeomorphic to the ball $B^{n+1} = \{x \in \mathbb{R}^{n+1} | \ d(x,0) \leq 1\}$. For, let $f : \mathbb{S}^n \times I \longrightarrow C\mathbb{S}^n$ the canonical surjection, and $g : \mathbb{S}^n \times I \longrightarrow B^{n+1}$ the map $g(x,t) = (1-t)x$; then $g \circ f^{-1}$ is bijective, and since $\mathbb{S}^n \times I$ is compact, the result follows. Similarly, the suspension $S\mathbb{S}^n$ of $\mathbb{S}^n$ is homeomorphic to $\mathbb{S}^{n+1}$.*

More generally, the **Tychonoff[2] Theorem** says that an arbitrary product of compact spaces is compact.

5.3.3 Bolzano-Weierstrass Property

Not all sequences are convergent, even Cauchy sequences may not converge, they converge in "good" spaces, i.e. complete spaces (cf. next chapter).
Consider the following sequence $\{1, -1, 1, -1, 1, -1, \cdots, 1, -1, 1, -1, \cdots\}$. This sequence does not converge, but there are some subsequences that converge, such as $\{1, 1, 1, \cdots\}$ or $\{-1, -1, -1, \cdots\}$.
The following result is called the **Bolzano[3]-Weierstrass[4]** theorem for real numbers: Every bounded sequence of real numbers have a convergent subsequence. Notice that this result remains valid in $\mathbb{R}^n, n \geq 1$.

Definition 5.3.26. *A topological space X has the Bolzano-Weierstrass property, if any infinite subset $A \in X$ has at least one limit point.*

Remark 5.3.27. *This property is different from the closeness. For example, the interval $[0, \infty[$ is closed in $\mathbb{R}$, but does not have the Bolzano-Weierstrass property. The infinite subset $\mathbb{N}$ has no limit point in $[0, \infty[$.*

Theorem 5.3.28. *A compact space X satisfies the Bolzano-Weierstrass property.*

Proof: Let X be a compact space and let S be a subset of X which has no limit point. We have to show that S is finite.
Given $x \in X$, we can find an open neighbourhood O_x of x such that

$$O_x \cap S = \begin{cases} \emptyset & \text{if } x \notin S \\ \{x\} & \text{if } x \in S \end{cases}$$

since otherwise x should be a limit point of S. By the compactness of X the open cover $\{O_x \mid x \in X\}$ has a finite subcover. But each O_x contains at most one point of S and therefore S must be finite. ■

5.3.4 Compactness in Metric Spaces

We already proved that a compact subset of a metric space is closed. Some more results for the particular case of the metric spaces.

Theorem 5.3.29. *Let (X, d) be a metric space. Then, the following properties are equivalent:*

1. *X is compact.*

[2]Andrey Nikolayevich Tikhonov (October 30, 1906 - October 7, 1993) was a Soviet and Russian mathematician and geophysicist known for important contributions to topology, functional analysis, mathematical physics, and ill-posed problems. He was also one of the inventors of the magnetotellurics method in geophysics.

[3]Bernhard Placidus Johann Nepomuk Bolzano (October 5, 1781 - December 18, 1848), was a Bohemian mathematician, logician, philosopher, theologian.

[4]Karl Theodor Wilhelm Weierstrass (October 31, 1815, February 19, 1897) was a German mathematician who is often cited as the "father of modern analysis".

2. *X has the Bolzano-Weierstrass property.*

3. *X is sequentially compact, i.e. every sequence has a convergent subsequence.*

Proof:

- 3) $\implies$ 2) Let Y be an infinite subset of X and let $\{y_n\}_{n\in\mathbb{N}}$ be a sequence of pairwise distinct points in Y. The space X is sequentially compact, so the sequence contains a convergent subsequence and the limit is a point of Y.

- 2) $\implies$ 3) Let X be a metric space with the Bolzano-Weierstrass property. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of points of X. If one point occurs infinitely many times in the sequence, then we have a constant subsequence which converges. If each point occurs finitely many times, we can choose a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ where the points are pairwise distinct. By assumption, this set has a limit and there is a subsequence which converges to this point.

- 1) $\implies$ 2) Suppose that the space X does not satisfy the Bolzano-Weierstrass property, i.e. there exists an infinite subset Y that has no limit point. So, for each $y \in Y$, there exists a ball B_y with center y such that B_y such that $B_y \cap Y$ contains no other point in Y.

- 3) $\implies$ 1) Let $(O_i, i \in I$ be an open cover of X. Then there exists $r > 0$ such that for each $x \in X$, the ball $B(x;r) \subseteq O_i$ for some $i \in I$. If not, choose a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that $B\left(x_n; \frac{1}{n}\right) \nsubseteq O - i, \forall i \in I$.
The space X is sequentially compact, so the sequence (x_n) has a convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$. Let $x \in X$ be the limit of this subsequence. For some i_0, we have $x \in O_{i_0}$, and there is $r_0 > 0$ such that $B(x;r_0) \subseteq O_{i_0}$. Choose N suvch that $d(x, x_N) < \frac{1}{2}r_0$ and $\frac{1}{N} < \frac{1}{2}r_0$. If $y \in B\left(x_N; \frac{1}{N}\right)$, then $d(x,y) \leq d(x,x_N) + d(x_N, y) < \frac{1}{2}r_0 + \frac{1}{2}r_0 = r_0$. Hence $y \in B(x;r_0) \subseteq O_{i_0}$. It follows that $B\left(x_N; \frac{1}{N}\right) \subseteq B(x;r_0) \subseteq O_{i_0}$ which is a contradiction.

- 1) $\implies$ 3) Assume there is a sequence $(x_n)_{n\in\mathbb{N}}$ with no convergent subsequence. Then no term in the sequence can occur infinitely many times, and we can assume that $x_i \neq x_j, i \neq j$. Each term of the sequence (x_n) is an isolated point of the set $\{x_n \mid n \in \mathbb{N}\}$ (otherwise (x_n) would have a convergent subsequence). Hence, for each i there is an open ball $B(x_i; r_i)$ such that $x_j \notin B(x_i; r_i), \forall i \neq j$. Denote $O_0 := X \setminus \{x_n \mid n \in \mathbb{N}\}$. Then O_0 is open since its complement consists only of isolated points, and so is closed. Then $\{O_0\} \cup \{B(x_n; r_n) \mid n \in \mathbb{N}\}$ is an open cover for X. This open cover has no finite subcover, since any subfamily of these sets would fail to include infinitely many terms for the sequence $(x_n)_{n\in\mathbb{N}}$ in its union.

■

Theorem 5.3.30 (Borel-Lebesgue)**.** [56] *Let $a, b \in \mathbb{R}$ such that $a < b$. Then $[a,b]$ is compact.*

Proof: Let $\mathcal{V} = (O_i)$ be an open cover of $[a,b]$, and $A = \{x \in \mathbb{R} \mid [a,x]$ is contained in a finite subcover of $\mathcal{V}\}$. Then $a \in A$, so $A \neq \emptyset$.
Let $m = \sup A$, then $a \leq m \leq b$, and there exists $O_{i_0} \in \mathcal{V}$ containing m.

[5] Félix Édouard Justin Émile Borel (7 January 1871 - 3 February 1956) was a French mathematician and politician. He was among the pioneers of measure theory and its application to probability theory. The concept of a Borel set is named in his honor. One of his books on probability introduced the amusing thought experiment that entered popular culture under the name infinite monkey theorem or the like. He also published a number of research papers on game theory.

[6] Henri Léon Lebesgue (June 28, 1875 - July 26, 1941) was a French mathematician most famous for his theory of integration, which was a generalization of the seventeenth century concept of integration summing the area between an axis and the curve of a function defined for that axis.

If $m \notin A$, then $]m-\varepsilon, m] \cap A \neq \emptyset$ for all $\varepsilon > 0$.
Let $\varepsilon > 0$ such that $]m-\varepsilon, m] \subset O_{i_0}$. Let $x \in A \cap]m-\varepsilon, m]$. Then there exists a finite subset $J \subset I$ such that $[a, x] \subset \bigcup_{j \in J} O_j$, so $[a, m] \subset \bigcup_{j \in J} O_j \cup O_{i_0}$, then $m \in A$.
If $m \leq b$, there exists $\varepsilon > 0$ such that $[m, m+\varepsilon] \subset O_{i_0}$ which is contradiction since $m = \sup A$, therefore $m = b$. ■

Lemma 5.3.31. *Let $f : X \longrightarrow \mathbb{R}$ be a continuous real-valued function on a compact topological space X. Then f is bounded above and below on X.*

Proof: The range $f(X)$ of the function f is compact so it is covered by some finite collection of open sets. Therefore, $f(X)$ is covered by some finite collection $\{I_1, I_2, \ldots, I_k\}$ of open intervals of the form $]-m; m[$, where $m \in \mathbb{N}$ and $\mathbb{R}$ is covered by the collection of all intervals of this form. It follows that $f(X) \subset]-M; M[$, where $]-M; M[$ is the largest of the intervals $I_1, I_2, \ldots, I_k$. Thus the function f is bounded above and below on X, as required. ■

Remark 5.3.32. *In fact, this result can be extended as follows: The metric space (X, d) is compact iff every continuous function $f : X \longrightarrow \mathbb{R}$ is bounded.*

Lemma 5.3.33. *Let $f : X \longrightarrow \mathbb{R}$ be a continuous map and suppose X compact. Then there exist $u, v \in X$ such that $f(u) \leq f(x) \leq f(v)$ for all $x \in X$.*

Proof: Let $m = \inf\{f(x) \mid x \in X\}$ and $M = \sup\{f(x) \mid x \in X\}$. Then there exists $v \in X$ such that $f(v) = M$.
If $f(x) < M$ for all $x \in X$, the map $x \longmapsto \dfrac{1}{M - f(x)}$ could be continuous and not bounded.
Similarly, there exists $u \in X$ such that $f(u) = m$, $\left(\text{choose the map } x \longmapsto \dfrac{1}{f(x) - m}\right)$ and the result follows. ■
This result plays an important role in linear programming.

Example 5.3.34. *The compactness of X is essential. For example, the function $f : \mathbb{R} \longrightarrow \mathbb{R}, x \longmapsto f(x) = \frac{1}{x}$ is continuous, but not bounded on $]0, 1]$.*
The function $f(x) = x$ is continuous and bounded on $]0, 1[$, but does not attain its bounds 0 and 1.

Theorem 5.3.35 (Heine-Borel criterion). [7] *The subset $K \subset \mathbb{R}^n$ is compact iff K is both closed and bounded.*

Proof: $\Longrightarrow$) $\mathbb{R}^n$ is Hausdorff then K is closed (cf. Corollary 5.3.17).
For each $m > 0$, let $B(0; m)$ be the ball of center 0 and radius m. Then the set of all such balls for all $m > 0$ is a cover of $\mathbb{R}^n$. There exist $m_1 > 0, \ldots, m_k > 0$ such that $K \subset \bigcup_{i=1}^{k} B(0; m_i)$ and $K \subset B(0; M)$ where $M = \max_{i=1,\ldots,k} m_i$ thus K is bounded.
$\Longleftarrow$) Suppose K bounded and closed subset of $\mathbb{R}^n$. There exists $L \geq 0$ such that $K \subset C = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid -L \leq x_j \leq L, j = 1, \ldots, n\}$. The interval $[-L, L]$ is compact and C is the product of n copies of $[-L, L]$ so C is compact. But K is a closed subset of C so it is compact (cf lemma 5.3.7.). ■

Remark 5.3.36. *It is shown that the closed unit ball in a normed linear space is compact iff the space is finite-dimensional. In an infinite-dimensional normed linear space, the closed unit ball is never compact.*

[7] Heinrich Eduard Heine (March 15, 1821 - October 21, 1881) was a German mathematician. Heine was born in Berlin, and became known for results on special functions and in real analysis. In particular, he authored an important treatise on spherical harmonics and Legendre functions

Example 5.3.37. *The interiors of the cylinder and the Möbius strip are not compact, because they are not closed subsets of* $\mathbb{R}^3$.

Example 5.3.38. *Heine-Borel theorem is valid in the Euclidean spaces, but it does not remain valid in metric spaces.*
For example, in $\mathbb{R}^2$, *let* d_R *be the distance defined in 2.3.1.e. The circle of center* $(1,0)$ *and radius* 2 *is* $\{(3,0)\} \cup \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\} \setminus \{(1,0)\}$. *It is not compact because it does not have the Bolzano-Weierstrass property (the distance between two points is at least* 2; *in fact, the induced topology is discrete. However, it is bounded and closed.*

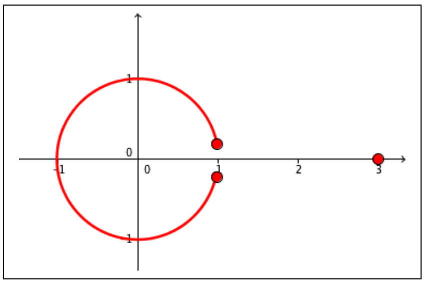


Figure 5.2

Now we introduce a new criterion of compactness in a metric space. Let A be a non-empty subset of the metric space X. We defined the **diameter** of A as $\sup\{d(x,y) \mid x,y \in A\}$. It is denoted $\delta(A)$. It is said that A is **bounded** if it is no empty and its diameter is finite.

Lemma 5.3.39 (Lebesgue)**.** *Let* (X,d) *be a compact metric space. Let* $\mathcal{V}$ *be an open cover of* X. *Then there exists* $\delta > 0$ *such that each subset of* X *whose the diameter is* $< \delta$ *is contained wholly within one of the open sets of the cover* $\mathcal{V}$.

Proof: Suppose Lebesgue's lemma is false, we can find a sequence $A_1, A_2, \dots$ of subsets of X, none of which are contained inside a member of the cover $\mathcal{V}$, and whose diameters tend to zero as we proceed along the sequence. For each n choose a point $x_n \in A_n$. Either the sequence (x_i) contains only finitely many distinct points, in which case some point repeats infinitely often; or it is infinite, in which case it must have a limit point since X is compact (Bolzano-Weierstrass property). denote the repeated point, or limit point by p. Let $O \in \mathcal{V}$ which contains p. Choose $\varepsilon > 0$ such that $B(p;\varepsilon) \subset O$, and choose an integer N large enough so that:

1. the diameter of A_N is less than $\dfrac{\varepsilon}{2}$, and
2. $x_N \in B\left(p; \dfrac{\varepsilon}{2}\right)$.

Then $d(x_N,p) < \dfrac{\varepsilon}{2}$ and $d(x,x_N) < \dfrac{\varepsilon}{2}$ for any point $x \in A_N$. Therefore $d(x,p) < \varepsilon$ if $x \in A_N$, showing $A_N \subset O$. This contradicts our initial choice of the sequence (A_n). ■

In the last part of this section, we shall see the properties of compactness in terms of sequences.

Proposition 5.3.40. *Let* X *be a compact topological space. Then any sequence* $(x_i)_{i\in\mathbb{N}}$ *of* X *has an adherent point.*

Proof: Let $A_n = \{x_i \mid i \geq n\}$. If $\bigcap\limits_n \mathrm{Cl}(A_n) = \emptyset$ then there exist $n_1, \dots, n_p$ such that $\bigcap\limits_i^p \mathrm{Cl}(A_{n_i}) = \emptyset$ because X is compact. But there exists n_0 such that $\mathrm{Cl}(A_{n_0}) \subset \bigcap\limits_i^p \mathrm{Cl}(A_{n_i})$ which is not, so $x_i)$ has an adherent point. ■

Proposition 5.3.41. *Let* (X,d) *be a metric space. Then* X *is compact iff any sequence has at least an adherent point.*

Proof:
$\Longrightarrow$) It follows from the previous proposition.
$\Longleftarrow$) Let $(O_i)_{i\in I}$ be an open cover of X and let $B(x;r) = \{y \mid d(x,y) < r\}$ be the open ball of center x and radius r.

1. There exists $\alpha > 0$ such that $B(x;\alpha) \subset O_i$ for some $i \in I$.
   If not, for any $n = 1, 2, \ldots$, there exists $x_n \in X$ such that $B\left(x_n; \frac{1}{n}\right) \not\subset O_i$ for any $i \in I$. Let $x \in X$ be an adherent point of (x_i). Then there exists $i \in I$ such that $x \in O_i$ and $B\left(x; \frac{1}{N}\right) \subset O_i$ for some N. Thus there exists $n \geq 2N$ such that $x_n \in B\left(x; \frac{1}{2N}\right)$ and $B\left(x_n; \frac{1}{n}\right) \subset B\left(x_n; \frac{1}{2N}\right) \subset B\left(x; \frac{1}{2N} + \frac{1}{2N}\right) \subset O_i$ which is not.

2. There exists some finite subcover with balls $B(x;\alpha)$.

   If $B(x_1;\alpha) = X$, then there is such finite subcover.

   If not, let $x_2 \in X \setminus B(x_1;\alpha)$. If $B(x_1;\alpha) \cup B(x_2;\alpha) = X$, then there is such finite subcover.

   If not let $x_3 \in X \setminus (B(x_1;\alpha) \cup B(x_2;\alpha))$ and so on.
   If the process does not stop, there exists a sequence (x_i) such that $x_n \notin B(x_1;\alpha) \cup \cdots \cup B(x_{n-1};\alpha)$ for any n. But $d(x_i, x_j) \geq \alpha$. Let x be an adherent point of the sequence (x_i). Then there exists n such that $x_n \in B\left(x; \frac{1}{2}\right)$ and there exists $n' > n$ such that $x_{n'} \in B\left(x; \frac{1}{2}\right)$. So $d(x_n, x_{n'}) < \alpha$, which is a contradiction.

■

Proposition 5.3.42. *Let (X,d) be a metric space. Then X is compact iff every infinite subset of X has at least one accumulation point.*

Proof: $\Longleftarrow$) Let $\mathcal{V}$ be an open cover of X. We have to find a finite subcover of $\mathcal{V}$.
From Lebesgue lemma, there exists $< \delta$ such that all open balls $B(x;\delta)$ is contained in an open set of the cover $\mathcal{V}$.. Let $x_1 \in X$. If $B(x_1;\delta)$ does not cover X, there exists x_2 such that $d(x_1, x_2) \geq \delta$. More generally, suppose we have define the points $x_1, x_2, \ldots, x_p$ of mutual distances $\geq \delta$. If the union of the balls $B(x_i;\delta), i = 1, \ldots, p$ does not cover X, there exists x_{p+1} such that the distances $d(x_{p+1}, x_i) \geq \delta$. But the sequence (x_i) cannot be infinite which is impossible by hypothesis (the sequence has at least an accumulation point).
$\Longrightarrow$) It follows from the Bolzano-Weierstrass property. ■

Remark 5.3.43. *Recall that an accumulation point is either an adherent point or an isolated point. So the previous proposition could be written as X is compact iff every infinite subset of X has at least one adherent point.*

Corollary 5.3.44. *Let (X,d) be a metric space and $A \subset X$. Then $\mathrm{Cl}(A)$ is compact iff any sequence of points of A has a subsequence with a limit point in X.*

Proof: $\Longrightarrow$) Let (x_n) be some sequence of points of $A \subset \mathrm{Cl}(A)$. Let (X,d) be a metric space. This sequence has an adherent point $x \in \mathrm{Cl}(A)$. Then there exists a subsequence which converges to x.
$\Longleftarrow$) Let $(y_1, y_2, \ldots)$ be some sequence of points of $\mathrm{Cl}(A)$. Some subsequence $(x_{n_1}, x_{n_2}, \ldots)$ converges to some point $x \in X$. Let $x_i \in A$ such that $d(y_{i_n}, x_i) \leq \dfrac{1}{i}$. Then $x \in \mathrm{Cl}(A)$. Moreover, $d(y_{n_i}, x) \leq d(y_{n_i}, x_{n_i}) + d(x_{n_i}, x) \leq \dfrac{1}{n_i} + d(x_{n_i}, x) \longrightarrow 0$ so (y_{n_i}) converges to x. Then the sequence $(y_1, y_2, \ldots)$ has an adherent point in $\mathrm{Cl}(A)$, hence $\mathrm{Cl}(A)$ is compact. ■

Compactness and Uniform Continuity

Let $f : X \longrightarrow Y$ be a map between two metric spaces. Recall that f is **uniformly continuous** if

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } x, x' \in X \text{ and } d_X(x, x') < \delta \Longrightarrow d_Y\big(f(x), f(x')\big) < \varepsilon$$

Remark 5.3.45. *Notice that δ is independent of x and x'.*

Example 5.3.46. *An isometry is uniformly continuous (exercise).*

We will give two examples that will help us to have a better understanding of the continuity and uniform continuity.

Example 5.3.47.

1. *The function* $\sin : [0, 2\pi] \longrightarrow \mathbb{R}, x \longmapsto \sin x$ *is uniformly continuous.*
   $\forall \varepsilon > 0, \exists \delta = \varepsilon > 0$*, (that does not depend on x, x', but only depends on ε).*
   Using the Mean Value Theorem[8]*, we have*

$$\begin{aligned} |x - x'| < \delta \Longrightarrow |\sin x - \sin x'| &\leq \left| \frac{d \sin x}{dx} \right|_{x=\zeta} |x - x'| \\ &= |\cos \zeta| |x - x'| \\ &\leq |x - x'| < \delta = \varepsilon \end{aligned}$$

2. *The map $f : \mathbb{R} \longrightarrow \mathbb{R} : x \mapsto x^2$ is continuous but not uniformly continuous. However, the restriction $f_{|[a,b]}$ of f to any compact interval $[a, b]$ is uniformly continuous.*
   Let us show that f is not uniformly continuous.
   We must show the existence of an $\varepsilon > 0$ such that for all $\delta > 0$, there exists x, x' satisfying $|x - x'| < \delta$, but $|x^2 - x'^2| \geq \varepsilon$. Take $\varepsilon = 1$, then, we can find x, x' such that $|x - x'| < \delta$, and $|x^2 - x'^2| \geq 1$. Take $x = n, x' = n + \delta/2$ where n is sufficiently large, so that

$$|x - x'| = |n - n + \delta/2| = \frac{\delta}{2} < \delta$$

but

$$|x^2 - x'^2| = |x - x'|.|x + x'| = \frac{\delta}{2}\left(2n + \frac{\delta}{2}\right) > 1$$

and we proved that f is not uniformly continuous.

Theorem 5.3.48. *let X and Y be two metric spaces and suppose X compact. Then every continuous map $f : X \longrightarrow Y$ is uniformly continuous.*

Proof: Let $\varepsilon > 0$. For any $x \in X$, there exists $\delta_x > 0$ such that $x' \in X$, $d(x, x') < \delta_x \Longrightarrow d\big(f(x), f(x')\big) < \frac{1}{2}\varepsilon$. Let $B_x = B\left(x; \frac{1}{2}\delta_x\right)$, which is, for all $x \in X$ an open cover of X. X is compact, so there exists a finite subcover $B_{x_1}, \ldots, B_{x_n}$. Let $\delta = \inf\left(\frac{1}{2}\delta_{x_1}, \ldots, \frac{1}{2}\delta_{x_n}\right) > 0$. Let $x', x'' \in X$ such that $d(x', x'') < \delta$. There exists i such that $x' \in B_{x_i}$. Then $d(x_i, x') < \frac{1}{2}\delta_{x_i}$ and $d(x_i, x'') \leq d(x_i, x') + d(x', x'') < \frac{1}{2}\delta_{x_i} + \delta \leq \frac{1}{2}\delta_{x_i} + \frac{1}{2}\delta_{x_i}$ and $d(x_i, x'') < \delta_{x_i}$. Therefore $d\big(f(x'), f(x_i)\big) < \frac{1}{2}\varepsilon$, $d\big(f(x''), f(x_i)\big) < \frac{1}{2}\varepsilon$, then $d\big(f(x'), f(x'')\big) < \varepsilon$. ■

[8] If a function $f : [a, b] \longrightarrow \mathbb{R}$ is continuous and differentiable on $]a, b[$, then there at least $\zeta \in]a, b[$ such that $f(x) - f(x') = f'(\zeta)(x - x')$.

5.3.5 Compact Subsets of Euclidean Spaces

The next result is a characterization of compact subsets of Euclidean spaces given by some remarkable property of the Cantor set.
Let us recall some properties of the Cantor set (cf. 1.3.9). Let $[0,1]$ be the unit interval as a subspace of the space of real numbers with the standard topology. Define the sequence of subsets $F_1 \supset F_2 \supset \cdots F_n \supset \cdots$ where $F_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1], F_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1], \ldots$.
The **Cantor set** is defined by

$$C = \bigcap_{n=1}^{\infty} F_n$$

The set G is an intersection of closed sets, so it is a closed subset of $[0,1]$ that is compact, so C is compact.

Lemma 5.3.49. *Let X be a metric space, A some closed subset, and K some compact subset of X. Then there exists $x_0 \in K$ such that $d(K,A) = d(x_0, A)$. If A is also compact, then there exists $y_0 \in A$ such that $d(K,A) = d(x_0, y_0)$.*

Proof: (to be done) ■

Lemma 5.3.50. *Every nonempty closed subset A of the Cantor space C is a retract of C.*

Proof: (to be done) ■

Corollary 5.3.51. *For each $k \in \mathbb{N}$, there exists a continuous map of the closed unit interval I onto I^k.*

Proof: (to be done) ■

Theorem 5.3.52. *Let X be a non-empty compact subset of an Euclidean space. Then there exists some Cantor set C and a continuous surjective map $f : C \longrightarrow X$.*

Proof: (to be done) ■

5.3.6 Exercises

1. Prove that every finite subset of a topological space is compact.

2. In a compact space, any infinite set has an accumulation point.

3. Let X be a Hausdorff space and let $(x_i)_{i \in \mathbb{N}}$ be a sequence of points of X which converges w.r.t. the filterbase $\mathcal{B}_{\mathbb{N}}$, to one point x. (Why does the assumption "Hausdorff" is necessary?)
   Show that $\{x_i \mid i \in \mathbb{N}\} \cup \{x\}$ is compact.

4. Let X and Y be two topological spaces and let $f : X \longrightarrow Y$ be a continuous map. Suppose X compact and Y Hausdorff.
   Show that for any closed set $A \subset X, f(A)$ is closed in Y.

5. The union of two compact sets is a compact. What about any union of compact?

6. Show that the intersection of any family of compact subsets of a Hausdorff space is compact.

7. Let $X = [0, +\infty[$ and $\tau = \{\emptyset, X,]a, +\infty[, \text{ for any } a \in X\}$.
   Characterize the compact subsets of X.

8. Let $X_i = \mathbb{S}^1 \times \{i\}, i = 1, 2$ be the topological spaces where $\mathbb{S}^1$ is the subspace of $\mathbb{R}^2$ and $Y = X_1 \cup X_2$. Define the equivalence relation on Y by $(z, 1) \sim (z, 2)$ for any $z \neq 1$, and $Z = Y/\sim$. Denote $p : Y \longrightarrow Z$ be the canonical projection.
   Show that Z is not Hausdorff.
   Show that $p(X_i)$ are compact.
   Show that $p(X_1) \cap p(X_2)$ is not compact.

9. For the metric d_R on the real plane, let the circle C of radius 2 with center $(1, 0) \in \mathbb{R}^2$ (cf. exercises 2.3.1.1. and 2.8.3.1.).
   Show that C is closed and bounded but it is not compact.

10. Let (X, d) be a compact metric space, and $f : X \longrightarrow X$ a map such that $d(f(x), f(y)) < d(x, y)$ for any $x, y \in X, x \neq y$.
    Show that f has a unique fixed point, i.e. x_0 such that $x_0 = f(x_0)$. (Hint: consider $x \longmapsto d(x, f(x))$.)

11. Prove that the Bolzano-Weierstrass property is topological, i.e. Let $X \cong Y$ be two topological spaces, then X has the Bolzano-Weierstrass property iff Y has the Bolzano-Weierstrass property.

12. Show that the function $f(x) = \cos\dfrac{1}{x}$ is continuous on $]0, 1]$. However, f is not uniformly continuous on $]0; 1]$. Notice that the function $f(x) = \cos\dfrac{1}{x}$ is uniformly continuous on $[a, 1]$ for any a such that $0 < a < 1$.

13. Let f be a real-valued continuous function defined on a compact set, say a closed and bounded interval $[a, b]$, then show that f is uniformly continuous on $[a, b]$. (We require the proof using ε and δ).

14. Recall that the real projective space $\mathbf{P}_n(\mathbb{R})$ can de defined as $\mathbb{S}^n/\sim$ where the equivalence relation $\sim$ is spanned by $x \sim -x$. Let $\pi_n : \mathbb{S}^n \longrightarrow \mathbf{P}_n(\mathbb{R})$ be the canonical surjection.
    (a) Show that for any open set O of $\mathbb{S}^n$, $\pi_n(O)$ is open.
    (b) Show that $\mathbf{P}_n(\mathbb{R})$ is compact.
    (c) Let $i_n : \mathbb{R}^n \longrightarrow \mathbb{R}^{n+1}$ be the map given by $i_n(x_1, \ldots x_n) = (x_1, \ldots x_n, 0)$. Show that i_n is compatible with the equivalence relation $\sim$ and let $r : \mathbf{P}_{n-1}(\mathbb{R}) \longrightarrow \mathbf{P}_n(\mathbb{R})$ be the map obtained from i_n.
    (d) Show that r is injective and identifies $\mathbf{P}_{n-1}(\mathbb{R})$ to a closed subset of $\mathbf{P}_n(\mathbb{R})$;
    (e) Show that $\mathbf{P}_n(\mathbb{R}) \cong B_n \coprod_{\iota, \pi_{n-1}} \mathbf{P}_{n-1}(\mathbb{R})$ where ι is the inclusion of $\mathbb{S}^{n-1}$ into B_n.
    (f) Let X, Y and Z be three topological spaces.
        i. Suppose X and Y are compact and X is Hausdorff, then prove that $(X \wedge Y) \wedge Z \cong X \wedge (Y \wedge Z)$.
        ii. Suppose Y and Z are compact and Z is Hausdorff, then prove that $(X \wedge Y) \wedge Z \cong X \wedge (Y \wedge Z)$.
        iii. Prove that $s(sX) \cong X \wedge S^2$ for any base point space X (where sX denotes the reduced suspension of X.

15. Let $\mathcal{C}$ be the algebraic curve defined by the polynomial $P(X, Y) \in \mathbb{C}[X, Y]$.
    Show that $\mathcal{C}$ is not a compact subset of $\mathbb{C}^2$. (Hint: Use Heine-Borel theorem).

5.4 Local Compactness

The topological space $\mathbb{R}$ is not compact but "locally", it looks like a compact space.

Definition 5.4.1. *A topological space X is said to be* **locally compact** *if each point has a neighbourhood with compact closure.*

Example 5.4.2.
1. *X compact $\Longrightarrow X$ locally compact.*
2. *X with the discrete topology $\Longrightarrow X$ locally compact. (For example, the space $\mathbb{Z} \subset \mathbb{R}$).*
3. *$\mathbb{R}$ is locally compact but it is not compact.*
4. *$\mathbb{Q} \subset \mathbb{R}$ is neither compact nor locally compact.*
   We have proved that $\mathbb{Q}$ is not compact.
   Suppose, for example, that 0 has in $\mathbb{Q}$ a compact neighbourhood V; then V contains a neighbourhood $\mathbb{Q} \cap [-a, a]$ for some $a > 0$. But $\mathbb{Q} \cap [-a, a]$ is closed in $\mathbb{Q}$, then it is compact. But it is not because for any $x \in [-a, a] \cap (\mathbb{R} \setminus \mathbb{Q})$ the sequence of decreasing closed sets $\mathbb{Q} \cap [-a, a] \cap \left[x - \frac{1}{n}, x + \frac{1}{n}\right]$ has an empty intersection which is a contradiction because any finite intersection is not empty.

Proposition 5.4.3. *Let X be a locally compact space and let Y be some subset of X where Y is either open or a closed. Then Y is locally compact.*

Proof: Let $y \in Y$. There exists a compact neighbourhood V_y of y in X. Then $V_y \cap Y$ is a neighbourhood of y in Y.

If Y is closed, then $V_y \cap Y$ is closed in Y and therefore compact.

If Y is open, then $V_y \cap Y$ is compact in Y if V_y is compact.

■

Corollary 5.4.4. *Let X be a locally compact space and let $x \in X$. Then $Y = X \setminus \{x\}$ is locally compact.*

Proof: Straightforward from the proposition. ■

Proposition 5.4.5. *Let $X_1, \ldots, X_n$ be n locally compact spaces. Then $\prod_{i=1}^{n} X_i$ is locally compact.*

Proof: (exercise) ■

Proposition 5.4.6. *Let $f : X \longrightarrow Y$ be a quotient map. If Z is a locally compact Hausdorff space. Then*

$$f \times i_Z : X \times Z \longrightarrow Y \times Z$$

is a quotient map.

Proof: (to be done) ■

Corollary 5.4.7. *If $f : X \longrightarrow Y$ and $g : Z \longrightarrow T$ are quotient maps, and if the domain of f and the range of g are locally compact Hausdorff spaces, then*

$$f \times g : X \times Z \longrightarrow Y \times T$$

is a quotient map.

Proof: (to be done) ■

5.4.1 Exercises

1. Let X be a compact space. We know that for any $x \in X$, the subspace $X \setminus \{x\}$ is locally compact. Find some example such that $X \setminus \{x\}$ is not compact and some example where it is compact.

2. Let A and B two locally compact subspaces of a Hausdorff space X.
Show that $A \cap B$ is locally compact.

3. Let A and B two locally compact subspaces of a Hausdorff space X.
Is $A \cup B$ locally compact?

4. Let $f : X \longrightarrow Y$ be a continuous map where X is locally compact and Y is Hausdorff.
Show that $f(X)$ could be not locally compact. (Hint: Consider a surjection $f : \mathbb{Z} \longrightarrow \mathbb{Q}$).

5.5 Mapping Spaces: Compact-Open Topology

Let X and Y be two topological spaces. Let denote $\mathcal{C}(X, Y)$ be the set of all continuous maps from X to Y.

Definition 5.5.1. *The* **compact-open topology** *on $\mathcal{C}(X, Y)$ is generated by the sets*

$$O(K, U) = \{f \in \mathcal{C}(X, Y) \mid f(K) \subset U\}$$

for K compact of X and U open in Y.

This topology characterizes the nearness of two maps by their nearness on the compact subsets of X.

Proposition 5.5.2. *The set of all $O(K, U)$ as above, are the open sets of a topology on $\mathcal{C}(X, Y)$.*

Proof: ■

Some useful subspaces of $\mathcal{C}(X, Y)$:

Proposition 5.5.3.
- *The map $j : Y \longrightarrow \mathcal{C}(X, Y)$ given by $y \longmapsto c_y$ where $c_y(x) = y$ for any $x \in X$, is a homeomorphism from Y onto a subspace of $\mathcal{C}(X, Y)$.*
- *Let $Y' \subset Y$, then $\mathcal{C}(X, Y')$ is homeomorphic to the subspace $\{f \in \mathcal{C}(X, Y) \mid f(X) \subset Y'\}$.*
- *$\mathcal{C}(X, Y)$ is Hausdorff iff Y is Hausdorff.*

Proof:

- $c_y \in O(K, Y)$ iff $y \in Y$, so the injection j is a homeomorphism.
- Let $\varphi : \mathcal{C}(X, Y') \longrightarrow \{f \in \mathcal{C}(X, Y) \mid f(X) \subset Y'\} := \mathcal{C}'$ be the identity map. Let U be an open set of Y and $V = U \cap \mathcal{C}'$. Then $\varphi(O(K, V)) = O(K, U) \cap \mathcal{C}'$ and φ is a homeomorphism.
- $\Longrightarrow$) It follows from the previous result.
$\Longleftarrow$) Let $f \neq g$, then for some $x_0 \in X$, $f(x_0) \neq g(x_0)$. Y being Hausdorff, there is open disjoint neighbourhoods U of $f(x_0)$ and U' of $g(x_0)$. Then $O(x_0, U)$ and $O(x_0, U')$ are disjoint neighbourhoods of f and g. ■

This compact-open topology has important properties whenever X contains enough compact subsets. So, we will mainly consider that X is a locally compact space.
Let X, Y and Z be three topological spaces and let $f \in \mathcal{C}(X,Y),\ g \in \mathcal{C}(Y,Z)$, then $g \circ f \in \mathcal{C}(X,Z)$ that defines a map $\mathcal{C}(X,Y) \times \mathcal{C}(Y,Z) \longrightarrow \mathcal{C}(X,Z)$.

Proposition 5.5.4. *The maps $f \longmapsto g \circ f$ and $g \longmapsto g \circ f$ are continuous.*

Proof: ■

So, the map $(f,g) \longmapsto g \circ f$ is continuous in each argument separately, but, in general, it is not continuous in both variables. However,

Proposition 5.5.5. *Let X, Z be two Hausdorff spaces and Y be a locally compact space. Then the map $\mathcal{C}(X,Y) \times \mathcal{C}(Y,Z) \longrightarrow \mathcal{C}(X,Z),\ (f,g) \longmapsto g \circ f$ is continuous.*

Proof: ■

Corollary 5.5.6. *Suppose Y is a locally compact space. Then the map $\mathcal{C}(Y,Z) \times Y \longrightarrow Z$ given by $(f,y) \longmapsto f(y)$ is continuous.*

Proof: The map $(f,y) \longmapsto f(y)$ is precisely the composition map $\mathcal{C}(X,Y) \times \mathcal{C}(Y,Z) \longrightarrow \mathcal{C}(X,Z)$ whenever X is a one-point set. Hence, the result follows from the previous one. ■

The above map $\mathcal{C}(Y,Z) \times Y \longrightarrow Z$ given by $(f,y) \longmapsto f(y)$ is called the **evaluation map** of $\mathcal{C}(Y,Z)$.
Given three topological spaces X, Y, Z, a map $\varphi : X \times Y \longrightarrow Z$ can be also regarded as a family of maps $\Phi_x : Y \longrightarrow Z$ with X as a parameter space, i.e. $\Phi : X \longrightarrow \mathcal{C}(Y,Z)$.

Proposition 5.5.7.
1. *If φ is continuous, then Φ is continuous.*
2. *If Φ is continuous and if Y is locally compact, then φ is continuous.*

Proof: ■

5.6 Compactification

5.6.1 Generalities

Let look at an elementary example. The space $\mathbb{R}$, with the standard topology, is no compact. But it can be embedded in a compact space by at least two distinct methods:

1. Identifying $\mathbb{R}$ with $]-1,1[\subset [-1,1]$ by the map $x \mapsto \dfrac{x}{(1+|x|)}$.
2. Identifying $\mathbb{R}$ with $S^1 \setminus \{\text{north pole}\}$ by stereographic projection.

The first process can be regarded as compactifying $\mathbb{R}$ by the addition of two new points, whereas the second does so by adding only one.

Definition 5.6.1. *A* **compactification** *of a non compact topological space X is a pair $(\widehat{X}, h)$ consisting of a compact space $\widehat{X}$ and a homeomorphism h from X onto a dense subset of $\widehat{X}$.*

There are several ways to compactify some topological spaces as we shown in the previous example; however, we will introduce the easiest method which is suitable for the locally compact spaces.

5.6.2 One-point Compactification

Let X be a compact topological space and let $x \in X$. Then $Y = X \setminus \{x\}$ is locally compact. Starting from a locally compact space, is it possible to construct a compact space by adding one point? This process will be called **one-point compactification** or **Alexandroff** [9]**compactification**. More precisely, let (X, τ) be a locally compact topological space and let $\widehat{X} = X \cup \{\omega\}$ be the set obtained by adding one point $\omega \notin X$.
Define the topology $\widehat{\tau}$ on $\widehat{X}$ as follows:

$$O \in \widehat{\tau} \text{ if either } O \in \tau \text{ or if } O = (X \setminus K) \cup \{\omega\} \text{ where } K \text{ is a compact set of } X$$

The point ω is called the point at the infinity and it is also denoted ∞.

Theorem 5.6.2. *$(\widehat{X}, \widehat{\tau})$ is a compact topological space and (X, τ) is a subspace of $(\widehat{X}, \widehat{\tau})$.*

Proof:
Show that $\widehat{\tau}$ is a topology.

1. $\emptyset$ and $\widehat{X} \in \widehat{\tau}$ is trivial.
2. Let $(O_i)_{i \in I}$ be a family of sets of $\widehat{\tau}$ and let $O = \bigcup_{i \in I} O_i$. Let $I = J \cup J'$ where
   - for $i \in J$, $O_i \in \tau$
   - for $i \in J'$, $O_i = (X \setminus K_i) \cup \{\infty\}$ with K_i compact of X.

   (a) If $J' = \emptyset$ then $O \in \tau$.

   (b) Suppose $J' \neq \emptyset$, then $\infty \in O$ and
   $$\widehat{X} \setminus O = \bigcap_{i \in I} (\widehat{X} \setminus O_i) = \left(\bigcap_{i \in J'} K_i \right) \cap \left(\bigcap_{i \in J} (X \setminus O_i) \right)$$
   But $\left(\bigcap_{i \in J'} K_i \right)$ is compact in X and $\bigcap_{i \in J} (X \setminus O_i)$ is closed in X, so $\widehat{X} \setminus O$ is a compact K of X, and $O = (X \setminus K) \cup \{\infty\}$.
3. Let O_1 and O_2 be two open sets of $\widehat{X}$. Show that $O_1 \cap O_2 \in \widehat{\tau}$.

   (a) $O_1 \cap O_2 \in \tau'$ if O_1 and O_2 be two open sets of X.

   (b) If $O_1 \in \tau$ and $O_2 = (X \setminus K) \cup \{\infty\}$ with K compact of X, then $O_1 \cap O_2 = O_1 \cap (X \setminus K)$ and $X \setminus K \in \tau$ so $O_1 \cap O_2 \in \widehat{\tau}$.

   (c) If $O_1 = (X \setminus K_1) \cup \{\infty\}$ and $O_2 = (X \setminus K_2) \cup \{\infty\}$, with K_1 and K_2 compact sets in X, then $O_1 \cap O_2 = \big(X \setminus (K_1 \cup K_2)\big) \cup \{\infty\}$ and $K_1 \cup K_2$ is a compact of X.

Show that $\widehat{X}$ is compact.
Let $(O_i)_{i \in I}$ be an open cover of $\widehat{X}$. There exists some $i_0 \in I$ such that $\infty \in O_{i_0}$. Then $O_{i_0} = (X \setminus K) \cup \{\infty\}$ where K is compact in X. Moreover, there exists a finite set J of I such

[9] Pavel Sergeyevich Alexandrov (sometimes romanized Alexandroff or Alexandrov) (November 16, 1896 - May 7, 1982) was a Soviet Russian mathematician. He wrote about three hundred papers, making important contributions to set theory and topology.

that $(O_i)_{i\in J}$ is a finite open cover of K. Then $\widehat{X} = O_{i_0} \cup \left(\bigcup_{i\in J} O_i\right)$.

Finally, any open set of X is an open set of $\widehat{X}$. The intersections with X of open sets of $\widehat{X}$ are open sets of X. So the induced topology on X by the topology $\widehat{\tau}$ is the topology τ. ■

Corollary 5.6.3. *Let X be a locally compact space which is not compact. Then $(\widehat{X}, \widehat{\tau})$ is a compactification of X.*

Proof: It remains to prove that X is dense into $\widehat{X}$. Assume now that X is not compact. Let O be a neighbourhood of ∞. Since X is not compact, in particular $X \neq K$, for some compact subset K, there must be some point(s) $x \in X \setminus K$. That is, the neighbourhood O of ∞ contains point(s) of X. ■

Example 5.6.4. *1. If $X =]0, 1] = \{x \in \mathbb{R} \mid 0 < x \leq 1\}$, then $\widehat{X} = [0, 1]$.*

2. *$X = \mathbb{R}$ with the standard topology τ. Then $\widehat{X} = X \cup \{\infty\}$ with the topology $\widehat{\tau}$, is homeomorphic to the circle S^1 (as subspace of the Euclidean real plane $\mathbb{R}^2$). (Hint: $\widehat{X} \cong]0, 1[\cup \{\infty\}$).*

3. *Let $X = \mathbb{R}^2$ be the Euclidean real plane with the topology τ. $\widehat{X} = \mathbb{R}^2 \cup \{\infty\}$ with the topology $\widehat{\tau}$, is homeomorphic to the sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. (cf. 4.4.1.2)*
   $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$.

$$\begin{array}{rcl} f : \mathbb{R}^2 \cup \{\infty\} & \longrightarrow & \mathbb{S}^2 \\ \infty & \longmapsto & (0, 0, 1) \\ (x, y) & \longmapsto & p_{(x,y)} \end{array}$$

   where $p_{(x,y)}$ is the intersection point of the line joining the points $(0, 0, 1)$ and $(x, y, 0)$ with the sphere $\mathbb{S}^2$, (stereographic projection). Then the map f is a homeomorphism (exercise).

4. *Let X be a locally compact space which is not compact. Let $f : X \longrightarrow \mathbb{C}$ be a map. Let $\mathcal{B}$ be the filterbase of all the sets $X \setminus K$ where K is compact in X. Then the followings are equivalent*

   - *f converges to 0 w.r.t. $\mathcal{B}$,*
   - *if $\widehat{f}$ is the map which extends f to $\widehat{X}$ and $\widehat{f}(\infty) = 0$, we have $\lim_{x\to\infty} \widehat{f}(x) = 0$.*

   Then we say that f converges to 0 at infinity.

5.6.3 Exercises

1. Suppose that X is a locally compact Hausdorff space. Then prove that $\widehat{X}$ is Hausdorff.

2. Let X be a compact space so it is locally compact. Describe the one-point compactification of X. (Show that it is the topological sum of X with the one-point set $\{\infty\}$).

3. If X is an infinite set with the discrete topology, then every neighbourhood of ∞ in $\widehat{X}$ contains all but a finite number of points of X.

4. Give explicit description of the one-point compactification of the set $\{(x, y) \in \mathbb{R}^2 \mid 1 < x^2 + y^2 < 2\}$.

5. Let X, Y be two Hausdorff spaces. Any map $f : X \longrightarrow Y$ extends to a map $f^* : X \cup \{\infty\} \longrightarrow Y \cup \{\infty\}$ by $f^*(\infty) = \infty$. Prove that f^* is continuous iff f is continuous and each compact subset of Y has a compact preimage. (Such a map f is called *proper*).

6. Characterize the closed sets of $\widehat{X} = X \cup \{\infty\}$, the one-point compactification of X.

7. Define $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ by adjoining two elements to the real line. Let $x, y \in \overline{\mathbb{R}}$ and define the following linear order $\leq$:

   (a) if $x, y \in \mathbb{R}$, then $x \leq y$ with the usual meaning in $\mathbb{R}$.

   (b) For any $x \in \mathbb{R}$, let $x < +\infty, -\infty < x$.

   (c) Assume $-\infty < +\infty$.

   Then we get a linear order on $\overline{\mathbb{R}}$. Let f be the map from $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ in $\overline{\mathbb{R}}$ as follows :

$$f(x) = \tan x \quad \text{if} \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$
$$f\left(\frac{\pi}{2}\right) = +\infty \quad \text{and} \quad f\left(-\frac{\pi}{2}\right) = -\infty$$

   Show that f is bijective, $x \leq y \Longrightarrow f(x) \leq f(y)$ for any $x, y \in \overline{\mathbb{R}}$.
   Define the topology on $\overline{\mathbb{R}}$ from the induced topology on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ via the bijection f. Deduce that $\overline{\mathbb{R}}$ is compact and show that the induced topology on $\mathbb{R}$ is the usual one.

5.7 Paracompactness

Recall that a **cover** of a space X is a family $\mathcal{A} = \{A_i\}_{i \in I}$ of subsets of X such that $X = \bigcup_{i \in I} A_i$. The cover is **open** if all A_i are open.
A **subcover** of the cover $\mathcal{A}$ is a subset of $\mathcal{A}$ that still covers X.
A family $\{A_i\}_{i \in I}$ of subsets of a space X is called **neighbourhood-finite**, (shortly written **nbd-finite** (also called **locally finite**) if each point of X has a neighbourhood V such that $V \cap A_i \neq \emptyset$ for at most finitely many $i \in I$.

Example 5.7.1. *The cover of the space* $\mathbb{R}$ *by the open intervals* $]n-1, n+1[, n \in \mathbb{Z}$ *is locally finite.*
The cover of the space $]-1, +1[$ *by the open intervals* $\left]-\frac{1}{n}, +\frac{1}{n}\right[$ *fails to be locally finite because of the point* 0.

A family $\{A_i\}_{i \in I}$ of subsets of a space X is called **point-finite cover** if for any point $x \in X$, there is at most finitely many $i \in I$ such that $x \in A_i$. Notice that a point-finite cover need not be nbd-finite.
Let $\{A_i\}_{i \in I}$ and $\{B_j\}_{j \in J}$ be two covers of the space X. The cover $\{A_i\}_{i \in I}$ is a **refinement** of the cover $\{B_j\}_{j \in J}$ if for any A_i, there is B_j such that $A_i \subseteq B_j$.
Recall that a space X is compact iff every open cover of X has a finite open subcover. For a given open cover of X, let be a finite open refinement of the given cover, then each element of the refinement is a subset of an element of the cover and we have a finite subcover of X. Hence we can say that a space X is compact iff every open cover of X has a finite open refinement that covers X.

Definition 5.7.2. *A space X is* **paracompact** *if every open cover has a locally finite open refinement that covers X.*

Remark 5.7.3. *It is often required that a paracompact space has to be Hausdorff.*

There is a similarity between compactness and paracompactness. For compactness, we need "subcover", resp. "finite" and for paracompactness, we need "open refinement", resp. "locally finite". If we change any one of these assumptions in the definition of a paracompact space, the space becomes compact. For example, we will see that the space $\mathbb{R}$ is paracompact. The space $\mathbb{R}$ has the open cover $\{A_n\}_{n\in\mathbb{N}_{>0}}$, where $A_n =]-\infty, n[$. Any subfamily of A_n's that covers $\mathbb{R}$ has to be infinite since each A_n is bounded on the right. So, this open cover of the paracompact space $\mathbb{R}$ does not admit locally finite subcovers even if it has locally finite refinements.

Example 5.7.4.
- *Every compact space is paracompact, since a finite open cover is trivially a locally finite open cover.*
- *A discrete space is paracompact since the open cover of all sets $\{x\}, x \in X$is locally finite and a refinement of any cover of X.*
- *A metrizable space is paracompact.*
- *The space $\mathbb{R}^p$, $p \in \mathbb{N}_{>0}$, is paracompact (but we already know that it is not compact).*
  Let $\{A_i\}_{i\in I}$ be an open cover of $\mathbb{R}^p$. We have to find a locally finite refinement of $\{A_i\}_{i\in I}$ that covers $\mathbb{R}^p$. Consider the balls $B_n = B(o;n), n \in \mathbb{N}$ where $B_0 = \emptyset$. The closure $\mathrm{Cl}(B_n)$ is compact. Choose finitely many $A_i, i \in I_n$ that cover $\mathrm{Cl}(B_n)$ and such that $A_i \cap (\mathbb{R}^p \setminus \mathrm{Cl}(B_n) \neq \emptyset$. Denote $\mathcal{A}'_n$ the family of such open sets, each is an open subset of an A_k. So, $\bigcup_{n\in\mathbb{N}} \mathcal{A}'_n$ is an open refinement of $\{A_i\}_{i\in I}$.
  For any $x \in \mathbb{R}^p$, there is a smallest $n \in \mathbb{N}$ such that $x \in \mathrm{Cl}(B_n)$, $(|x| \leq n < |x|+1)$ and x belongs to an element of $\mathcal{A}'_n$.
  The family $\{\mathcal{A}'_n\}$ is locally finite since for given $x \in \mathbb{R}^p$, the ball B_n intersects only finitely many $\mathcal{A}'_m$, i.e. those in $\mathcal{A}'_1 \cup \mathcal{A}'_2 \cup \cdots \cup \mathcal{A}'_n$. So, the family $\{\mathcal{A}'_m\}_{m\in\mathbb{N}}$ is a locally open refinement of $\{A_i\}_{i\in I}$ that covers $\mathbb{R}^p$ and hence $\mathbb{R}^p$ is paracompact.

Properties of Paracompact Spaces

Proposition 5.7.5. *A space homeomorphic to a paracompact space is paracompact.*

Proof: (Exercise). ■

So, paracompactness is a topological property.

Proposition 5.7.6. *A closed subset of a paracompact space is paracompact.*

Proof: Let A be a closed set of the paracompact space X, and $\{A_i\}$ an open cover of A. Then $A_i = O_i \cap A$, where O_i is an open set of X. Consider the cover of X formed with $X \setminus A$ and the A_i's. There exists an open locally finite refinement of the cover of X and the intersections with A form a locally finite refinement of the given cover of A. ■

Remark 5.7.7. *A paracompact subspace need not to be closed.*
Consider the subspace $]0,1[\subset \mathbb{R}$. $]0,1[$ is homeomorphic to $\mathbb{R}$, so it is paracompact but it is not closed.

Proposition 5.7.8. *The product of a paracompact space and a compact space is paracompact.*

Proof: Let X be a paracompact space and Y a compact space, $\mathcal{A}$ an open cover of $X \times Y$. For any $(x, y) \in X \times Y$, there exists an open neighbourhood $V(x, y)$ of x in X, and a neighbourhood $W(x, y)$ of y in Y such that $V(x, y) \times W(x, y) \subset A$ for $A \in \mathcal{A}$.
For any $x \in X$, the sets $W(x, y)$ where $y \in Y$ form an open cover of Y. Hence, there exists finitely many points $y_i, 1 \leq i \leq n(x)$ such that the $W(x, y_i)$ form an open cover of Y.
Set $U(x) = \bigcap_{i=1}^{n(x)} V(x, y_i)$. Each open set $U(x) \times W(x, y_i)$ is contained in a set of the cover $\mathcal{A}$. Let $\{B_i\}_{i \in I}$ be an open locally finite refinement of the cover formed by the $U(x)$. For any $i \in I$, let $x_i \in X$ such that $B_i \subset U(x_i)$, and denote $S_{i,k}$ the sets $W(x_i, y_k), 1 \leq k \leq n(x_i)$. Then the sets $B_i \times S_{i,k}$ form an open refinement of the cover $\mathcal{A}$.
This cover is locally finite, because for any (x, y), there exists a neighbourhood N of x meeting finitely man sets B_i. Then the neighbourhood $N \times Y$ of (x, y) intersects finetely many sets $B_i \times S_{i,k}$.. ■

Proposition 5.7.9. *The sum of paracompact spaces is paracompact.*

Proof: Let $\{A_i\}_{i \in I}$ be an open cover of the space $X = \coprod_{j \in J} X_j$. The cover $\{A_i \cap B_j\}_{i,j}$ is a refinement of $\{A_i\}_{i \in I}$. If for any $j \in J, \{B_{i,j}\}_{i \in I_j}$ is an open refinement of the cover $\{A_i \cap X_J\}_{i \in I}$ of X_j, the open cover of the space X formed of the B_{ij}'s, $j \in J, i \in I_j$ is a locally finite refinement of $\{A_i\}_{i \in I}$. ■

5.8 First and Second-Countable Spaces

Definition 5.8.1. *A Hausdorff space is* **second-countable** *if it has a countable basis.*

Proposition 5.8.2.
1. *Second-countability is invariant under continuous open surjections.*
2. *Every subspace of a second-countable space is second-countable.*
3. *A product $\prod_{i \in I} X_i$ is second-countable iff each X_i is second-countable and $\aleph(I) \leq \aleph_0$.*

Proof:

1. Let $\{O_n\}_{n \in \mathbb{Z}}$ be a basis for the space X and $p : X \longrightarrow Y$ be a continuous open surjection. Then $\{p(O_n)\}_{n \in \mathbb{Z}}$ is a basis for Y.
2. It's trivial.
3. Second-countability of $\prod_{i \in I} X_i$ implies second-countability of each X_i.
   Assume that each X_i is second-countable. The cardinal of a basis for $\prod_{i \in I} X_i$ is $\aleph_0.\aleph(I)$. Then, second countability occurs iff $\aleph(I) \leq \aleph_0$.

■

Definition 5.8.3. *A Hausdorff space is* **Lindelöf** *if each open cover contains a countable subcover.*

Theorem 5.8.4 (K. Morita[10]). *In Lindelöf spaces, regularity and paracompactness are equivalent.*

[10]Kiiti Morita (February 11, 1915 - August 4, 1995) was a Japanese mathematician working in algebra and topology.

Proof: The result follows from E. Michael (A note on Paracompact spaces. Proceedings of the American Mathematical Society. 4 (5): 831 - 838.). ■

Lemma 5.8.5. *A second-countable Hausdorff and locally compact space admits a countable basis of open sets with compact closure.*

Proof: Let $\{O_n\}_{n\in\mathbb{N}}$ be a countable basis of open sets in the space X. For any $x \in X$, there exists an open set U_x containing x with compact closure and some $O_{n(x)}$ contains x and is contained in U_x. The closure of $O_{n(x)}$ is a closed subset of the compact $\mathrm{Cl}(U_x)$ and so, $\mathrm{Cl}(O_{n(x)})$ is also compact. Thus, the O_n's with compact closure are a countable basis of open sets with compact closure. ■

Proposition 5.8.6. *Any second-countable Hausdorff and locally compact space X is paracompact.*

Proof: Let $\{O_n\}_{n\in\mathbb{N}}$ be a countable basis of open sets in the space X, and $\{U_i\}_{i\in I}$ an open cover of X. Let $x \in X$, then $x \in U_i$ for some $i \in I$ and so, there exist a $O_{n(x)}$ such that $x \in O_{n(x)}$ and $V_{n(x)} \subseteq U_i$. Thus $\{O_{n(x)}\}$ form a countable refinement of $\{U_i\}_{i\in I}$.
By the previous lemma, we can assume that all $\mathrm{Cl}(O_n)$ are compact. Therefore, we can consider countable covers of open sets with compact closures.
Closure commutes with finite unions, replacing U_n by $\cup_{j\leq n} U_j$, the compactness condition is preserved and (U_n) is an increasing set of open sets with compact closures. For m sufficiently large, we have $\mathrm{Cl}(U_n) \subseteq U_m$. Replace, for each n, U_{n+1} by a U_m, we get $\mathrm{Cl}(U_n) \subseteq U_{n+1}$ for each n.
Let $K_0 = \mathrm{Cl}(U_0)$ and for $n \geq 1$, let $K_n = \mathrm{Cl}(U_n) \setminus U_{n-1} = \mathrm{Cl}(U_n) \cap (X \setminus U_{n-1})$, so K_n is compact for every n and for any fixed m, we have $U_m \cap K_n = \emptyset$ for all $n > m$.
For $n > 1$, the open set $W_n = U_{n+1} \setminus \mathrm{Cl}(U_{n-2})$ contains K_n, so for each $x \in K_n$, there exists some $O_m \subseteq W_n$ such that $x \in O_m$. There are finitely many such O_m's that cover the compact K_n and $\{O_m\}$ is a locally finite family of open sets in X whose union contains $X \setminus \mathrm{Cl}(U_0)$.
Consider finitely many O_m's contained in U_1 that cover the compact $\mathrm{Cl}(U_0)$, we get an open and locally finite cover of X that refines $\{U_i\}$. ■

Definition 5.8.7. *A space X is* **first-countable** *if with each $x \in X$, there is an at most countable family $\{O_n(x)\}_{n\in\mathbb{Z}_{>0}}$ of neighbourhoods such that for each open set V with $x \in V$, there is some $O_n(x) \subset V$, i.e. if X has a countable basis at each point.*

Example 5.8.8. *Each metric space is first-countable.*
Second-countable spaces are always first-countable.

Proposition 5.8.9.

1. *A subspace of a first-countable space is first-countable.*
2. *A countable product of first-countable spaces is first-countable.*
3. *A subspace of a second-countable space is second-countable.*
4. *A countable product of second-countable spaces is second-countable.*

Proof: We give the proofs of 3 and 4.
3. If $\mathcal{B}$ is a countable basis for the space X, then $\{B \cap A \mid B \in \mathcal{B}\}$ is a countable basis for the subspace A of X.
4. If $\mathcal{B}_i$ is a countable basis for the space X_i, then the family of all products $\prod U_i$, where $U_i \in \mathcal{B}_i$ for finitely many values i and $X_i = U_i$ for all other values of i, is a countable basis for $\prod X_i$.
The proof for 1 and 2 is similar. ■

5.9 Partitions of Unity

Partitions of unity play an important role in topology, geometry and analysis. They allow to pass from "local to global".
A **partition of unity** in topology is a decomposition of the constant function on the topological space X into a sum of continuous functions

$$\sum_{i\in I} \eta_i = 1$$

and each η_i is concentrated on an "small" open set O_i where the O_i's form a cover of the space. In geometry, more is required and, for example, in differentiable geometry, the functions have to be differentiable.
If the family $\{O_i\}_{i\in I}$ is locally finite, then so is the family $\{\mathrm{Cl}(O_i)\}_{i\in I}$ and

$$\bigcup_{i\in I} \mathrm{Cl}(O_i) = Cl\left(\bigcup_{i\in I} O_i\right)$$

Let $f : X \longrightarrow \mathbb{R}$ be a function where X is a topological space.

Definition 5.9.1. *The* **support** *of the function f is defined by*

$$supp(f) = \mathrm{Cl}(\{x \in X \mid f(x) \neq 0\})$$

Notice that a point $x \notin \mathrm{supp}(f)$ iff x has a neighbourhood on which f vanishes identically.
We can give the precise definition of partition of unity.

Definition 5.9.2. *Let X be a topological space. A family $\{\eta_i\}_{i\in I}$ of continuous maps $\eta_i = X \longrightarrow \mathbb{R}_{\geq 0}$ is called* **partition of unity** *on X if*

1. *The family $\{supp(\eta_i)\}_{i\in I}$ form a locally finite cover of X.*
2. $\sum_{i\in I} \eta_i(x) = 1$ *for each $x \in X$.*

The sum in 2. is well-defined because each $x \in X$ lies in the support of at most finitely many η_i.

Definition 5.9.3. *A partition of unity $\{\eta_i\}_{i\in I}$ is* **subordinated** *to an open cover $\{U_i\}_{i\in I}$ of X if for each η_i, there is an open set U_i of the cover such that $supp(\eta_i) \subset U_i$.*

Each space has a partition of unity subordinated to the cover by the single set itself.
The space X admits partitions of unity iff for every open cover of X there is a partition of unity subordinate to the cover.

Proposition 5.9.4. *A Hausdorff space admits a partition of unity iff it is paracompact.*

The proposition can be written as follows: A Hausdorff space is paracompact iff for each cover, there is a partition of unity subordinated to the cover.
Proof:

■

5.10 Separability II

5.10.1 Separable Spaces

Definition 5.10.1. *A Hausdorff space is* **separable** *if it contains a countable dense subset.*

Some Invariance properties

1. The continuous image of a separable space is separable.
2. An open subspace of a separable space is separable.
3. If X is second-countable space, then every subspace is separable.
4. A product of space is separable iff each factor is separable.
5. Let X be a paracompact space. Then if X is separable, it is Lindelöf.

5.10.2 Hausdorff, Regular & Normal spaces

We already defined several kinds of separation. This section completes the study of separation for topological spaces obtained following the several constructions defined in this chapter.

Hausdorff Spaces

Recall that a space is Hausdorff if each two distinct points have disjoint neighbourhoods.
Some invariance properties of Hausdorff spaces:

1. Hausdorff is invariant under closed bijections.
2. A subspace of a Hausdorff space is Hausdorff.
   Let x, y be two distinct points of $Y \subset X$ and O_x, O_y be two disjoint neighbourhoods of x and y respectively. Then $O_x \cap Y$ and $O_y \cap Y$ are disjoint neighbourhoods of x and y in the subspace Y.
3. A product of spaces is Hausdorff iff each space is Hausdorff.
   Let $X = \prod_i X_i$ a product of Hausdorff spaces and $x = (x_i), y = (y_i)$ two distinct points of X. There exist i such that $x_i \neq y_i$. Let $O_{x,i}, O_{y,i}$ be two disjoint neighbourhoods of x_i and y_i respectively. Then $p_i^{-1}(O_{x,i})$ and $p_i^{-1}(O_{y,i})$ are disjoint open sets in X containing x and y respectively.

The next result is frequently used.

Proposition 5.10.2. *For any pair f, g of continuous mappings from a space X into a Hausdorff space Y, the set $\{x \in X \mid f(x) = g(x)\}$ is closed in X.*

Proof: It's equivalent to show that the set $A = \{x \in X \mid f(x) \neq g(x)\}$ is open in X.
For any $x \in A$, we have $f(x) \neq g(x)$. Hence, there exist two open sets O and O' in Y such that $f(x) \in O, g(x) \in O'$ and $O \cap O' = \emptyset$, the space Y being Hausdorff. The set $f^{-1}(O) \cap g^{-1}(O')$ is a neighbourhood of x and it is contained in A. So, A is open. ■

Regular Spaces

Recall that a space is regular if one-points sets are closed and if each point and each closed set not containing the point have disjoint neighbourhoods.
Some properties of regular spaces:

1. A subspace of a regular space is regular.
   Let Y be a subspace of the regular space X. Let $x \in Y$ and B be a closed subset of Y such that $x \notin B$. We have $\mathrm{Cl}(B) \cap Y = B$, where $\mathrm{Cl}(B)$ is the closure of B in X. Therefore, $x \notin \mathrm{Cl}(B)$, and by regularity of X, there exist two open sets $U, V \subset X$ with $x \in U, \mathrm{Cl}(B) \subset V$.

2. A product of spaces is regular iff each space is regular.
   Let $X = \prod_i X_i$ be a product of regular spaces. X is Hausdorff, so the one-point sets are closed in X. Let $x = (x_i) \in X$ and let O be a neighbourhood of x in X. Let $\prod_i O_i$ be a basis element about x contained in O. For each i, let V_i be a neighbourhood of x_i in X_i such that $\mathrm{Cl}(V_i) \subset O_i$. If $O_i = X_I$, take $V_i = X_i$. Then $V = \prod_i V_i$ is a neighbourhood of x in X. Since $\mathrm{Cl}(V) = \prod_i \mathrm{Cl}(V_i)$, it follows that $\mathrm{Cl}(V) \subset \prod_i O_i \subset U$, so X is regular.

3. Let X be regular and $p : X \longrightarrow X/\sim$ where $\sim$ is an equivalence relation. Suppose the map p closed and open. Then the quotient space $X/\sim$ is Hausdorff.

4. Let X be regular and $A \subset X$ be closed. Then X/A is Hausdorff.

Normal Spaces

Recall that a space is normal if one-points sets are closed and if each pair of disjoint closed sets have disjoint neighbourhoods.
Some invariance properties of normal spaces:

1. Normality is invariant under continuous closed surjections.

2. A closed subspace of a normal space is normal.

3. If a product of spaces is normal, then each space is normal.

Proposition 5.10.3. *A second-countable regular space is normal.*

Proof: Consider the space X that is regular with a countable basis $\mathcal{B}$.
Let C, C' be two disjoint closed subsets of the space X. For each $x \in C \subset X \setminus C'$, let O_x be a neighbourhood of x such that $\mathrm{Cl}(O_x) \subset X \setminus C'$, X being regular. Take some $B_x \in \mathcal{B}$ such that $x \in B_x \subset O_x$. The family of all such B_x covers C. This family of open sets can be indexed, $\{O_n\}$, whose closure $\mathrm{Cl}(O_n) \subset X \setminus C'$. Choose a countable family $\{V_n\}$ of open sets that covers C' such that $\mathrm{Cl}(V_n) \subset X \setminus C$ for all n.
Let $O'_n = O_n \setminus \bigcup_{i=A}^n \mathrm{Cl}(V_i)$ and $V'_n = V_n \setminus \bigcup_{i=1}^n \mathrm{Cl}(O_i)$. If $n \leq m$, then O'_n and V'_m are disjoint. Similarly, if $n \geq m$, then O'_n and V'_m are disjoint. Hence O'_n and V'_m are disjoint for all n, m.
The family of open sets $\{O'_n\}$ covers C.
The family of open sets $\{V'_n\}$ covers C'.
Then $O = \bigcup O'_n$ and $V = \bigcup V'_n$ are disjoint and they are neighbourhoods of C and C' respectively.

■

Proposition 5.10.4. *A compact Hausdorff space is normal.*

Corollary 5.10.5. *A locally compact Hausdorff space is regular.*

Corollary 5.10.6. *A second-countable locally compact Hausdorff space is normal.*

The family of normal spaces contains the second-countable regular spaces and the compact Hausdorff spaces. The next theorem gives a characterization of the normal spaces.

Theorem 5.10.7 (Urysohn[11]'s lemma). *A topological space (X, τ) is normal iff for every disjoint closed subsets C, C' of X, there exists a continuous function $f : X \longrightarrow [0,1]$ such that $f(x) = 0$ for $x \in C$ and $f(x) = 1$ for $x \in C'$.*

Proof: $\Longleftarrow$) Let C, C' be two disjoint closed subsets of X and a continuous function $f : X \longrightarrow [0,1]$ such that $f(x) = 0$ for $x \in C$ and $f(x) = 1$ for $x \in C'$. Then $C \subset f^{-1}\left([0, \frac{1}{2}[\right)$ and $C' \subset f^{-1}\left(]\frac{1}{2}, 1]\right)$. Then $f^{-1}\left([0, \frac{1}{2}[\right)$ and $f^{-1}\left(]\frac{1}{2}, 1]\right)$ are two disjoint sets that are open since f is continuous.
$\Longrightarrow$) Let C, C' be two disjoint closed subsets of the normal space X. We have to construct a family of open sets $\{O_q \mid q \in [0,1] \cap \mathbb{Q}\}$, and we denote such a q as q_n where $n \in \mathbb{N}$ with $q_0 = 1, q_1 = 0$, moreover the family of open sets must satisfy the property

$$q < q' \Longrightarrow \mathrm{Cl}(O_q) \subseteq O_{q'} \qquad (*)$$

Start with $O_1 = X \setminus C'$, and using normality of X, let O_0 be an open set such that

$$C \subseteq O_0 \subseteq \mathrm{Cl}(O_0) \subseteq O_1$$

Suppose we defined $O_{q_k}, k = 0, 1, 2, \ldots n$. Denote $Q_n = \{q_1, q_2, \ldots, q_n\}, r = q_{n+1}$.
By inductive hypothesis, we have $\mathrm{Cl}(O_{q_i}) \subseteq O_{q_j}$ where $q_i < r < q_j$ and there is no other elements of Q_n between q_i and q_j. Then, by mormality of X, we get an open set O_r such that

$$\mathrm{Cl}(O_{q_i}) \subseteq O_r \subseteq \mathrm{Cl}(O_r) \subseteq O_{q_j}$$

We have defined O_q for all $q \in [0,1] \cap \mathbb{Q}$. Extend to the whole set $\mathbb{Q}$ by setting $O_q = \emptyset$ for $q \in \mathbb{Q}_{<0}$ and $O_q = X$ if $q \in \mathbb{Q}_{>0}$.
For $x \in X$, define $Q_x = \{q \in \mathbb{Q} \mid x \in O_q\}$.
Q_x is bounded below by 0, if $q < 0, O_q = \emptyset$ and so, $x \notin O_q$.
$Q_x \neq \emptyset$, since $x \in X = O_q$ for all $q > 1$.
Hence, for every $x \in X, Q_x$ contains every rational number larger than 1, no rational number less than zero, and some rational numbers numbers in between.
So, for all $x \in X, Q_x$ has a greater lower bound. Define the function $f : X \longrightarrow [0,1]$ by $f(x) = \inf Q_x = \inf\{q \in \mathbb{Q} \mid x \in O_q\}$.

Let A, A' be two subsets of the topological space X and $f : X \longrightarrow [0,1]$ a continuous function such that $f(A) = \{0\}, f(A') = \{1\}$. Then the subsets A, A' are said to be **separated by a continuous function**.

- The function f separates C and C', i.e. $f(x) = 0$ for all $x \in C$ and $f(x) = 1$ for all $x \in C'$. Let $x \in C$, then $x \in C \subseteq O_0 \subseteq O_q$ for all $q \geq 0$. Therefore $Q_x = [0, \infty[\cap \mathbb{Q}$, and $f(x) = 0$. Let $x \in C'$, then $x \notin O_1$, and so, $x \notin O_1$ and $x \notin O_q$ for all $q \leq 1$. Therefore $Q_x =]1, \infty[\cap \mathbb{Q}$, and $f(x) = 1$.

[11] Pavel Samuilovich Urysohn (February 3, 1898 - August 17, 1924) was a Soviet mathematician who is best known for his contributions in dimension theory, and for developing Urysohn's Metrization Theorem and Urysohn's Lemma, both of which are fundamental results in topology. His name is also commemorated in the terms Urysohn universal space, Fréchet-Urysohn space, Menger-Urysohn dimension and Urysohn integral equation. He and Pavel Alexandrov formulated the modern definition of compactness in 1923.

- Prove that if $x \in \mathrm{Cl}(O_q)$, then $f(x) \leq q$.
  Suppose $x \in \mathrm{Cl}(O_q)$. Then $x \in \mathrm{Cl}(O_q) \subseteq O_{q'}$ for some $q' > q$. Therefore $]q, \infty[\cap \mathbb{Q} \subset Q_x$ and $\inf Q_x \leq q$.
  Prove that if $x \notin O_q$, then $f(x) \geq q$.
  Suppose $x \notin O_q$, then $x \notin O_{q'}$ for any $q' > q$, i.e. $]-\infty, q] \cap Q_x = \emptyset$, hence $\inf Q_x \geq q$.
- Prove that f is continuous.
  Let $O =]a, b[\in \mathbb{R}$, open interval that intersects $[0, 1]$. Show that $f^{-1}(O)$ is open in X.
  Let $x \in f^{-1}(O)$. We have to find an open set V of X such that $x \in V \subseteq f^{-1}(O)$, i.e. $f(x) \in f(V) \subseteq O$. Then $f(x) \in]a, b[$, and there exists rational numbers q, q' such that

  $$a < q < f(x) < q' < b$$

  Since $q < f(x)$, we have $x \notin \mathrm{Cl}(O_q)$. Since $f(x) < q'$, we have $x \in O_{q'}$. Hence, $x \in O_{q'} \setminus \mathrm{Cl}(O_q)$. It will be the open set V.
  Let us show that $f(V) \subseteq]a, b[$.
  Let $y \in V$, then $y \in O_{q'} \subseteq \mathrm{Cl}(O_{q'})$, and $f(y) \leq q' < b$. Since $y \notin \mathrm{Cl}(O_q) \supseteq O_q$, we have $f(y) \geq q > a$. Therefore

  $$f(y) \in [q, q'] \subseteq]a, b[$$

  Thus, Uryshon's lemma is proved.

■

The normal spaces are so-called because they have nice properties. We already mentioned some of them and the next theorem which follows Uryshon's lemma is one such properties of normal spaces.

Theorem 5.10.8 (Tietze). *Let X be a normal space and Y a closed subset of X. Then any continuous function $f : Y \longrightarrow [0, 1]$ has an extension to X, i.e. there is a continuous function $g : X \longrightarrow [0, 1]$ such that $g_{|Y} = f$.*

5.10.3 Tychonoff Spaces

Definition 5.10.9. *A space X is* **completely regular** *if one-points sets are closed in X and if for each point x_0 and each closed set C not containing x_0, there is a continuous map $f : X \longrightarrow I$ such that $f(x_0) = 1$ and $f(C) = 0$.*
The map f is said to ***separate points and closed sets****.*

By the Uryshon lemma, a normal space is completely regular.
A completely regular space is regular, since given a map f, the sets $f^{-1}\left([0, \frac{1}{2}[\right)$ and $f^{-1}\left(]\frac{1}{2}, 1]\right)$ are disjoint open sets about C and x_0, respectively.
Complete regularity fits between regularity and normality. So, complete regularity is called $T_{3\frac{1}{2}}$.

Remark 5.10.10. *The conditions $T_i, 1 \leq i \leq 3$ need only set-theoretic notions and open set notion. The condition $T_{3\frac{1}{2}}$ needs the notion of continuous real-valued maps.*

Example 5.10.11. *A metric space is completely regular.*

Proposition 5.10.12. *1. A subspace of a completely regular space is completely regular.*

2. A product of completely regular spaces is completely regular.

Proof:

1. Let X be a completely regular space, $Y \subset X, x_0 \in Y$ and C a closed set of Y with $x_0 \notin A$. Then $C = \mathrm{Cl}(C) \cap Y$, $\mathrm{Cl}(C)$ is the closure of C in X. Therefore $x_0 \notin \mathrm{Cl}(C)$. We can choose a continuous map $f : X \longrightarrow I$ such that $f(x_0) = 1$ and $f(\mathrm{Cl}(C)) = \{0\}$. Hence, $f_{|Y}$ is the continuous map on Y satisfying the desired conditions.

2. Let $X = \prod_{\alpha \in A} X_\alpha$ be a product of completely regular spaces. Let $\mathbf{x} = (x_\alpha) \in X$ and C be a closed set of X such that $\mathbf{x} \notin C$. Take a basis element $\prod_{\alpha} O_\alpha$ containing $\mathbf{x}$ that does not intersect C. Then $O_\alpha = X_\alpha$ but for finitely many α, say $\alpha_1, \ldots, \alpha_n$. Choose the continuous maps $f_i : X_{\alpha_i} \longrightarrow I$ such that $f_i(x_{\alpha_i}) = 1$ and $f_i(X \setminus O_{\alpha_i}) = \{0\}$. Let $\phi_i(\mathbf{x}) = f_i(\pi_{\alpha_i}(\mathbf{x}))$. Then ϕ_i maps X continuouslly into $\mathbb{R}$ and vanishes outside $\pi_{\alpha_i}^{-1}(O_{\alpha_i})$. The map f such that $f(\mathbf{x}) = \phi_1(\mathbf{x}).\phi_2(\mathbf{x}) \cdots .\phi_n(\mathbf{x})$ is the desired continuous map on X.

■

Definition 5.10.13. *A completely regular* $\mathbf{T}_1$*-space is said to be a* **Tychonoff space**.

Some properties

- A product of Tychonoff spaces is Tychonoff space.
- A subspace of a Tychonoff space is a Tychonoff space.
- A Tychonoff space is Hausdorff.

5.10.4 Exercises

1. Let $X = (\mathbb{R}, \tau_\ell)$ and $Y = (\mathbb{R}, \tau_K)$ be two topological spaces (cf. Ex. 2.4.1.3). Show that
   (a) X is normal, X^2 is regular and not normal.
   (b) Y is Hausdorff and not regular.

2. A Hausdorff space X is **completely regular** if for each point $x \in X$ and closed set A not containing x, there is a continuous function $\varphi : X \longrightarrow [0,1]$ such that $\varphi(x) = 1$ and $\varphi(y) = 0$ for any $y \in A$.
   (a) Show that every subspace of a completely regular space is completely regular.
   (b) Show that a product of spaces is completely regular iff each factor is completely regular.

5.11 Metrization of Topological Spaces

A topological space is called **metrizable** if its topology is induced by a metric. The metric spaces have many nice properties, so, given a topological space, it is important to know if the topology is induced by a metric, i.e. if the space is metrizable.
Under the assumptions that the topological space X is regular and has a countable basis, we show that X can be embedded in a metric space. This metric space is a "cube". Therefore, X is homeomorphic to a subspace of a metric space. A subspace of a metric space is metrizable, and metrizability is a topological property, so X is metrizable.

5.11.1 Some properties of Metric Spaces

Proposition 5.11.1. *1. Metrizability is a topological property.*

2. A subspace of a metric space is a metric space.

3. (**A.H. Stone**)[12] *A metric space is paracompact.*

4. A metric space is first-countable.

Proof:

1. Let $f : X \longrightarrow Y$ be a homeomorphism from the topological space X onto the metric space Y. Then the map d defined by $d_X(x, x') = d_Y(f(x), f(x'))$ is clearly a distance on X.

2. The proof of this item is clear.

3. Let $\{O_s\}_{s\in S}$ be an open cover of the metric space (X, d) where $<$ is a well-ordered relation on the set S. A well-order (or well-ordering or well-order relation) on a set S is a total order on S with the property that every non-empty subset of S has a least element in this ordering.
   For each $n \in \mathbb{N}_{>0}$, define $D_{s,n}$ to be the union of all spheres $S(x; 2^{-n})$ such that:
   (a) s is the smallest element of S with $x \in O_s$.
   (b) $x \notin D_{s',j}$ if $j < n$.
   (c) $S(x; 3.2^{-n}) \subset O_s$.

   Then $\{D_{s,n}\}$ is a locally refinement of $\{O_s\}$ which covers X, hence X is paracompact. (exercise)

4. The set of open balls centered at x with radius $1/n$ for integers $n > 0$ form a countable local base at x.

■

5.11.2 Embedding in Cubes

Definition 5.11.2. *Let A be a set and for each $\alpha \in A$, let (X_α, τ_α) be a topological space homeomorphic to the interval $[0, 1]$ with the standard topology. Then the product space $\prod_{\alpha\in A} X_\alpha$ is denoted I^A and called a* **cube***.*

Suppose $\text{card}(A) = \aleph_0$, then a point of I^A is a sequence $\mathbf{x} = (x_1, x_2, \ldots, x_n, \ldots)$ where $x_i \in I = [0, 1]$. We define two topologies on the cube I^A as follows:

1. The product topology where the topology on I is the standard topology.

2. A metric topology where $d(\mathbf{x}, \mathbf{y}) = \displaystyle\sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i}$.
   d is a metric for I^A: We have to prove the triangle inequality. Let $\mathbf{x}, \mathbf{y}$ and $\mathbf{z}$ be three points of I^A. Let $n > 0$. For $1 \le i \le n$, we have $|x_i - z_i| \le |x_i - y_i| + |y_i - z_i|$, so $\displaystyle\sum_{i=1}^{n} \frac{|x_i - z_i|}{2^i} \le \sum_{i=1}^{n} \frac{|x_i - y_i|}{2^i} + \sum_{i=1}^{n} \frac{|y_i - z_i|}{2^i} \le d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z})$, and the result follows.

[12] Arthur Harold Stone (30 September 1916 - 6 August 2000) was a British mathematician born in London, who worked mostly in topology.

Proposition 5.11.3. *The two topologies on the cube I^A are equivalent.*

Proof: Let $O = \prod_{i>0} O_i$, where $O_i = I$ for $i > n$, be an element of the basis for the product topology. and $\mathbf{x} \in O$. For each $i \leq n$, let $r_i > 0$ such that if $t \in I$ and $|t - x_i| < r_i$, then $t \in O_i$. Let $r = \inf\left\{\frac{r_1}{2}, \frac{r_2}{4}, \ldots, \frac{r_n}{2^n}\right\}$. Take $\mathbf{y} \in I^A \setminus O$, there exists an integer $j \leq n$ such that $y_j \notin O_j$. Then $|y_j - x_j| \geq r_j$, so $\frac{|y_j - x_j|}{2^j} \geq r$ and we have $d(\mathbf{x}, \mathbf{y}) \geq r$. Hence, $B(\mathbf{x}, r) \subseteq O$ and an open set for the product topology is open for the metric topology.
Conversely, let $\mathbf{x} \in I^A$ and $r > 0$. There exists a positive integer n such that $\sum_{i>n} 2^{-i} < \frac{r}{2}$. For each positive integer $i \leq n$, let $O_i = I \cap \left]x_i - \frac{r}{2}, x_i + \frac{r}{2}\right[$ and $O_i = I$ for $i > n$. Let $O = \prod_{i>0} O_i$. If $\mathbf{y} \in O, d(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} \frac{|x_i - y_i|}{2^i} + \sum_{i>n} \frac{|x_i - y_i|}{2^i} < \sum_{i=1}^{n} \frac{(\frac{r}{2})}{2^i} + \frac{r}{2} < r$. Thus $O \subseteq B(\mathbf{x}, r)$. ■

Proposition 5.11.4. *The cube I^A is a compact space.*

Proof: A product of compact spaces is compact, and the space I is compact. ■

Theorem 5.11.5 (Embedding Lemma). *Let X and $\{Y_\alpha\}_{\alpha \in A}$ be topological spaces. Let F be a family of maps $X \longrightarrow Y_\alpha, \alpha \in A$. Then*

1. *The evaluation map $e : X \longrightarrow \prod_{\alpha \in A} Y_\alpha,\ x \longmapsto (\pi_i(x))$ is a continuous map.*
2. *The evaluation map e is an open map onto its image if separates points and closed sets.*
3. *The evaluation map e is one-to-one iff separates points.*

Proof:

■

Theorem 5.11.6 (Embedding Theorem). *A topological space is Tychonoff iff it is homeomorphic to a subspace of a cube.*

Proof: The unit interval I is a Tychonoff space, and hence a cube, a product of Tychonoff spaces being Tychonoff space.
A subspace of a Tychonoff space is a Tychonoff space.
If X is a Tychonoff space and let F the family of all continuous maps from X to I, then, using the embedding lemma, the evaluation map is a homeomorphism form X into the cube I^F. ■

5.11.3 Metrization

The next lemma gives some properties of metrizable spaces.

Lemma 5.11.7. *Let X be a metrizable space and $\mathcal{A}$ an open cover of X. Then there is an open countably locally finite refinement of $\mathcal{A}$.*

Proof: Choose a well-order $<$ for the family $\mathcal{A}$. Choose a metric on X. Let $n \in \mathbb{N}_{>0}$ and $O \in \mathcal{A}$. Define the subset $S_n(O) \subset O$ as $\left\{x \in O \mid B\left(x, \frac{1}{n}\right) \subset O\right\}$ and define

$$T_n(O) = S_n(O) \setminus \bigcup_{O' < O} O'$$

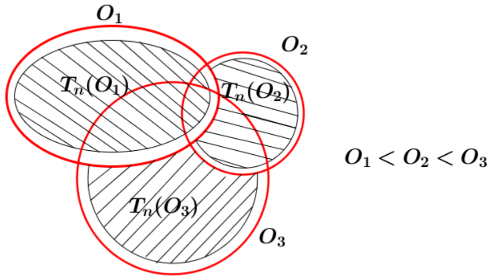


Figure 5.3

Let $O_1 < O_2 < O_3$ be three open sets of the cover, then the three sets $T_n(O_1), T_n(O_2)$ and $T_n(O_3)$ are disjoint. Let $x \in T_n(O_1)$ and $y \in T(O_2)$, then $d(x, y) \geq \frac{1}{n}$. To prove it, $x \in S_n(O_1)$, so $B(x, \frac{1}{n}) \subset O_1$. But $y \in T_n(O_2)$, so, by definition of $T_n, y \notin B\left(x, \frac{1}{n}\right)$.
It remains to modify these sets in order to make them open.
Let $E_n(O) = \left\{x \in T_n(O) \mid B\left(x, \frac{1}{3n}\right) \subset T_n(O)\right\}$. If we have $O_1 < O_2 < O_3$, then the three sets $E_n(O_1), E_n(O_2)$ and $E_n(O_3)$ are disjoint.
We have $d(x, y) \geq \frac{1}{3n}$ whenever $x \in E_n(O), y \in E_n(O')$ where $O < O'$.
The family $\{E_n(O) \mid O \in \mathcal{A}\}$ is a locally finite family of open sets that is a refinement of $\mathcal{A}$.
It is a locally finite family because for any $x \in X, B\left(x, \frac{1}{6n}\right) \cap E_n(O) \neq \emptyset$ for at most one O.
The family $\{E_n(O) \mid O \in \mathcal{A}, n \in \mathbb{N}_{>0}\}$ covers X.
Let $x \in X$ and choose the first (for $<$) $O \in \mathcal{A}$ containing x. The set O is open, so choose n such that $B\left(x, \frac{1}{n}\right) \subset O$. Hence, $x \in S_n(O)$ and $x \in T_n(O)$, so $x \in E_n(O)$. ■

Urysohn Metrization Theorem

The next theorem gives conditions under which a space is metrizable.

Theorem 5.11.8 (Urysohn Metrization). *Every regular space with a countable basis is metrizable.*

Proof: The proof consists to show that the space is homeomorphic to a subspace of a metrizable space.
Let $\mathcal{B} = \{B_i\}$ be a countable basis for the space X. For each pair $(B_m, B_n) \in \mathcal{B} \times \mathcal{B}$, such that $\mathrm{Cl}(B_m) \subset B_n$, the two closed subsets $\mathrm{Cl}(B_m)$ and $X \setminus B_n$ are disjoint, so, by Urysohn's lemma, there exists a function $f_{mn} : X \longrightarrow [0, 1]$ such that $f_{mn}(\mathrm{Cl}(B_m)) = \{1\}$ and $f_{mn}(X \setminus B_n) = \{0\}$.
Given $x \in X$, and an arbitrary open neighbourhood O_x of x, one can choose a basis element B_n containing x and contained in O_x. Using regularity of X, choose B_m so that $x \in B_m$ and $\mathrm{Cl}(B_m) \subset B_n$.
Then, there exists a function f_{mn} such that $f_{mn}(x) > 0$ and $f_{mn}(y) = 0$ for $y \notin O_x$.
The family $\{f_{mn}\}$ is indexed by a subset of $\mathbb{N}^2$, so we can reindexed so that to have the family $\{g_n\}_{n \in \mathbb{N}}$.
Then we define the map

$$\begin{aligned} G : X &\longrightarrow I^\infty \\ x &\longmapsto (g_1(x), g_2(x), \ldots) \end{aligned}$$

The map G is continuous because I^∞ has the product topology and each component g_n is continuous.

The map G is injective because given $x \neq y$, there is an index n such that $g_n(x) > 0$ and $g_n(y) = 0$. Therefore $G(x) \neq G(y)$.
The map G is a homeomorphism of X onto its image $G(X)$ in I^∞. We know that G defines a continuous bijection of X with $G(X)$. We have to show that for each open set O in X, the set $G(O)$ is open in $G(X)$. Let $z = G(x) \in G(X)$ such that the point $x \in O$. We have to find an open set contains in $G(O)$ and containing the point z. Choose $n \in \mathbb{N}$ such that $g_n(x) > 0$ and $g_n(X \setminus O) = \{0\}$.
Let $V = p_n^{-1}(]0, +\infty[)$ where p_n is the n^{th} projection. The set V is an open set of I^∞. Let $W = V \cap G(X)$. By definition of the subspace topology, W is open in $G(X)$.
We have to show that $z \in W \subset G(O)$.
- $z \in W$ because $p_n(z) = p_n(G(x)) = g_n(x) > 0$.
- $W \subset G(O)$. Let $u \in W$, then $u = G(v)$ for some $v \in X$ and $p_n(u) \in]0, +\infty[$. Since $p_n(u) = p_n(G(v)) = g_n(v)$ and g_n vanishes outside O, the point v must belong to O. Then $v = G(u) \in G(O)$.
Thus, G is an embedding of X in I^∞. ∎

This theorem which is a great step toward the metrization question does not give necessary and sufficient conditions, the countable basis condition is not sufficient.

Nagata-Smirnov Metrization Theorem

The next theorem gives necessary and sufficient conditions for a space to be metrizable.
We know that a countable intersection of open sets need not to be open, such sets occur frequently in analysis.
A subset of a space is called **G_δ-set** in X if it is the intersection of a countable family of open sets in the space.
In a metric space, each closed set is a G_δ-set. Let $A \subset X$ be a closed set of the metric space (X, d), then $A = \bigcap_{n \in \mathbb{N}_{>0}} \left\{ x \mid d(x, A) < \frac{1}{n} \right\}$.
Each open subset of a G_δ-set is a G_δ-set.

Lemma 5.11.9. *Let X be a regular space with a countable locally finite basis. Then X is normal, and every closed set in X is a G_δ-set in X.*

Proof: (to be done)

∎

Theorem 5.11.10 (Nagata[13]-Smirnov[14]). *A topological space is metrizable iff it is regular and has a basis that can be decomposed into an at most countable set of locally finite families.*

Proof: $\Longrightarrow$).
Assume the topological space X is regular with a countably locally finite basis $\mathcal{B}$. Then X is normal, and every closed set in X (to be completed)

∎

Remark 5.11.11. *Let (X, τ) be a metrizable topological space and $\sim$ an equivalence relation on X. Then the quotient space $(X/\sim, \tau_\sim)$ is not necessarily metrizable.*

[13] Masayoshi Nagata; February 9, 1927 - August 27, 2008) was a Japanese mathematician, known for his work in the field of commutative algebra.
[14] Vladimir Ivanovich Smirnov) (10 June 1887 - 11 February 1974) was a Russian mathematician who made significant contributions in both pure and applied mathematics, and also in the history of mathematics.

Chapter 6

Complete Metric Spaces

The complete metric spaces play a fundamental role in Analysis. They are at the origin of important theorems as, for example, fixed-point theorem in topology from which follow the theorem of local inversion in differential calculus and of Cauchy[1]-Lipschitz[2] in the theory of the differential equations), the theorem of Baire[3] (always in Topology) (and from which follow the theorem of Banach[4]-Steinhaus[5], ...).
In this chapter, we will only consider metric spaces.

6.1 Cauchy Sequences

Definition 6.1.1. *Let (X, d) be a metric space. The sequence $(x_1, x_2, x_3, \ldots)$ of points of X is said to be a* **Cauchy sequence**, *if for every positive real number $\varepsilon > 0$ there is a positive integer N such that for all natural numbers $m, n > N$, the distance $d(x_m, x_n) < \varepsilon$.*

Recall that the sequence $(x_i)_{i\in\mathbb{N}}$ of the topological space X converges to $x \in X$ if for any neighbourhood V_x of x, there exists $i_0 \in \mathbb{N}$ such that for any $i > i_0,\ x_i \in V_x$.
In the metric space (X, d), the sequence $(x_i)_{i\in\mathbb{N}}$ converges to $x \in X$ if for any $\varepsilon > 0$, there exists $i_0 \in \mathbb{N}$ such that for any $i > i_0,\ d(x_i, x) < \varepsilon$.
Roughly speaking, the terms of the sequence are getting closer and closer together in a way that suggests that the sequence ought to have a limit in X. Nevertheless, such a limit does not always exist within X.

[1] Augustin-Louis Cauchy (21 August 1789 - 23 May 1857) was a French mathematician who was an early pioneer of analysis. He started the project of formulating and proving the theorems of infinitesimal calculus in a rigorous manner. He also gave several important theorems in complex analysis and initiated the study of permutation groups in abstract algebra. A profound mathematician, Cauchy exercised a great influence over his contemporaries and successors. His writings cover the entire range of mathematics and mathematical physics.

[2] Rudolf Otto Sigismund Lipschitz (14 May 1832 - 7 October 1903) was a German mathematician and professor at the University of Bonn from 1864. Peter Gustav Dirichlet was his teacher. He supervised the early work of Felix Klein. While Lipschitz gave his name to the Lipschitz continuity condition, he worked in a broad range of areas. These included number theory, algebras with involution, mathematical analysis, differential geometry and classical mechanics.

[3] René-Louis Baire (21 January 1874 - 5 July 1932) was a French mathematician.

[4] Stefan Banach (March 30, 1892 - August 31, 1945) was a Polish mathematician. Banach was the founder of modern functional analysis

[5] Hugo Dionizy Steinhaus (January 14, 1887 - February 25, 1972) was a Polish mathematician and educator.

Example 6.1.2. *Let* $(]0,1[,d)$ *be the metric space where* $d(x,y) = |x-y|$*. Then the sequence* $\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right)$ *is a Cauchy sequence which does not converge in* $]0,1[$.

Remark 6.1.3. *The notion of Cauchy sequence depends on the metric used. The same sequence can be Cauchy for one metric, but not Cauchy for an equivalent metric. For example, let* $(\mathbb{R}, d)$ *be the metric space where* $d(x,y) = |x-y|$*. This metric is equivalent to the metric* $d'(x,y) = \left|\frac{x}{1+\mid x \mid} - \frac{y}{1+\mid y \mid}\right|$ *since the latter is derived from the homeomorphism* $x \mapsto \frac{x}{1+\mid x \mid}$ *of* $\mathbb{R}$ *onto* $]-1,+1[$*. The sequence* $\{n \mid n = 1,2,\ldots\}$ *in* $\mathbb{R}$ *is not Cauchy for the metric* d *and it is Cauchy for the metric* d'*.*

Proposition 6.1.4. *Let* (X,d) *be a metric space and let* $(x_1, x_2, \ldots)$ *be a sequence of points of* X*. If this sequence has a limit, then it is a Cauchy sequence.*

Proof: Suppose the sequence (x_i) converges to x. Let $\varepsilon > 0$, there exists an integer N such that $n > N \Longrightarrow d(x_n, x) < \frac{\varepsilon}{2}$. Then $m,n > N \Longrightarrow d(x_m,x) < \frac{\varepsilon}{2}$ and $d(x_n,x) < \frac{\varepsilon}{2}$, then $d(x_m,x_n) < \varepsilon$. ■

Let (X,d) be a metric space and let $(x_1, x_2, \ldots)$ be a sequence of points of X. Recall that $a \in X$ is an adherent point of the sequence if there exists a subsequence which admits a as a limit. Moreover, if the sequence has a limit then this limit is an accumulation point but the converse is false.
For example, the sequence (x_n) such that $x_{2n} = \frac{1}{n}$ and $x_{2n+1} = n, n \geq 1$, admits 0 as adherent value but it does not converge.

Proposition 6.1.5. *Let* (X,d) *be a metric space and let* $(x_1, x_2, \ldots)$ *be a Cauchy sequence of points of* X*. Any accumulation point of the sequence* $(x_1, x_2, \ldots)$ *is a limit point of the sequence.*

Proof: Let a be an accumulation point of the sequence (x_n). Given $\varepsilon > 0$ there exists n_0 such that for $p > n_0$, $q > n_0$ we have $d(x_p, x_q) < \frac{\varepsilon}{2}$. Moreover, there exists $p_0 > n_0$ such that $d(a, x_{p_0}) < \frac{\varepsilon}{2}$. From the triangle inequality, $d(a, x_n) < \varepsilon$ for all $n \geq n_0$. ■

6.2 Complete Metrics and Complete Spaces

Definition 6.2.1. *A metric space* X *is said to be* **complete** *(or* **Cauchy***) if every Cauchy sequence of points in* X *has a limit that is also in* X *or alternatively if every Cauchy sequence in* X *converges in* X*.*

Intuitively, a space is complete if there are no "points missing" from it (inside or at the boundary).

Example 6.2.2. *1. The space* $\mathbb{Q}$ *of rational numbers, with the standard metric given by the absolute value, is not complete. Consider, for instance, the sequence defined by* $x_1 = 1$ *and* $x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}$*. This is a Cauchy sequence of rational numbers, but it does not converge towards any rational limit: Such a limit* x *of the sequence would have the property that* $x^2 = 2$*, but no rational numbers have that property. However, considered as a sequence of real numbers* R *it converges towards the irrational number* $\sqrt{2}$*, the square root of two.*

2. *The space $\mathbb{R}$ of real numbers and the space $\mathbb{C}$ of complex numbers (with the metric given by the absolute value) are complete.*

Remark 6.2.3. *Note that completeness is a property of the metric and not of the topology, meaning that a complete metric space can be homeomorphic to a non-complete one. An example is given by the space of real numbers, which is complete and homeomorphic to the open interval $]0,1[$, which is not complete.*

Proposition 6.2.4. *Let (X,d) be a metric space. Assume that there exists $\varepsilon > 0$ such that for any $x \in X, \mathrm{Cl}\big(B(x;\varepsilon)\big)$ is compact. Then X is complete.*

Proof: Let (x_n) be a Cauchy sequence in X. For any $\varepsilon > 0$, there exists n such that $d(x_p, x_q) < \varepsilon/2$ for any $p, q > n$. Then for any $p > n$, $x_p \in \mathrm{Cl}\big(B(x;\varepsilon)\big)$ and therefore (x_n) has an accumulation point x_0 which is a limit point of (x_n). ■

Lemma 6.2.5. *Let (X,d) be a metric space and let $A \subset X, x \in X$. Then the two followings are equivalent:*

1. *$x \in \mathrm{Cl}(A)$.*

2. *There exists a sequence $(x_1, x_2, \ldots)$ of points of A which converges to x.*

Proof: $\Longrightarrow$) Let $x \in \mathrm{Cl}(A)$. Then, for any integer $n \geq 1$, there exists a point $x_n \in A$ which belongs to $\mathrm{Cl}\big(B(x;\frac{1}{n})\big)$. Then the sequence (x_n) converges to x.
$\Longleftarrow$) Suppose there exists a sequence $(x_1, x_2, \ldots)$ of points of A which converges to x. Then any neighbourhood of x contains some x_n so it intersects A and $x \in \mathrm{Cl}(A)$. ■

Proposition 6.2.6. *Let X be a complete metric space and let $Y \subset X$. Then Y is complete iff Y is closed.*

Proof: $\Longrightarrow$) Let $y \in \mathrm{Cl}(Y)$. There exists a sequence $(y_1, y_2, \ldots)$ in Y which converges to y. This is a Cauchy sequence which converges to z in Y which is complete. In X, the sequence (y_i) converges both to y and z so $y = z \in Y$ and $\mathrm{Cl}(Y) = Y$.
$\Longleftarrow$) Let $(y_1, y_2, \ldots)$ be a Cauchy sequence in Y. This is a Cauchy sequence in X so it converges to $x \in X$. But $y_i \in Y$ for any i and $x \in \mathrm{Cl}(Y) = Y$ so (y_i) converges in Y. ■

Definition 6.2.7. *Let $\mathcal{B}$ be a filterbase on the metric space X. $\mathcal{B}$ is said to be a* **Cauchy filterbase** *if for any $\varepsilon > 0$, there exists $A \in \mathcal{B}$ such that its diameter $\delta(A) < \varepsilon$.*

Proposition 6.2.8. *Let $\mathcal{B}$ be a Cauchy filterbase on the complete metric space X. Then $\mathcal{B}$ has a limit point.*

Proof: For any $n \in \mathbb{N}^*$, there exists $A_n \in \mathcal{B}$ such that $\delta(A_n) < \dfrac{1}{n}$. Let x_n be an arbitrary point of A_n. Then $A_p \cap A_q \neq \emptyset$ for any p, q so $\delta(A_p \cup A_q) < \dfrac{1}{p} + \dfrac{1}{q}$ and $d(x_p, x_q) < \dfrac{1}{p} + \dfrac{1}{q}$. The sequence (x_n) is Cauchy and, X being complete, (x_n) has a limit x. Thus for any $\varepsilon > 0$, there exists n such that $d(x_n, x) < \dfrac{\varepsilon}{2}$ and $\delta(A_n) < \dfrac{\varepsilon}{2}$. Hence $A_n \subset B(x;\varepsilon)$ and $\mathcal{B}$ converges to x. ■

Proposition 6.2.9. *Let $X_1, \ldots, X_n$ be n complete metric spaces. Then the metric space $X = X_1 \times \cdots \times X_n$ is complete.*

Proof: Let $(y_1, y_2, \ldots)$ be a Cauchy sequence in X. Notice that $y_i = (y_{i1}, \ldots, y_{in})$ where $y_{ij} \in X_j$. We have $d(y_{m1}, y_{n1}) \leq d(y_m, y_n) \longrightarrow 0$ when $m, n \longrightarrow \infty$. Then $(y_{11}, y_{21}, y_{31}, \ldots)$ is a Cauchy sequence of X_1 so it converges to l_1 in X_1. Similarly for the other components. Therefore, the sequence $(y_1, y_2, \ldots)$ converges to the point $(l_1, l_2, \ldots)$ in X. ■

Example 6.2.10. *The Euclidean space $\mathbb{R}^n$ is complete.*

Proposition 6.2.11. *Let X be a metric space. The two following conditions are equivalent:*

1. *X is compact.*

2. *X is complete and, for all $\varepsilon > 0$, there exists a finite cover of X with balls of radius ε.*

Proof: $1 \Longrightarrow 2)$: Suppose X compact. Let $(x_1, x_2, \ldots)$ be a Cauchy sequence in X i.e. for any $\varepsilon > 0$, there exists N_0 such that for any $i, j \geq N_0,\ d(x_i, x_j) \leq \dfrac{\varepsilon}{2}$.
X compact, then there exists a subsequence (x_{n_i}) which has a limit x in X, i.e. for any $\varepsilon > 0$, there exists N_1 such that for any $n_i \geq N_1,\ d(x_{n_i}, x) \leq \dfrac{\varepsilon}{2}$.
Thus the sequence $(x_1, x_2, \ldots)$ has also a limit in X, i.e. let $N \geq \sup(N_0, N_1)$, then for any $i \geq N$, for any $n_j \geq N$, $d(x_i, x) \leq d(x_i, x_{n_j}) + d(x_{n_j}, x) \leq \dfrac{\varepsilon}{2} + \dfrac{\varepsilon}{2}$.
Then X is complete.
Let $\varepsilon > 0$, the open balls of radius $\varepsilon > 0$ define a cover of X. But X is compact, there exists a finite subcover of X.
$2 \Longrightarrow 1)$: Let $(x_1, x_2, \ldots)$ be a sequence in X. Let $\mathcal{V}_{1/2}$ be a finite cover of X with balls of radius $1/2$. One of these balls, B, contains infinitely many x_i. Let $(y_1^1, y_2^1, y_3^1, \ldots)$ a subsequence of $(x_1, x_2, \ldots)$ such that all the $y_i^1 \in B$ so the distances of any two points is ≤ 1. Now let begin again this process replacing $\dfrac{1}{2}$ by $\dfrac{1}{4}$, by $\dfrac{1}{6}, \dfrac{1}{8}, \ldots$ and we get infinitely many finite covers $\mathcal{V}_{1/2n},\ n \geq 1$ and sequences

$$\left(y_1^1, y_2^1, y_3^1, \ldots\right)$$
$$\left(y_1^2, y_2^2, y_3^2, \ldots\right)$$
$$\left(y_1^3, y_2^3, y_3^3, \ldots\right)$$

where each one is a subsequence of the previous one and such that $d(y_i^n, y_j^n) \leq \dfrac{1}{n}$ for any i and j. Let $\left(y_1^1, y_2^2, y_3^3, \ldots\right)$ be the diagonal sequence . Then $d\left(y_m^m, y_n^n\right) \leq \dfrac{1}{m}$ for any $m \leq n$. So the sequence (y_i^i) is Cauchy and it has a limit. Then X is compact. ■

Notice that every compact metric space is complete but the converse is not true; for example, $\mathbb{R}$ with the standard topology.

6.3 Function Spaces

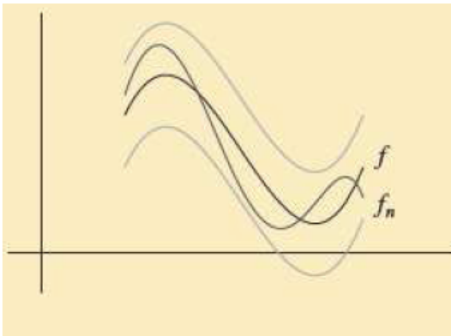
The function spaces play an important role in modern topology. Recall that we introduced a metric topology on some function spaces as follows. Let X be a set and (Y, d) a metric space. Let $C(X, Y; d)$ be the set of bounded continuous map of X into Y i.e.

$$C(X, Y; d) = \left\{f \in Y^X \mid f \text{ continuous and } \delta f(X) < \infty\right\}$$

where δ is the diameter.
We defined the distance $D(f, g) = \sup\{d\left(f(x), g(x)\right) \mid x \in X\}$ for any $f, g \in C(X, Y; d)$.

Definition 6.3.1. *Let X be a set and (Y,d) a metric space and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of maps $f_n : X \longrightarrow Y$. The sequence (f_n) is said to be* **uniformly convergent** *with limit the map $f : X \longrightarrow Y$, if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $x \in X$ and all $n > n_0$, we have $d\left(f_n(x), f(x)\right) < \varepsilon$.*

Remark 6.3.2. *The sequence (f_n) is uniformly convergent to the map f iff for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, we have $D(f_n, f) < \varepsilon$.*



Lemma 6.3.3. *Let $\big(C(X,Y;d), D\big)$ be the metric space. Then*

1. *Any sequence (f_n) converges to f iff $f_n \to f$ uniformly on Y.*
2. *If $f_n \to f$ uniformly on Y, then f is continuous and $f \in C(X,Y;d)$.*

Proof:

1. It follows from the inequality $d\big(f(x), f_n(x)\big) \le D(f, f_n)$.
2. f is continuous at x_0: For any $\varepsilon > 0$, there exists n such that $d\big(f(x), f_n(x)\big) \le \varepsilon$ for any $x \in X$. From

$$d\big(f(x), f(x_0)\big) \le d\big(f(x), f_n(x)\big) + d\big(f_n(x), f_n(x_0)\big) + d\big(f_n(x_0), f(x_0)\big) \le 3\varepsilon$$

$f \in C(X,Y;d)$: Note that $d\big(f(x), f_n(x)\big) \le 1$ for any $x \in X$ and sufficiently large n so that $\delta f(X) \le \delta f_n(X) + 2$.

■

Example 6.3.4. *Let (f_n) be the sequence of continuous functions $f_n : [0,1] \longrightarrow \mathbb{R}$, where $f_n(x) = x^n$. Then $\lim_{n\to\infty} f_n = f$ where f is not continuous. (exercise).*

Proposition 6.3.5. *Let Y be a complete metric space. Then $\big(C(X,Y;d), D\big)$ is complete.*

Proof: Let (f_n) be a Cauchy sequence of $\big(C(X,Y;d), D\big)$ so that for any $\varepsilon > 0$, there exists N such that for any $m, n \ge N, D(f_m, f_n) \le \varepsilon$. Since $d\big(f_m(x), f_n(x)\big) \le D(f_m, f_n)$, it follows that $\big(f_n(x)\big)$ is a Cauchy sequence in Y for any $x \in X$ and therefore converges to some element which we denote $f(x) \in Y$. Furthermore, we have $f_n(x) \in B\big(f_m(x); \varepsilon\big)$ for any x and any $m, n \ge N$; consequently, $f(x) \in \mathrm{Cl}(B\big(f_m(x); \varepsilon\big)$ which shows that (f_n) converges to f uniformly in Y. Using the previous lemma, f is continuous and belongs to $C(X,Y;d)$. Since $f_n \to f$, the proof is complete. ■

Example 6.3.6. *The metric spaces $\big(C([a,b], \mathbb{R}), D\big)$ and $\big(C(\mathbb{R}, \mathbb{R}), D\big)$ are complete.*

6.4 Extension of Uniformly Continuous Maps

Although the image under a continuous map of a convergent sequence is a convergent sequence, the image of a Cauchy sequence need not be Cauchy.

Proposition 6.4.1. *A uniformly continuous function maps Cauchy sequences into Cauchy sequences.*

Proof: Let (X, d_X) and (Y, d_Y) be two metric spaces and let $f : X \longrightarrow Y$ be a uniformly continuous map. Let (x_n) be a Cauchy sequence in X. Let $\varepsilon > 0$, then there exists $\delta > 0$ such that for any $x, x' \in X, d_X(x, x') < \delta \Longrightarrow d_Y\big(f(x), f(x')\big) < \varepsilon$. Thus there exists $N \in \mathbb{N}$ such that $d_X(x_m, x_n) < \delta$ for any $m, n \geq N$. It follows that $d_Y(f(x_m), f(x_n)) < \varepsilon$ for any $m, n \geq N$. Hence $\big(f(x_n)\big)$ is a Cauchy sequences in Y. ■

Remark 6.4.2. *If f is not uniformly continuous, then the result may not be true. For example, let $f :]0, \infty[\longrightarrow \mathbb{R}; x \mapsto 1/x$. f is continuous and the sequence $(1/n)$ is Cauchy. But the sequence $\big(f(1/n)\big) = (n)$ is not a Cauchy sequence.*

Theorem 6.4.3. *Let A be a dense subset of a metric space (X, d_X). Let f be a uniformly continuous map from A into a complete metric space (Y, d_Y). Then there exists a unique uniformly continuous map g from X into Y which extends f, i.e. $g|_A = f$.*

Proof:

1. Define a map $g : X \longrightarrow Y$
   For each $x \in X = \mathrm{Cl}(A)$, there exists a sequence (x_n) in A which converges to x. Then (x_n) is a Cauchy sequence in X. Thus $\big(f(x_n)\big)$ is a Cauchy sequence in Y which converges. Set $g(x) = \lim_{n\to\infty} f(x_n)$ for any $x \in X$.

2. The map g does not depend upon the sequence (x_n).
   Let (x_n) and (x'_n) be two sequence in A which converge to $x \in X$. Then the sequence $(x_1, x'_1, x_2, x'_2, \ldots, x_n, x'_n, \ldots)$ must converge to x. Hence the sequence $\big(f(x_1), f(x'_1), \ldots, \ldots, f(x_n), f(x'_n), \ldots\big)$ converges to some point $y \in Y$. Since $\big(f(x_1), f(x_2), \ldots\big)$ and $\big(f(x'_1), f(x'_2), \ldots\big)$ are the subsequences, they must also converge to y. Hence $y = g(x)$ does not depend on the choice of the sequences.

3. The map g is an extension of f.
   Let $a \in A$ and let $a_n = a$ for each n. Then (a_n) is a sequence in A which converges to a. Hence $g(a) = \lim_{n\to\infty} f(a_n) = f(a)$. This shows that g is an extension of f.

4. The map g is uniformly continuous on X.
   Let $\varepsilon > 0$. Then there exists $\alpha > 0$ such that for any $a, b \in A, d_X(a, b) \leq \alpha \Longrightarrow d_Y\big(f(a), f(b)\big) \leq \dfrac{\varepsilon}{3}$. Let $x, x' \in X$ be such that $d_X(x, x') \leq \alpha$. Then there exist sequences (x_n) and $(x'n)$ in A which converge to x and x'. Hence $\big(f(x_n)\big)$ converges to $g(x)$ and $\big(f(x'_n)\big)$ converges to $g(x')$. Choose N such that

$$d_X(x_N, x) \leq \frac{\alpha - d_X(x, x')}{2}, \qquad d_X(x'_N, x') \leq \frac{\alpha - d_X(x, x')}{2}$$

$$d_Y(\big(f(x_N), g(x)\big) \leq \frac{\varepsilon}{3} \qquad \text{and} \qquad d_Y(\big(f(x'_N), g(x')\big) \leq \frac{\varepsilon}{3}$$

Then $d_X(x_N, x'_N) \leq d_X(x_N, x) + d_X(x, y) + d_X(x', x'_N) \leq \alpha$ which implies that $d_Y\big(f(x_N), f(x'_N)\big) \leq \varepsilon$, be the uniform continuity of f on A. Hence,

$$d_Y\big(g(x), d(x')\big) \leq d_Y\big(g(x), f(x_N)\big) + d_Y\big(f(x_N), f(x'_N)\big) + d_Y\big(f(x'_N), d(x')\big) \leq 3\frac{\varepsilon}{3} = \varepsilon$$

This shows that g is uniformly continuous on X.

5. The map g is unique.
Let g and h be two uniformly continuous maps on X which extends f. Let $x \in X$. Then there is a sequence (x_n) in A which converges to x. By continuity of g and h, $g(x) = \lim\limits_{n\to\infty} g(x_n) = \lim\limits_{n\to\infty} h(x_n) = h(x)$. Hence $g = h$.

■

Corollary 6.4.4. *Let (x, d_X) and (Y, d_Y) be two complete metric spaces, and let $A \subset X$ and $B \subset Y$ be dense. Then each uniform isomorphism $h : A \longrightarrow B$ has an extension $H : X \longrightarrow Y$ that is also a uniform isomorphism. Furthermore, if h is an isometry, then so also H.*

Proof: Since h is uniformly continuous, it is extendable to a uniformly continuous $H : X \longrightarrow Y$. Since $g = h^{-1}$ is also uniformly continuous, it has also a uniformly extension $G : Y \longrightarrow X$; Because $G \circ H|_A = \mathrm{Id}_A$ and A is dense in X, we have that $G \circ H = \mathrm{Id}_X$ and , similarly, that $H \circ G = \mathrm{Id}_Y$. It follows that H is a uniform isomorphism. The second part is immediate from the manner in which the extension H is defined. ■

6.5 Completion of a Metric Space

In this section, a non complete metric space will be embedded in a complete metric space. The first example is the embedding of the rational numbers $\mathbb{Q}$ into the space of real numbers $\mathbb{R}$ which is complete.

Definition 6.5.1. *A* **completion** *of a non complete metric space (X, d) is a pair consisting of a complete metric space $(\widehat{X}, \widehat{d})$ and an isometry f from X into $\widehat{X}$ such that $f(X)$ is dense in $\widehat{X}$.*

Theorem 6.5.2. *Every non complete metric space has a completion.*

Proof: Let (X, d) be a non complete metric space. Let $\mathcal{S}_X$ be the set of all Cauchy sequences on X. Define the relation $\sim$ on $\mathcal{S}_X$ by

$$(x_n) \sim (y_n) \iff \lim_{n\to\infty} d(x_n, y_n) = 0$$

Notice that the both sequences have limits in X is irrelevant.
It is easy to check that this is an equivalence relation on $\mathcal{S}_X$ (exercise).
Denote $[(x_n)]$ be the equivalence class of the Cauchy sequence (x_n), so it is an element of the quotient set $\mathcal{S}_X/\sim$. Denote $\widehat{X} = \mathcal{S}_X/\sim$ which is the space we are looking for.

1. Define a metric $\widehat{d}$ on $\widehat{X}$.

   (a) First, define a function $\Delta : \mathcal{S}_X \times \mathcal{S}_X \longrightarrow \mathbb{R}$ by $\Delta\big((x_n), (y_n)\big) = \lim\limits_{n\to\infty} d(x_n, y_n)$. Let prove that Δ is a well-defined function, i.e. the limit exists.
   By the triangular inequality

$$d(x_n, y_n) \leq d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n)$$

We have $d(x_n, y_n) - d(x_m, y_m) \leq d(x_n, y_m) + d(y_m, y_n)$ and reversing the roles of m and n we find that

$$|\, d(x_n, y_n) - d(x_m, y_m)\,| \leq \; d(x_n, y_m) + d(y_m, y_n)$$

Since both sequences are Cauchy, for any $\varepsilon > 0$, we can choose N such that for any $m, n \geq N, d(x_n, x_) \leq \varepsilon/2$ (choose N_1 for the sequence (x_n) and N_2 for the sequence (y_n) and $N = \sup(N_1, N_2)$. Thus we have $|\, d(x_n, y_n) - d(x_m, y_m)\,| \leq \varepsilon$ for all $m, n \geq N$, i.e. the sequence (α_n) where $\alpha_n = d(x_n, y_n)$ is a Cauchy sequence in $\mathbb{R}$ which is complete and the limit exists.

(b) Second, construct a metric $\widehat{d}$ on $\widehat{X}$ from the map Δ.
Define
$$\widehat{d}\big([(x_n)], [(y_n)]\big) = \Delta\big((x_n), (y_n)\big)$$
In order to show that $\widehat{d}$ is well-defined, we have to prove that it does not depend on the choice of the Cauchy sequence in its equivalence class, i.e. $\Delta\big((x_n), (y_n)\big) = \Delta\big((x'_n), (y'_n)\big)$ where $(x_n) \sim (x'_n)$ and $(y_n) \sim (y'_n)$.

$$\Delta\big((x_n), (y_n)\big) \leq \Delta\big((x_n), (x'_n)\big) + \Delta\big((x'_n), (y'_n)\big) + \Delta\big((y'_n), (y_n)\big) = \Delta\big((x'_n), (y'_n)\big)$$

since $\Delta\big((x_n), (x'_n)\big) = \Delta\big((y_n), (y'_n)\big) = 0$ so that $\Delta\big((x_n), (y_n)\big) \leq \Delta\big((x'_n), (y'_n)\big)$. A similar argument also shows that $\Delta\big((x'_n), (y'_n)\big) \leq \Delta\big((x_n), (y_n)\big)$ so that $\Delta\big((x_n), (y_n)\big) = \Delta\big((x'_n), (y'_n)\big) = \widehat{d}\big([(x_n)], [(y_n)]\big)$. It remains to check that $\widehat{d}$ is a metric which is straightforward.

2. There exists an isometry f from (X, d) into $(\widehat{X}, \widehat{d})$.
For each $x \in X$, let $\widehat{x} = [(x_n)] \in \widehat{X}$ such that $x_n = x$ for all n, i.e. the equivalence class of the constant (Cauchy) sequence $(x, x, x, \ldots) = (x)$. Define $f : X \longrightarrow \widehat{X}$ by $f(x) = [(x)] = \widehat{x}$. Then for any $x, y \in X$,
$$\widehat{d}\big(f(x), f(y)\big) = \widehat{d}(\widehat{x}, \widehat{y}) = \lim_{n\to\infty} d(x, y) = d(x, y)$$
Hence f is an isometry from X into $\widehat{X}$.

3. $f(X)$ is dense in $\widehat{X}$.
Let $[(x_n)] \in \widehat{X}$ and let $\varepsilon > 0$. Since (x_n) is a Cauchy sequence, there exists N such that for any $m, n \geq N$, $d(x_m, x_n) < \dfrac{\varepsilon}{2}$. Let $y = x_N$. Then $\widehat{y} \in f(X)$ and
$$\widehat{d}([(x_n)], \widehat{y}) = \lim_{n\to\infty} d(x_n, y) = \lim_{n\to\infty} d(x_n, x_N) \leq \frac{\varepsilon}{2} < \varepsilon$$
Thus $\widehat{y} \in B([(x_n)]; \varepsilon) \cap f(X)$ where $B([(x_n)]; \varepsilon)$ is the open ball of center $[(x_n)]$ and radius ε in $(\widehat{X}, \widehat{d})$. Hence, $f(X)$ is dense in $\widehat{X}$.

4. $(\widehat{X}, \widehat{d})$ is complete.
We have to consider Cauchy sequences in $\widehat{X}$ i.e. $\big([(x_n)]_k\big)$. For each k, choose a representative (x_{nk}) which is a Cauchy sequence of X. Notice that $x_{nk} \in X$ for any n, k. To say that $\big([(x_n)]_k\big)$ is a Cauchy sequence means that for any $\varepsilon > 0$, there exists N such that $\Delta\big((x_{nk_1}), (x_{nk_2})\big) \leq \varepsilon$ for any $k_1, k_2 \geq N$ i.e. $\lim\limits_{n\to\infty} d(x_{nk_1}, x_{nk_2}) \leq \varepsilon$ for any $k_1, k_2 \geq N$.

Recall that for each k, the sequence $(x_{1k}, x_{2k}, \dots)$ (w.r.t. the first index) is Cauchy in X. Thus there exists N_k such that $d(x_{pk}, x_{qk}) < \dfrac{1}{k}$ for any $p, q \geq N_k$.
For each k, choose some x_{pk} with $p \geq N_k$ and let the constant sequence $(x_{pk}, x_{pk}, \dots)$, denoted c_k, which is clearly Cauchy. For simplicity, denote $y_k = x_{pk}$ the chosen element, so $c_k = (y_k, y_k, \dots)$. Let $\widehat{c}_k$ be its equivalence class in $\widehat{X}$. Then for each k, $\Delta\big((x_{nk}), c_k\big) \leq \dfrac{1}{k}$, and hence $\widehat{d}\big([(x_{nk})], \widehat{c}_k\big) \leq \dfrac{1}{k}$.
Consider the sequence $\widehat{c}_k$ and find its limit. Let $c = (y_1, y_2, y_3, \dots)$ which is a Cauchy sequence of X. To prove this,

$$d(y_m, y_n) \leq d(y_m, x_{jm}) + d(x_{jm}, x_{jn}) + d(x_{jn}, y_n)$$

There exists N' such that $d(y_m, x_{jm}) \leq \dfrac{\varepsilon}{3}$ for any $m \geq N'$.
There exists N'' such that $d(x_{jm}, x_{jn}) \leq \dfrac{\varepsilon}{3}$ for any $m, n \geq N''$.
There exists N''' such that $d(x_{jn}, y_n) \leq \dfrac{\varepsilon}{3}$ for any $n \geq N'''$.
So $d(y_m, y_n) \leq \varepsilon$.
$\Delta\big((y_k), c\big) = \lim_{j\to\infty} d(y_k, y_j) = 0$ because $c = (y_1, y_2, y_3, \dots)$ is a Cauchy sequence.
Finally the Cauchy sequence $\big([(x_n)]_k\big)$ converges to $\widehat{c} \in \widehat{X}$ because $\widehat{d}\big([(x_{nk})], \widehat{c}\big) \leq \dfrac{1}{k}$ for any k. ■

Example 6.5.3. *Starting from the set $\mathbb{N}$ of natural integers, using algebraic processes, we can construct the set $\mathbb{Z}$ of integers, then the set $\mathbb{Q}$ of rational numbers. The completion of the metric space $(\mathbb{Q}, d)$ where $d(x, y) = |x - y|$, is the space of real numbers $\mathbb{R}$.*

Corollary 6.5.4. *Let (X, d) and (X', d') be two non complete metric spaces, and $g : X \longrightarrow X'$ a uniformly continuous map. Then there exists a unique uniformly continuous map $\widehat{g} : \widehat{X} \longrightarrow \widehat{X'}$ such that the following diagram is commutative.*

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\
\widehat{X} \arrow[r, "\widehat{g}"] & \widehat{X'}
\end{tikzcd}$$

Proof: The map $f' \circ g \circ f^{-1} : f(X) \longrightarrow \widehat{X'}$ is uniformly continuous; since the space $\widehat{X'}$ is complete and $f(X)$ is dense in $\widehat{X}$, then $\widehat{g} : \widehat{X} \longrightarrow \widehat{X'}$ is uniformly continuous and the diagram is commutative. ■

6.6 Baire's Theorem for Complete Metric Spaces

We have sufficient condition for topological completeness, the following proposition is a necessary condition which is one of the most important and useful.

Definition 6.6.1. *A space X is a* **Baire** *space if the intersection of each countable family of open dense sets is dense.*

Theorem 6.6.2. *Any topological complete space is a Baire space.*

Proof: Let (O_n) be a countable family of dense open sets and let $A = \bigcap_n O_n$. To show that A is everywhere dense it suffices to prove that for every non-empty open ball B in the space X we have $B \cap \bigcap_n O_n \neq \emptyset$ Let (r_n) be a sequence of strictly positive numbers converging to 0. If B is a non-empty open ball and O_1 an every where dense open set, $B \cap O_1$ is non-empty and open, and so contains a non-empty open ball. Choose this ball B_1 to have radius less than r_1. Since B_1 is non-empty and open, $B_1 \cap O_2$ is also non-empty and open and contains a non-empty open ball B_2 whose radius we may suppose to be less than r_2. In this way we construct, step by step, a countable family of non-empty open balls B_n with radii less than r_n, respectively, and such that $B_n \subset B_{n-1} \cap O_{n-1}$ which implies that$B_n \subset B_{n-1}$ and $\bigcap_n B_n \subset B \cap \bigcap_n O_n$. It remains to prove that the B_n have a non-empty intersection. Let x_n denote the center of B_n. For integers p such that $0 \leq p \leq q$, $x_q \in B_q$ so that $d(x_p, x_q) < r_p$. Since $\lim r_p = 0$, $d(x_p, x_q)$ tends to 0, and so the sequence (x_n) is a Cauchy sequence. Since X is complete (x_n) converges to a point $x \in X$. It follows that for an arbitrary integer p, and q tending to infinity:

$$d(x_p, x) \leq d(x_p, x_q) + d(x_q, x) < r_p + d(x_q, x)$$

and so, since $\lim_{q \to \infty} d(x_q, x) = 0$, it follows that $d(x_p, x) \leq r_p$ for all p and so $x \in B_p$ for all p, or $x \in \bigcap_p B_p$. The intersection of the balls B_n is thus non-empty and so $B \cap \bigcap_n O_n \neq \emptyset$. ■

Remark 6.6.3. *The completeness assumption is necessary in this theorem as the following example illustrates. Let $X = \mathbb{Q}$. Write $\mathbb{Q} = \{r_n \mid n \in \mathbb{N}\}$ and let $O_n = \mathbb{Q} \setminus \{r_n\}$ for each $n \in \mathbb{N}$. Then O_n is open and dense in $\mathbb{Q}$ for each n, but $\bigcap_{n=1}^{\infty} O_n = \emptyset$.*

Corollary 6.6.4. *If a complete metric space is a union of countable many closed sets, then at least one of the closed sets has nonempty interior.*

Proof: Let X be a complete metric space. Assume that $X = \bigcup_{n=1}^{\infty} C_n$, where each C_n is closed. For each $n \in \mathbb{N}$, let $O_n = X \setminus C_n$. Then $\bigcap_{n=1}^{\infty} O_n = \emptyset$. By Baire's theorem, there exists an open set O_n which is not dense in X. Thus, $\mathrm{Cl}(O_n) \neq X$. But $\mathrm{Int}(C_n) = X \setminus \mathrm{Cl}(O_n)$, and hence C_n has nonempty interior. ■

6.7 Fixed-Point Theorem for Complete Spaces

Definition 6.7.1. *Let $f : X \longrightarrow X$ be a map of the space X to itself. A point x_0 is called a* **fixed point** *for f if $f(x_0) = x_0$.*

Not every map has a fixed point; for example, the map $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by $x \mapsto x + 1$ has no fixed point.

Proposition 6.7.2. *Let (X, d) be a complete metric space, let $f : X \longrightarrow X$ be a map. Assume there exists $\lambda \in [0, 1[$ such that $d\big(f(x), f(x')\big) \leq \lambda\, d(x, x')$ for all $x, x' \in X$. Such a map is called* **contractive**. *Then*

1. *There exists only one point $a \in X$ such that $f(a) = a$.*

2. *For all $x_0 \in X$, the sequence of points, $x_{n+1} = f(x_n), n \geq 0$ converges towards a.*

Proof: 1. Let $a, b \in X$ such that $a = f(a)$ and $b = f(b)$. Then $d(a,b) = d\big(f(a), f(b)\big) \leq \lambda\, d(a,b)$. So $(1-\lambda)\, d(a,b) \leq 0$. But $1 - \lambda > 0$, then $d(a,b) \leq 0$ and finally $d(a,b) = 0$ thus $a = b$.
2. Let $x_0 \in X$, define $x_1 = f(x_0),\ x_2 = f(x_1), \ldots$. Let us show that $d(x_n, x_{n+1} \leq \lambda^n\, d(x_0, x_1)$. It is clear for $n = 0$. Suppose it is true for n. We deduce that for any integers $n, p \geq 0$

$$\begin{aligned} d(x_n, x_{n+p}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+p-1}, x_{n+p}) \\ &\leq (\lambda^n + \lambda^{n+1} + \cdots + \lambda^{n+p-1})\, d(x_0, x_1) \\ &\leq \frac{\lambda^n}{1-\lambda}\, d(x_0, x_1) \end{aligned}$$

But $0 \leq \lambda < 1$ then $\lambda^n \longrightarrow 0$ for $n \longrightarrow \infty$. Therefore (x_n) is a Cauchy sequence which converges to the point a. Moreover $d(x_n, a) \longrightarrow 0$ so $d\big(f(x_n), f(a)\big) \longrightarrow 0$ which means $d(x_{n+1}, f(a)\big) \longrightarrow 0$. So the sequence (x_n) also converges to $f(a)$ and finally $a = f(a)$. ■

The most interesting applications of this proposition arise when the space X is a function space. We can use it to prove a number of existence and uniqueness theorems for differential and integral equations.

Proposition 6.7.3 (Picard)**.** [6] *Given a function $f(x,y)$ defined and continuous on a plane domain G containing the point (x_0, y_0) suppose f satisfies a Lipschitz condition of the form $|f(x,y) - f(x,y')| \leq M|y - y'|$ in the variable y. Then there is an interval $|x - x_0| \leq \alpha$ in which the differential equation $\dfrac{dy}{dx} = f(x,y)$ has a unique solution $y = \varphi(x)$ satisfying the initial condition $\varphi(x_0) = y_0$*

By an n-dimensional domain we mean an open connected set in Euclidean n-space.
Proof: Together the differential equation and the initial condition are equivalent to the integral equation $\varphi(x) = y_0 + \displaystyle\int_{x_0}^{x} f(t, \varphi(t))dt$. By the continuity of f, we have $|f(x,y)| \leq K$ in some domain $G' \subset G$ (in fact f is bounded on $G' \subset G$) containing the point (x_0, y_0). Choose $\alpha > 0$ such that

1. $(x,y) \in G'$ if $|x - x_0| \leq \alpha,\ |y - y_0| \leq K\alpha$

2. $M\alpha < 1$

and let C be the space of continuous functions φ defined on the interval $|x - x_0| \leq \alpha$ and such that $\big|\varphi(x) - y_0\big| \leq K\alpha$, equipped with the metric $d(\varphi, \varphi') = \max_x |\varphi(x) - \varphi'(x)|$. The space C is complete, since it is closed subspace of the space of all continuous functions on $[x_0 - \alpha, x_0 + \alpha]$. Consider the mapping $\psi = A\varphi$ defined by the integral equation

$$\psi(x) = y_0 + \int_{x_0}^{x} f(t, \varphi(t))dt, \quad (|x - x_0| \leq \alpha)$$

Clearly A is a contraction mapping carrying C into itself. In fact, if $\varphi \in C, |x - x_0| \leq \alpha$ then $\big|\psi(x) - y_0\big| = \big|\displaystyle\int_{x_0}^{x} f(t, \varphi(t))dt\big| \leq \int_{x_0}^{x} \big|f(t, \varphi(t))\big|dt \leq K\big|x - x_0\big| \leq K\alpha$ and hence $\psi = A\varphi$ also

[6]Charles Émile Picard (24 July 1856 - 11 December 1941) was a French mathematician.

belongs to C. Moreover,

$$\big|\psi(x) - \psi'(x)\big| \leq \int_{x_0}^{x} \big|f(t,\varphi(t)) - f(t,\varphi'(t))\big| dt \leq M\alpha \big|\varphi(t) - \varphi'(t)\big|$$

and hence $d(\psi, \psi') \leq M\alpha . d(\psi, \psi')$ after maximizing with respect to x. But $M\alpha < 1$, so that A is a contraction mapping. It follows that the equation $\varphi = A\varphi$, i.e. the integral equation has a unique solution in the space C. ■

6.7.1 Exercises

1. Let $X = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right\} \subset \mathbb{R}$ and define the metric d on X as follows: $d(x, y) = |x - y|$.
   (a) Show that (X, d) is not complete.
   (b) Let $x_n = 1/n$. Show that $d(x_n, x) \geq \dfrac{1}{n^2 + n}$ for any $x \in X$ with $x \neq x_n$.
   (c) Show that the only Cauchy sequences (y_n) in X are
   - either $y_n = x_p$ for some p and all $n \geq N$.
   - or for any N, there exists M such that for each $n \geq M$ we have $y_n = x_p$ for some $p \geq N$.

2. Let X be a metric space and let $f : X \longrightarrow X$ be a map such that for any $x, y \in X, x \neq y$ then $d\big(f(x), f(y)\big) < d(x, y)$. Suppose X complete. Find out an example such that f has no fixed point. (Hint: Consider the function $f : [0, +\infty[\longrightarrow [0, +\infty[; x \mapsto \sqrt{x^2 + 1}$).

3. Let X be a metric space and let $f : X \longrightarrow X$ be a map. Assume there exists $\lambda \in [0, 1[$ such that $d\big(f(x), f(x')\big) \leq \lambda\, d(x, x')$ for all $x, x' \in X$. Suppose X is not complete. Find out an example such that f has no fixed point. (Hint: Consider the function $f :]0, 1] \longrightarrow]0, 1]; x \mapsto \dfrac{x}{2}$).

4. Let (X, d) be a metric space. The distance between a point $x \in X$ and a subset $A \subset X$ is $d(x, A) = \inf_{y \in A}(x, y)$. Let $\mathcal{H}(X)$ be the set of non-empty compact subspaces of X and let A, B be two elements of $\mathcal{H}(X)$. The **Hausdorff metric** on $\mathcal{H}(X)$ is defined by

$$h(A, B) = \sup\left\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\right\}$$

   (a) Show that h is a metric on $\mathcal{H}(X)$ and $(\mathcal{H}(X), h)$ is complete.
   (b) If X is compact, show that $\mathcal{H}(X)$ is compact.
   (c) Show that the topology of $\mathcal{H}(X)$ depends only on the topology of X and not on the metric d.

Part II

Continuous Deformations

Chapter 7

Homotopy

7.1 Fundamental Group

7.1.1 Introduction

The title of this chapter is "continuous deformation". What is it? Let consider some examples: Is it possible to have a continuous deformation

- from $Id : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ into the constant map to 0?
- from $Id : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ into the map $\mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ x \longmapsto -x$, where $\mathbb{S}^1 = \{x \in \mathbb{C} \mid |x| = 1\}$?
- from $Id : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ into the constant map $\mathbb{S}^1 \longrightarrow \mathbb{S}^1,\ x \longmapsto 1$?
- from some continuous map $f : X \longrightarrow \mathbb{R}$ to some other continuous map $g : X \longrightarrow \mathbb{R}$?

A continuous deformation exists, means that there is a family of continuous maps depending of a parameter t such that for $t = 0$, the map is the first map and for $t = 1$, the map is the second map. In other words, if the maps are from X to Y, there is a continuous map $F : X \times [0,1] \longrightarrow Y$ such that $F(-,0)$ is the first map, $F(-,1)$ the second, and each $F(-,t)$ is continuous.

7.1.2 Homotopy of Paths

Let consider the two complex maps $z \longmapsto z$ and $z \longmapsto \dfrac{1}{z}$. Let $\mathcal{C}$ be a loop, i.e. a closed curve in $\mathbb{C}$ and the integrals $\displaystyle\int_{\mathcal{C}} zdz$ and $\displaystyle\int_{\mathcal{C}} \frac{dz}{z}$. The first integral is always $= 0$ because the curve $\mathcal{C}$ can be "shrunk" to a point in the domain of analyticity of the map, and the second integral can be $\neq 0$ because some curves in the domain of analyticity of the map, $\mathbb{C}^*$, for example, the unit circle, cannot be "shrunk" to a point.
These two integrals show that the topologies of the domains of analyticity of the two maps are different.
More precisely, let $\gamma_i : [0,1] \longrightarrow X, i = 1,2$, be two continuous maps, called **paths**, to a topological space X such that $\gamma_1(0) = \gamma_2(0) = x_0$ and $\gamma_1(1) = \gamma_2(1) = x_1$.

The two paths are **homotopic** with the end points held fixed if there exists a continuous map $F : [0,1] \times [0,1] \longrightarrow X$ such that:

- $F(s,0) = \gamma_1(s)$ for any $s \in [0,1]$.
- $F(s,1) = \gamma_2(s)$ for any $s \in [0,1]$.
- $F(0,t) = x_0$ for any $t \in [0,1]$.
- $F(1,t) = x_1$ for any $t \in [0,1]$.

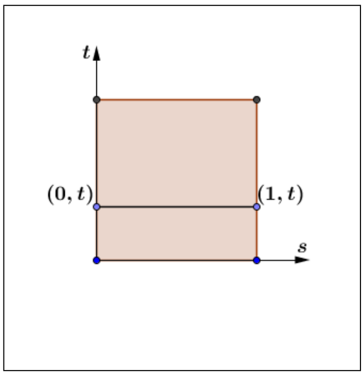


Figure 7.1

The second variable t is the variable of the deformation.

Definition 7.1.1. *The map F is called* **homotopy** *from γ_1 to γ_2.*

More generally, let $f, g : Y \longrightarrow X$ be two continuous maps. The map f is said to be **homotopic** to the map g if there exists a continuous map $F : Y \times [0,1] \longrightarrow X$ such that $F(y,0) = f(y)$ and $F(y,1) = g(y)$ for any $y \in Y$.
If the map g is a constant map, then the map f is said to be **null-homotopic**.
Let $Y = [0,1]$, we will be interested by some kinds of paths γ such that $\gamma(0) = x_0 = x_1 = \gamma(1)$. Such paths are called **loops** based at x_0.
For example, there exists a homotopy from any closed curve in the plane $\mathbb{R}^2$ to the constant map at the origin (and end) point of the curve, but there does not exist any homotopy from the unit circle with the origin point $(1,0)$ to the constant map in the punctered plane $\mathbb{R}^2 \setminus \{(0,0)\}$.
The homotopy relation between loops based at a given point, is an equivalence relation.

- ***Reflexivity***: γ is homotopic to γ; take $F(s,t) = \gamma(s)$.
- ***Symmetry***: γ_1 is homotopic to γ_2; so there exists a homotopy $F(s,t)$. Take $G(s,t) = F(s, 1-t)$ and we have γ_2 is homotopic to γ_1.
- ***Transitivity***: γ_1 is homotopic to γ_2 where F is the homotopy, and γ_2 is homotopic to γ_3 where G is the homotopy. Then
  $H(s,t) = \begin{cases} F(s,2t) & 0 \leq t \leq \frac{1}{2} \\ G(s,2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$ is a homotopy from γ_1 to γ_3.

We denote $[\gamma]$ the homotopy class of the loop γ.
Let $\gamma_i, i = 1,2$ be two loops based at $x_0 \in X$. The **product** $\gamma_1.\gamma_2$ is the loop given by

$$\gamma_1.\gamma_2(s) = \begin{cases} \gamma_1(2s) & 0 \leq s \leq \frac{1}{2} \\ \gamma_2(2s-1) & \frac{1}{2} \leq s \leq 1 \end{cases}$$

Notice that the product of loops is not the composition of the maps that has no meaning, a path being a map $[0,1] \longrightarrow X$. It traverses first γ_1 and then γ_2.
The set of loops based at a given point is not a group under the product, for example, the product is not associative.

Definition 7.1.2. *The set of homotopy classes of loops based at x_0 in the space X, is denoted $\pi_1(X; x_0)$.*

Remark 7.1.3. *A loop in X can be viewed as a continuous map $(\mathbb{S}^1;*) \longrightarrow (X;x_0)$. Then $\pi_1(X;x_0)$ is the set of homotopy classes of such continuous maps.*

Lemma 7.1.4. *Homotopy behaves well w.r.t. composition of maps*

Proof: If the loops based at x_0, γ_1 and γ_2, are homotopic, then $\gamma_1.\gamma$ and $\gamma_2.\gamma$ are homotopic as well as $\gamma.\gamma_1$ and $\gamma.\gamma_2$. ■

Theorem 7.1.5. *The set $\pi_1(X;x_0)$ is a group for the product $[\gamma_1].[\gamma_2] = [\gamma_1.\gamma_2]$.*
The neutral element is the class of the constant loop at x_0.
The inverse of the class $[\gamma]$ is the class of the loop γ^{-1} defined by

$$\gamma^{-1}(s) = \gamma(1-s)$$

Proof: ***Associativity***
Let $\gamma_i, i = 1,2,3$ be three loops based at x_0, we show that $[\gamma_1.\gamma_2].[\gamma_3] = [\gamma_1].[\gamma_2.\gamma_3]$.
The map F defined by

$$F(s,t) = \begin{cases} \gamma_1\left(\dfrac{4s}{t+1}\right) & 0 \le s \le \frac{1}{4}(t+1) \\ \gamma_2(4s-t-1) & \frac{1}{4}(t+1) \le s \le \frac{1}{4}(t+2) \\ \gamma_3\left(\dfrac{4s-t-2}{2-t}\right) & \frac{1}{4}(t+2) \le s \le 1 \end{cases}$$

is a homotopy from the loop $(\gamma_1.\gamma_2).\gamma_3$ to the loop $\gamma_1.(\gamma_2.\gamma_3)$ based at the same point x_0.
Explanation of the formulas above: $\gamma_1\left(\dfrac{4s}{t+1}\right)$, for $0 \le s \le \dfrac{1}{4}(t+1)$.
The condition $0 \le s \le \dfrac{1}{4}(t+1)$ can be written as $0 \le \dfrac{4s}{t+1} \le 1$. The other formulas are obtained similarly.

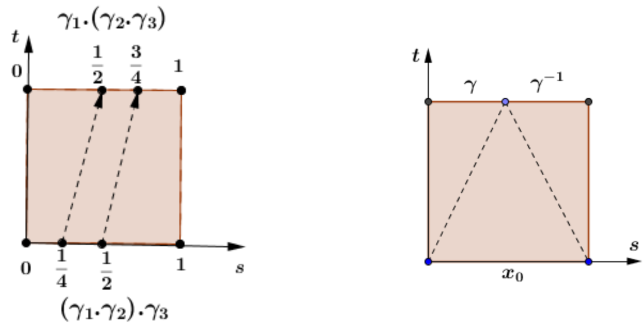


Figure 7.2

Neutral element
The map G defined by

$$G(s,t) = \begin{cases} \gamma\left(\dfrac{2s}{t+1}\right) & 0 \le s \le \frac{1}{2}(t+1) \\ x_0 & \frac{1}{2}(t+1) \le s \le 1 \end{cases}$$

Inverse elements
Let γ be a loop based at the point x_0. Then, the map H defined by

$$H(s,t) = \begin{cases} \gamma(2s) & 0 \leq 2s \leq t \\ \gamma(t) & t \leq 2s \leq 2-t \\ \gamma^{-1}(2s-1) & 2-t \leq 2s \leq 2 \end{cases}$$

is a homotopy from $\gamma.\gamma^{-1}$ to the constant x_0. ■

Definition 7.1.6. *The group $\pi_1(X;x_0)$ is called the* **fundamental group** *of X based at the point x_0.*

Remark 7.1.7. *The group $\pi_1(X;x_0)$ is not abelian, in general. However, in some cases, it is abelian.*

Is there any relation between $\pi_1(X;x_0)$ and $\pi_1(X;x_1)$, where x_0 and x_1 are two points of X?

Proposition 7.1.8. *Let α be a path from x_0 to x_1. The map $[\gamma] \longmapsto [\alpha^{-1}.\gamma.\alpha]$ is an isomorphism of the group $\pi_1(X;x_0)$ onto the group $\pi_1(X;x_1)$.*

Proof: (exercise) ■

Corollary 7.1.9. *If the space X is pathwise connected, the group $\pi_1(X;x_0)$ is independent of the point x_0 and it is called the* **fundamental group** *of the space X.*

Remark 7.1.10. *The fundamental group is the first group of a series of higher homotopy groups, which explains the notation π_1. These higher homotopy groups will be defined by replacing the circle $\mathbb{S}^1$ by the spheres $\mathbb{S}^n, n \in \mathbb{N}^*$.*

Definition 7.1.11. *A topological space is* **simply connected** *if it is pathwise connected and its fundamental group is trivial.*

7.1.3 The Functor π_1

Let X be a topological space, the category of paths in X, denoted $\mathcal{P}(X)$ has the points of X as objects and the morphisms from x to y are the homotopy classes of path from x to y. Let σ be a path from x to y, γ a path from y to z, then $[\sigma.\gamma] = [\sigma].[\gamma]$.

Let define the category of *pointed topological spaces* as the category with the objects $(X;x_0)$ where $x_0 \in X$ and the morphisms from $(X;x_0)$ to $(Y;y_0)$ are the maps $f : X \longrightarrow Y$ such that $f(x_0) = y_0$.
Let denote $\gamma \sim \gamma'$ when the two paths γ and γ' are homotopic. The compose $\gamma \longmapsto f \circ \gamma$ where γ is a path joining x to x' in X and $f\gamma$ is a path joining $f(x)$ to $f(x')$ in Y satisfies the following properties

- If $\gamma \sim \gamma'$, then $f \circ \gamma \sim f \circ \gamma'$ (homotopy of paths).
- $f \circ \gamma\gamma' \sim (f \circ \gamma)(f \circ \gamma')$ (compose of paths).

We define the functor π_1 from the category of pointed topological spaces $\mathcal{T}op_*$ to the category of groups $\mathcal{G}ps$ which sends the pointed space $(X;x)$ to the group $\pi_1(X;x)$ and the morphism $f : (X;x) \longrightarrow (Y;y)$ to the homomorphism of groups

$$\begin{array}{rcl} f_* : \pi_1(X;x) & \longrightarrow & \pi_1(Y;y) \\ [\gamma] & \longmapsto & [f \circ \gamma] = f_*([\gamma]) \end{array}$$

The continuous map f induces a homomorphism such that $f_*([\gamma\gamma']) = f_*([\gamma])f_*([\gamma'])$.
The identity map $Id : (X; x) \longrightarrow (X; x)$ induces the identity homomorphism Id_*.
For a continuous map $g : (Y; y) \longrightarrow (Z; z)$, we have $(g \circ f)_* = g_* \circ f_*$.

Example 7.1.12. *Let D_2 be the unit disk. There is no continuous map $r : D_2 \longrightarrow \mathbb{S}^1$ such that $r(p) = p$ for all $p \in \mathbb{S}^1$.*
Suppose there exists such map r. The point 1 will be the base point for both spaces D_2 and $\mathbb{S}^1$. Let $\iota : \mathbb{S}^1 \longrightarrow D_2$ be the inclusion map, thus $r \circ \iota = Id_{\mathbb{S}^1}$. Then $r_* \circ \iota_* = Id_{\pi(\mathbb{S}^1}$. So, let a be a generator of $\pi_1(\mathbb{S}^1)$, then $r_* \circ \iota_*(a) = a$.
We have $\iota_*(a) \in \pi_1(D_2)$ and D_2 is contractible, so $\pi_1(D_2)$ is trivial, so $\iota_*(a) = e$ and $r_* \circ \iota_*(a) = e \neq a$ and we get a contradiction.

7.1.4 Deformation Retracts

If $A \subset X$ is a retract of X, i.e. there is a continuous map $r : X \longrightarrow A$ such that $r(x) = x$ for all $x \in A$, then the induced homomorphism $\pi_1(A) \longrightarrow \pi_1(X)$ is injective. We have $r \circ \iota = Id_A$, i.e. r is a left inverse of the inclusion. For example, any point $x \in X$ is a retract of X.
$A \subset X$ is a retract of X iff for any space Y, any continuous map $A \longrightarrow Y$ extends to a continuous map $X \longrightarrow Y$.
A more interesting notion is the following.

Definition 7.1.13. *A* **deformation retract** *of a space X onto a subspace A is a family of maps $f_t : X \longrightarrow X, 0 \leq t \leq 1$, such that $f_0 = Id, f_1(X) = A$ and $f_{|A} = Id$ for all t. The family $X \times [0, 1] \longrightarrow X, (x, t) \longmapsto f_t(x)$ is continuous.*

A deformation retract of X onto A is a homotopy from the identity map of X to a retraction r of X onto A.

Proposition 7.1.14. *If A is a deformation retract of X, then the inclusion $\iota : A \hookrightarrow X$ induces an isomorphism from $\pi_1(A)$ onto $\pi_1(X)$ for any based point in A.*

Proof: From $r \circ \iota = Id_A$, we have $r_* \circ \iota_A = Id_{\pi_1(A)}$ and $\iota \circ r$ is homotopic to Id_X, so $\iota_* \circ r_* = Id_{\pi_1(X)}$. ■

Example 7.1.15. *The punctured plane $\mathbb{R}^2 \setminus \{(0, 0)\}$ retracts onto the unit circle. The retraction is given in polar coordinates by $F((e^s, \theta), t) = (e^{(1-t)s}, \theta)$.*

7.1.5 Examples

The definition of the fundamental group of a topological space is easy to understand, but its calculation could be difficult even for simple topological spaces.

Fundamental Group of the Real Line

Let 0 be the base point of $\mathbb{R}$, then $\pi_1(\mathbb{R}; 0) = 0$.
Define $F(s, t) = (1 - t)s$. Then F is a homotopy from the identity to the constant map at 0. Let γ be a loop at 0, define $G(s, t) = F(\gamma(s), t)$. The homotopy G shows that γ is homotopic to the constant map at 0.
The fundamental group of a disk is trivial. More generally, the fundamental group of any convex subset in $\mathbb{R}^n$ is trivial.

Fundamental Group of the Product Space

Let X, Y and Z be three topological spaces. Then the map $f = (f_X, f_Y) : Z \longrightarrow X \times Y$ is continuous iff the maps f_X and f_Y are continuous.
Let $Z = [0, 1]$, then f, f_X, f_Y are paths in $X \times Y, X, Y$ with the following properties:

- If f, g are two paths with $f(0) = g(0), f(1) = g(1)$, then f and g are homotopic iff f_X, g_X are homotopic , and f_Y, g_Y are homotopic.
- Let $h = f.g$, If f, g are two paths with $f(0) = g(0), f(1) = g(1)$, then $h_X = f_X.g_X$ and $h_Y = f_Y.g_Y$.

We assume that X and Y are two path-connected spaces.

Theorem 7.1.16. *The groups $\pi_1(X; x) \times \pi_1(Y; y)$ and $\pi_1(X \times Y; (x, y))$ are isomorphic.*

Proof: Let $(p_X)_* : \pi_1(X \times Y; (x, y)) \longrightarrow \pi_1(X; x)$ and $(p_Y)_* : \pi_1(X \times Y; (x, y)) \longrightarrow \pi_1(Y; y)$ denote the homomorphisms of groups induced by p_X and p_Y.
From the first property above, a loop f in $X \times Y$ based at (x, y) has two loops associated to it, $p_X \circ f$ and $p_Y \circ f$, which are continuous and properly based. Similarly, a homotopy of the loop translates to homotopies of the loops in X and Y. Thus there is a map

$$(p_X)_* \times (p_Y)_* : \pi_1(X \times Y; (x, y)) \longrightarrow \pi_1(X; x) \times \pi_1(Y; y)$$

which is a bijection and a homomorphism of groups from the second property above, so it is an isomorphism. ■

Fundamental Group of a Topological Group

The fundamental group of a topological space is not necessarily abelian, but when the topological space is a topological group, then it is abelian.
Let (G, τ) be a group G with a topology τ such that the group law $*$ is continuous as well as the inverse, i.e. the following maps are continuous

$$\begin{aligned} G \times G &\longrightarrow G \qquad & G &\longrightarrow G \\ (g, h) &\longmapsto g * h \qquad & g &\longmapsto g^{-1} \end{aligned}$$

Consider the map

$$\begin{aligned} F : [0,1] \times [0,1] &\longrightarrow \mathbb{R}^2 \\ (s, t) &\longmapsto \begin{cases} ((1-t)2s + st, st) & if \quad 0 \leq s \leq \frac{1}{2} \\ (st + 1 - t, (1-t)(2s-1) + st) & if \quad \frac{1}{2} \leq s \leq 1 \end{cases} \end{aligned}$$

The following figure shows the images of the horizontal segments $t = t_0$ for $0 \leq t_0 \leq 1$.

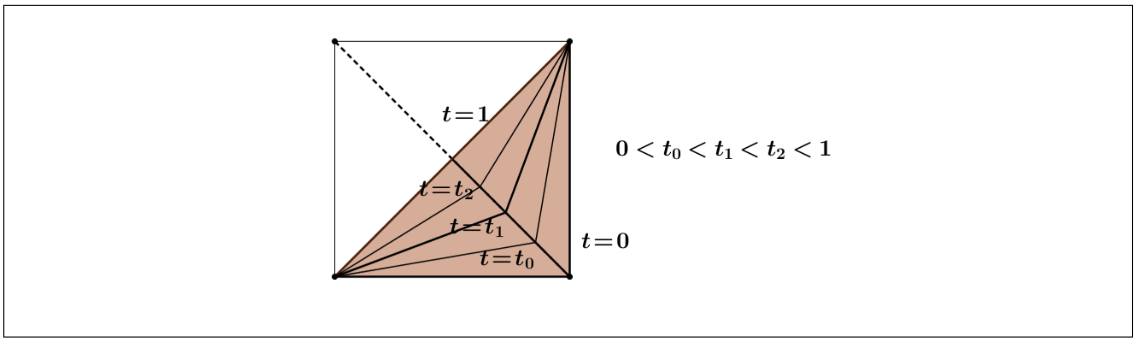


Figure 7.3

The set $\{F(s,0), 0 \leq s \leq 1\}$ is the union of the segment $[(0,0),(1,0)$ with the segment $[(1,0),(1,1)]$. The set $\{F(s,1), 0 \leq s \leq 1\}$ is the diagonal of the square, and the set $\{F(s,t_0), 0 \leq s \leq 1\}$ is the union of the segment $\left[(0,0), \left(1-\frac{t_0}{2}, \frac{t_0}{2}\right)\right]$ with the segment $\left[\left(1-\frac{t_0}{2}, \frac{t_0}{2}\right), (1,1)\right]$.

Proposition 7.1.17. *Let $((G,*),\tau)$ be a topological group and e its neutral element. Then the fundamental group $\pi_1(G;e)$ is an abelian group.*

Proof: Let γ and γ' be two paths in G, based at the neutral element e. Consider the map

$$\begin{aligned} M : [0,1] \times [0,1] &\longrightarrow G \\ (u,v) &\longmapsto \gamma(u) * \gamma'(v) \end{aligned}$$

Consider the loop $M_{|\{(u,u)\}} = \{\gamma(u) * \gamma'(u) \mid 0 \leq u \leq 1\}$, restriction to the diagonal.
There exists a homotopy H from the loop $\gamma.\gamma'$ to the loop $M_{|\{(u,u)\}}$:

$$H(s,-) = \begin{cases} H(s,0) & = \begin{cases} \gamma(2s) * \gamma'(0) = \gamma(2s) & \text{for} \quad 0 \leq s \leq \frac{1}{2} \\ \gamma(0) * \gamma'(2s-1) = \gamma'(2s-1) & \text{for} \quad \frac{1}{2} \leq s \leq 1 \end{cases} \\ H(s,t_0) & = \begin{cases} \gamma((1-t_0)2s + st_0) * \gamma'(st_0) & \text{for} \quad 0 \leq s \leq \frac{1}{2} \\ \gamma(st_0 + 1 - t_0) * \gamma'((1-t_0)(2s-1) + st_0) & \text{for} \quad \frac{1}{2} \leq s \leq 1 \end{cases} \\ H(s,1) & = \quad \gamma(s) * \gamma'(s) \quad \text{for} \quad 0 \leq s \leq 1. \end{cases}$$

It is clear that H is a continuous map. Moreover, $H(-,0) = \gamma.\gamma'$ and $H(-,1) = \gamma(-) * \gamma'(-)$, so it is a homotopy.
Similarly, we have a homotopy from the loop $\gamma'.\gamma$ to the loop $M_{|\{(u,u)\}}$. Then, the loops $\gamma.\gamma'$ and $\gamma'.\gamma$ are homotopic and the fundamental group $\pi_1(G;e)$ is abelian. ■

7.2 Homotopy Equivalence of Spaces

7.2.1 Homotopy of Maps

Recall the definition of the homotopy of maps given in 7.1.2. Two continuous maps $f, g : X \longrightarrow Y$ are said to be homotopic if there exists a continuous map

$$\begin{aligned} F : X \times [0,1] &\longrightarrow Y \\ (x,t) &\longmapsto F(x,t) \end{aligned}$$

such that $\begin{cases} F(x,0) = f(x) \\ F(x,1) = g(x) \end{cases}$ for any $x \in X$.

Exercice 7.2.1. *Let f and g be two maps of a singleton to the space X where*

1. $X = \mathbb{R}^2 \setminus (\mathbb{Q} \times \{0\})$. *Are these two maps homotopic?*
2. $X = \mathbb{R}^3 \setminus \{(x,y,z) \mid z = 0\}$. *Are these two maps homotopic?*
3. $X = \mathbb{R}^3 \setminus \{(x,y,z) \mid y = 0, z = 0\}$. *Are these two maps homotopic?*

7.2.2 Homotopy Equivalence

Two topological spaces X and Y are of the **same homotopy type** if there exist continuous maps, called **homotopy equivalences**, $f : X \longrightarrow Y, g : Y \longrightarrow X$, such that $g \circ f$ and $f \circ g$ are homotopic to the identity maps.

Proposition 7.2.2. *Let X and Y be two pathwise connected spaces. If they are homotopically equivalent, then their fundamental groups are isomorphic.*

7.2.3 Homotopy & Homeomorphism

Proposition 7.2.3. *Homeomorphic spaces are homotopy equivalent.*

Proof: Let $f : X \longrightarrow Y$ be a homeomorphism, then $f \circ f^{-1} = Id_Y$ and $f^{-1} \circ f = Id_X$. ■
The converse is not true, as the following example shows.

Example 7.2.4.

- *The space $\mathbb{R}$ is homotopic to a point but they are not homeomorphic.*
- *$\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for $n \neq 2$.*
  Suppose $\mathbb{R}^2 = \mathbb{R}^n$. If $n = 1$, then $\mathbb{R}^2 \setminus \{p\}$ is path-connected but $\mathbb{R}^n \setminus \{q\}$ is not path-connected when $n = 1$. When $n > 2$, the same argument does not work, but consider $\pi_1(\mathbb{R}^n \setminus \{p\})$. We can prove that $\mathbb{R}^n \setminus \{p\}$ is homotopy equivalent to $\mathbb{S}^{n-1} \times \mathbb{R}$. We have $\pi_1(\mathbb{R}^n \setminus \{p\}) = \mathbb{Z}$ for $n = 2$ and trivial for $n > 2$.

As a consequence of this result, there are fewer invariants and many homeomorphism invariants are not homotopy invariants. For example, compactness, connectedness after removing one or more points.

Definition 7.2.5. *A topological space is said to be* **contractible** *if it is homotopically equivalent to a point, or equivalently, if its identity map is null-homotopic.*

Example 7.2.6.

1. *$\mathbb{R}^n$ and $\mathbb{C}^n$ are contractible.*
2. *Any convex subset of $\mathbb{R}^n$ is contractible.*
3. *Any star-shaped set is contractible.*

Proposition 7.2.7. *A contractible space is simply connected.*

Proof: A contractible space has the homotopy type of a point, so its fundamental group is isomorphic to the fundamental group of a space which is a singleton and such a space has only one loop, the constant loop. Then the fundamental group is trivial. ■

Notice that the converse is not true. For example, the sphere $\mathbb{S}^2$ is simply connected but it is not contractible.

7.3 The Seifert-Van Kampen Theorem

The Seifert[1]-Van Kampen[2] theorem gives a method to compute the fundamental group of a space knowing the fundamental groups of some subspaces.
Let X be a topological space, U, V two open sets such that $U \cap V \neq \emptyset, U \cup V = X$, and suppose all spaces are arcwise connected and they have the same based point x_0 which will be omitted for simplicity. Then the inclusion maps in the left diagram induce the following commutative right diagram:

$$\begin{tikzcd} U \cap V \arrow[r, hook] \arrow[d, hook] \arrow[dr] & V \arrow[d, hook] \\ U \arrow[r, hook] & X \end{tikzcd} \qquad \begin{tikzcd} \pi_1(U \cap V) \arrow[r, "\iota_V"] \arrow[d, "\iota_U"'] \arrow[dr, "p"] & \pi_1(V) \arrow[d, "p_V"] \\ \pi_1(U) \arrow[r, "p_U"'] & \pi_1(X) \end{tikzcd}$$

[1]Herbert Karl Johannes Seifert (May 27, 1907, Bernstadt - October 1, 1996, Heidelberg), was a German mathematician known for his work in topology.
[2]Egbert Rudolf van Kampen (28 May 1908, Berchem, Belgium - 11 February 1942, Baltimore, Maryland) was a mathematician. He made important contributions to topology, especially to the study of fundamental groups.

Theorem 7.3.1. *Let G be a group and $p : \pi_1(U \cap V) \longrightarrow G$, $p_U : \pi_1(U) \longrightarrow G$ and $p_V : \pi_1(V) \longrightarrow G$ three homomorphisms such that the following diagram is commutative:*

$$\begin{tikzcd}
\pi_1(U \cap V) \arrow[r, "\iota_V"] \arrow[d, "\iota_V"'] \arrow[dr, "p"] & \pi_1(V) \arrow[d, "p_V"] \\
\pi_1(U) \arrow[r, "p_U"] & G
\end{tikzcd}$$

Then, there is a unique homomorphism $\pi_1(X) \longrightarrow G$ such that the following diagram is commutative:

$$\begin{tikzcd}
\pi_1(U \cap V) \arrow[r] \arrow[d] \arrow[dr] \arrow[ddrr, bend right, "p"] & \pi_1(V) \arrow[d, "j_V"] \arrow[ddr, bend left, "p_V"] & \\
\pi_1(U) \arrow[r, "j_U"'] \arrow[drr, bend right, "p_U"'] & \pi_1(X) \arrow[dr, dotted] & \\
 & & G
\end{tikzcd}$$

i.e. $\pi_1(X)$ is a fibered sum, i.e. an amalgamated free product $\pi_1(X) \cong \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V)$.

Proof: The amalgamated free product is given by the diagram:

$$\begin{tikzcd}
\pi_1(U \cap V) \arrow[r] \arrow[d] \arrow[dr] & \pi_1(V) \arrow[d] \\
\pi_1(U) \arrow[r] & \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V)
\end{tikzcd}$$

and for any group G such that the diagram is commutative

$$\begin{tikzcd}
\pi_1(U \cap V) \arrow[r] \arrow[d] \arrow[dr] & \pi_1(V) \arrow[d, "p_V"] \\
\pi_1(U) \arrow[r, "p_U"'] & G
\end{tikzcd}$$

In particular for the group $G = \pi_1(X)$, the following diagram is commutative

$$\begin{tikzcd}
\pi_1(U \cap V) \arrow[r] \arrow[d] \arrow[dr] & \pi_1(V) \arrow[d] \\
\pi_1(U) \arrow[r] & \pi_1(X)
\end{tikzcd}$$

and there exists a unique homomorphism Φ making commutative the diagram

$$\begin{tikzcd}
\pi_1(U \cap V) \arrow[r] \arrow[d] \arrow[dr] \arrow[ddrr, bend right] & \pi_1(V) \arrow[d] \arrow[ddr, bend left] & \\
\pi_1(U) \arrow[r] \arrow[drr, bend right] & \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V) \arrow[dr, dotted, "\Phi"] & \\
 & & \pi_1(X)
\end{tikzcd}$$

Let G be any group. We have to construct a unique homomorphism $\psi : \pi_1(X) \longrightarrow G$ such that the following diagrams commute. Let $G = \pi_1(U) *_{\pi_1(U\cap V)} \pi_1(V)$, then we get Φ is an isomorphism and $\Phi = \psi^{-1}$.

$$\begin{tikzcd}
\pi_1(U\cap V) \arrow[r] \arrow[d] \arrow[dr] \arrow[ddrr, bend right] & \pi_1(V) \arrow[d] \arrow[ddr, bend left] & \\
\pi_1(U) \arrow[r] \arrow[drr, bend right] & \pi_1(X) \arrow[dr, dotted, "\psi"] & \\
& & G
\end{tikzcd}
\qquad
\begin{tikzcd}
\pi_1(U\cap V) \arrow[r] \arrow[d] \arrow[dr] \arrow[ddrr, bend right] & \pi_1(V) \arrow[d] \arrow[ddr, bend left] & \\
\pi_1(U) \arrow[r] \arrow[drr, bend right] & \pi_1(X) \arrow[dr, dotted, "\Phi^{-1}"] & \\
& & \pi_1(U) *_{\pi_1(U\cap V)} \pi_1(V)
\end{tikzcd}$$

The point $x_0 \in U \cap V$ is the base point and $\gamma : [0,1] \longrightarrow X = U \cup V$ be a loop based at x_0. Then γ is homotopic to a product of loops based at x_0 contained in either U or V.
By the compactness of $[0,1]$, there is $0 = s_0 < s_1 < \cdots < s_{m-1} < s_m = 1$ such that $\gamma([s_i, s_{i+1}]) \subset U$ or V and $\gamma(s_i) \in U \cap V$ for any $i = 0, \dots, m-1$. Let γ_i denote the path from $\gamma(s_i)$ to $\gamma(s_{i+1})$ where $\gamma_i(s) = \gamma(s_i + s(s_{i+1} - s_i)), 0 \leq s \leq 1$. Then $\gamma = \gamma_0.\gamma_1.\cdots.\gamma_{m-1}$ (concatenation of paths).
$U \cap V$ is path-connected, so there are paths $\sigma_i, i = 1, \dots, m$, contained in $U \cap V$ from x_0 to $\gamma(s_{i-1})$. The loop

$$(\gamma_0.\sigma_1^{-1}).(\sigma_1.\gamma_1.\sigma_2^{-1}).\cdots.(\sigma_{m-1}.\gamma_{m-1}.\sigma_m^{-1}).(\sigma_m.\gamma_m) : [0,1] \longrightarrow X$$

is homotopic to γ, with for any $i, [\sigma_{i-1}.\gamma_{i-1}.\sigma_i^{-1}] \in \pi_1(U)$ or $\pi_1(V)$. Then

$$[\gamma] = [\gamma_0.\sigma_1^{-1}][\sigma_1.\gamma_1.\sigma_2^{-1}].\cdots.[\sigma_{m-1}.\gamma_{m-1}.\sigma_m^{-1}][\sigma_m.\gamma_m] \in \phi(\pi_1(U) * \pi_1(V)) \subset \pi_1(X)$$

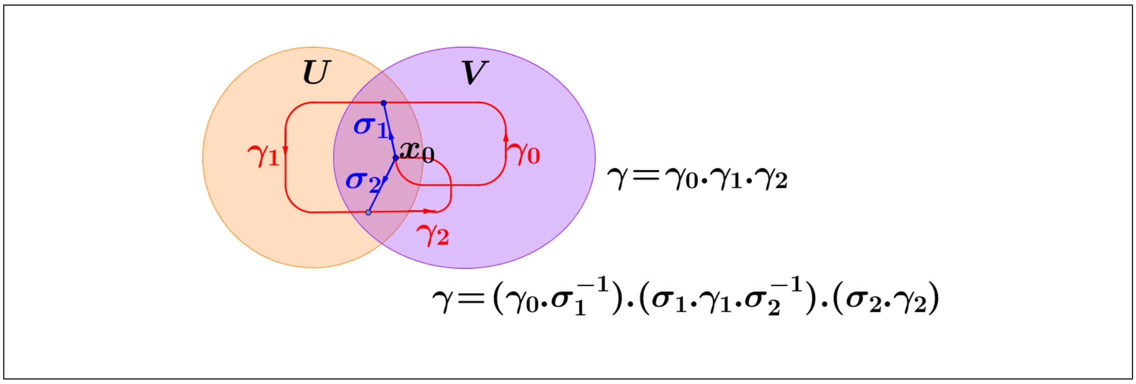


Figure 7.4

We define the map $\psi : \pi_1(X) \longrightarrow G$ by

$$\psi([\gamma]) := p_*([\gamma_0\sigma_1^{-1}])p_*([\sigma_1.\gamma_1.\sigma_2^{-1}])\cdots p_*([\sigma_{m-1}.\gamma_{m-1}.\sigma_m^{-1}])p_*([\sigma_m.\gamma_m])$$

where p_* means either p_U or p_V depending on whether $[\sigma_{i-1}.\gamma_{i-1}.\sigma_i^{-1}], i = 1, \dots, m$ belongs to $\pi_1(U)$ or $\pi_1(V)$. However, ψ is a well defined map if we have the following properties:

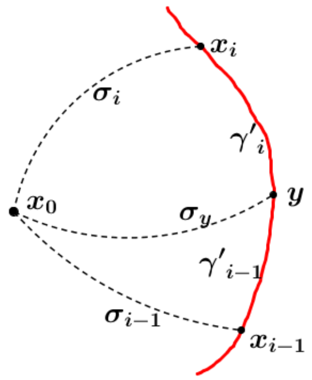
1. if $\sigma_{i-1}.\gamma_{i-1}.\sigma_i^{-1}$ lies in both U and V, we can choose either p_U or p_V. But $\sigma_{i-1}.\gamma_{i-1}.\sigma_i^{-1}$ lies in $\pi_1(U \cap V)$ and the commutativity of the following diagram shows that we get the same result because $p_U \circ \iota_U = p_V \circ \iota_V = p$.

$$\begin{tikzcd} \pi_1(U \cap V) \arrow[r, "\iota_V"] \arrow[d, "\iota_U"'] \arrow[dr, "p"] & \pi_1(V) \arrow[d, "p_V"] \\ \pi_1(U) \arrow[r, "p_U"'] & G \end{tikzcd}$$

2. The construction of ψ must not depend on the choice of the points $\gamma(s_i)$ that we will denote x_i.
   Suppose we add another point y along γ_i defining the two new paths γ'_{i-1} and γ'_i. Let σ_y be a path in $U \cap V$ joining y and x_0. Suppose that the loop $\sigma_{i-1}.\gamma_i.\sigma_i^{-1}$ is contained in U, then the same is true for the two new loops $\sigma_{i-1}.\gamma'_i.\sigma_y^{-1}$ and $\sigma_y.\gamma'_i.\sigma_i^{-1}$. We have

$$p_*([\sigma_{i-1}.\gamma'_{i-1}.\sigma_y^{-1}]).p_*([\sigma_y.\gamma'_i.\sigma_i^{-1}]) = p_*([\sigma_{i-1}.\gamma'_{i-1}.\sigma_y^{-1}.\sigma_y.\gamma'_i.\sigma_i^{-1}]) = p_*([\sigma_{i-1}.\gamma_i.\sigma_i^{-1}])$$

   where $p_* : \pi_1(W) \longrightarrow \pi_1(X)$ and W is either U or V or $U \cap V$.
   So, adding the point y does not change the value of $\psi([\gamma])$ and it will be the same if we add finitely many points. Thus, ψ is independent of our choice of subdivision and hence, it is well-defined.

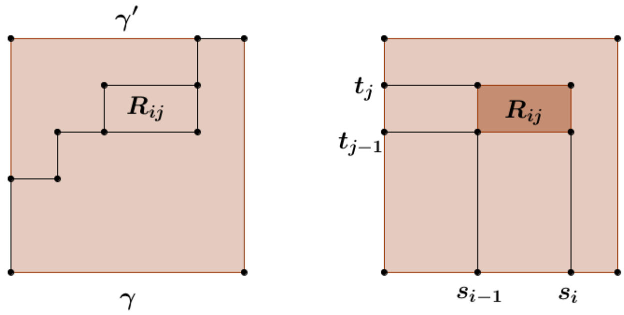


3. We have to check that if γ' is another loop homotopic to γ, then $\psi([\gamma]) = \psi([\gamma'])$.
   Let $H : [0,1] \times [0,1] \longrightarrow X$ be the homotopy realizing $\gamma \sim \gamma'$, i.e.

$$\begin{aligned} H(s,0) &= \gamma(s) \\ H(s,1) &= \gamma'(s) \\ H(0,t) = H(1,t) &= x_0 \end{aligned}$$

   By the Lebesgue Covering lemma[3], we can subdivide $[0,1] \times [0,1]$ into rectangles $R_{ij} = [s_{i-1}, s_i] \times [t_{j-1}, t_j]$ where $0 = s_0 < s_1 < \cdots < s_n = 1$ and $0 = t_0 < t_1 < \cdots < t_m = 1$ such that either $H(R_{ij}) \subset U$ or $H(R_{ij}) \subset V$.

[3]Let (X,d) be a compact metric space and an open cover of X. Then, there exists a number $\delta > 0$ such that every subset of X having a diameter $< \delta$, is contained in some element of the cover



Let γ_i, (resp. γ_i'), denote the restriction of γ (resp. γ') to $[s_{i-1}, s_i]$. The path γ_i need not be loop. Let $x_i = \gamma_i(s_i)$ for any i, then we defined the paths σ_i in either U or V and the loop $\sigma_{i-1}\gamma_i\sigma_i$.
Consider the effect of adding the rectangle R_{ij}.
Let h_{ij} be the path associated with the horizontal edge from (s_{i-1}, t_j) to (s_i, t_j), i.e. $h_{ij}(s) = H((1-s)s_{i-1} + ss_i, t_j)$, and v_{ij} the path associated with the vertical edge from (s_i, t_{j-1}) to (s_i, t_j), i.e. $v_{ij}(t) = H(s_i, (1-t)t_{j-1} + tt_j)$. Then in either U or V,we have $h_{ij-1}v_{ij}$ homotopic to $v_{i-1j}h_{ij}$, which gives the equalities $[h_{ij-1}][v_{ij}] = [v_{i-1j}][h_{ij}]$in either $\pi_1(U)$ or $\pi_1(V)$.
Hence, the value of ψ is unchanged.

4. We have to prove that $\psi([\gamma\gamma']) = \psi([\gamma)][\psi(\gamma'])$.
Suppose that the subdivision $0 = s_0 < s_1 < \cdots < s_n = 1$ is such that for some $k, s_k = \frac{1}{2}$.
Then, for $i = 0, \dots, k$,

$$s \in [0,1] \longmapsto u = (1-s)2s_{i-1} + s.2s_i \in [2s_{i-1}, 2s_i] \longmapsto \gamma(u) := \gamma_i(s)$$

and for $i = k+1, \dots, n$,

$$s \in [0,1] \longmapsto u = (1-s)(2s_{i-1} - 1) + s(2s_i - 1) \in [2s_{i-1} - 1, 2s_i - 1] \longmapsto \gamma'(u) := \gamma'_{i-k}(s)$$

Using this subdivision for the domain of $\gamma\gamma'$, we have

$$\psi([\gamma\gamma']) = \psi([\gamma_1]) \cdots \psi([\gamma_k]).\psi([\gamma'_1]) \cdots \psi([\gamma'_{n-k}])$$

Using the subdivision $2s_0, 2s_1, \dots, 2s_k$ for the path γ we have

$$\psi([\gamma]) = \psi([\gamma_1]) \cdots \psi([\gamma_k])$$

and using the subdivision $2s_k - 1, 2s_{k+1} - 1, \dots, 2s_n - 1$ for the path γ' we have

$$\psi([\gamma']) = \psi([\gamma'_1]) \cdots \psi([\gamma'_{n-k}])$$

Thus, we have $\psi([\gamma\gamma']) = \psi([\gamma]).\psi([\gamma'])$.

■

Remark 7.3.2. *The condition "$U \cap V$ is path-connected" is neceesary. For example, take $X = \mathbb{S}^1$ and $U = \mathbb{S}^1 \setminus \{1\}, V = \mathbb{S}^1 \setminus \{-1\}$. Both U and V are contractible, so their images in $\pi_1(X)$ are the trivial group. If the theorem applied, we conclude that $\pi_1(\mathbb{S}^1)$ is trivial, which is not because the space $U \cap V = \mathbb{S}^1 \setminus \{-1, 1\}$ is disconnected..*

Corollary 7.3.3. *If $U \cap V$ is simply connected, then $\pi_1(X)$ is the free product $\pi_1(U) * \pi_1(V)$ w.r.t. to the homomorphisms induced by the inclusion maps.*

Proof: As $\pi_1(U \cap V) = 0$, $U \cap V$ being simply connected, the amalgamated free product is a free product, i.e. $\pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V) = \pi_1(U) * \pi_1(V)$. ■

Corollary 7.3.4. *If V is simply connected,, and N is the normal subgroup generated by the image of $\pi_1(U \cap V)$in $\pi_1(U)$, then $\pi_1(X) \cong \pi_1(U)/N$.*

Example 7.3.5.

1. *The sphere $\mathbb{S}^n, n \geq 2$ is simply connected.*
   Let $U = \mathbb{S}^n \setminus \{n\}, V = \mathbb{S}^n \setminus \{s\}$ where n is the north pole and s the south pole. Thne U and V are contractible, so simply connected, $U \cap V$ is path-connected, and we have the result. Notice that if $n = 1$, then $U \cap V$ fails to be path-connected and the Seifert-Van Kampen theorem does not apply.

2. *Consider the space X formed by the number 8, i.e. two circles intersecting in one point. Then $\pi_1(X) = \mathbb{Z} * \mathbb{Z}$. Notice that it is a free group with two generators.*

3. *Consider the space X formed by three circles intersecting in one point. Then $\pi_1(X) = \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$ is a free group with three generators.*

The Seifert-Van Kampen theorem is a powerful tool to determine the fundamental group of some topological spaces.
Conversely, a "natural" question is the following: given a group, is there any topological space whose the fundamental group is the given group?
The answer is yes, let be a group G given by generators and relations, there is a topological space X such that $\pi_1(X) = G$ (cf. Hatcher's book, p.52).

7.4 Isotopy

A *homotopy* is a continuous one-parameter family of continuous functions.
An **isotopy** is a continuous one-parameter family of homeomorphisms.

Remark 7.4.1. *The words homotopy and isotopy have two parts; "topo" for one-parameter family, and "homo" for homomorphism or simply morphism, "iso" for isomorphism.*

Definition 7.4.2. *Let $f, g : X \longrightarrow Y$ be two homeomorphisms, they are called* **isotopic** *if they can be joined by a homotopy F (the* **isotopy***) which is a homeomorphism $F(t, .)$ for every $t \in [0, 1]$.*

Notice that the fact that X and Y are homeomorphic depends on X and Y themselves and not on their disposition in the space, which is not true for isotopy. For example, given a long rectangle, then glue the short opposite sides, first to obtain a cylinder, and secondly to obtain a Mobius strip. These two surfaces are homeomorphic but not isotopic.
Let take another example. The map $f : [-1, +1] \longrightarrow [-1, +1]$ defined by $f(x) = -x$ is not isotopic to identity, but it is homotopic to identity. The homotopy is given by the map $F : [-1, +1] \times [0, 1] \longrightarrow [-1, +1]$ given by $F(x, t) = 2tx - x$. It is clearly a homotopy from the function $x \longmapsto -x$ to Identity. But F is not a isotopy (for $0 < t < 1$, the function F is not a homeomorphism). However, it is not a proof. We have to notice that the function Id preserves the orientation and the function f reverse the orientation, that implies they are not isotopic.
Another example is two circles in $\mathbb{R}^3$ which are not linked, we call them X, and which are linked,

we call them Y. Then X and Y are homeomorphic but not isotopic.

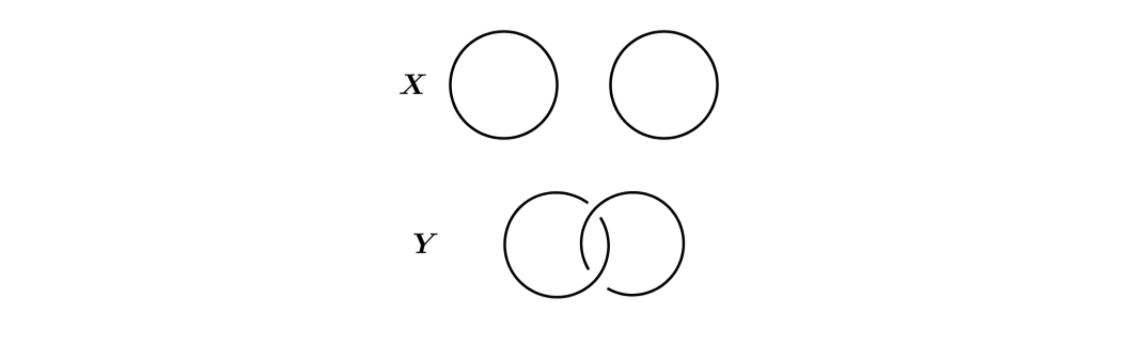


Figure 7.5

However, they are isotopic in $\mathbb{R}^4$.
In the figure below, X and Y are not isotopic in $\mathbb{R}^2$, but they are isotopic in $\mathbb{R}^3$.

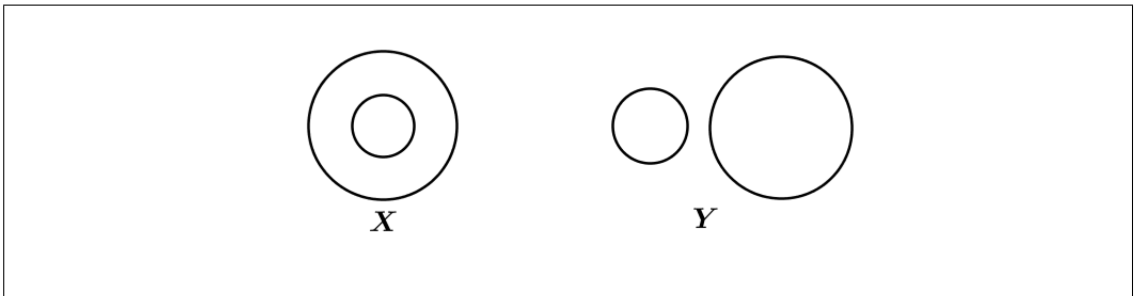


Figure 7.6

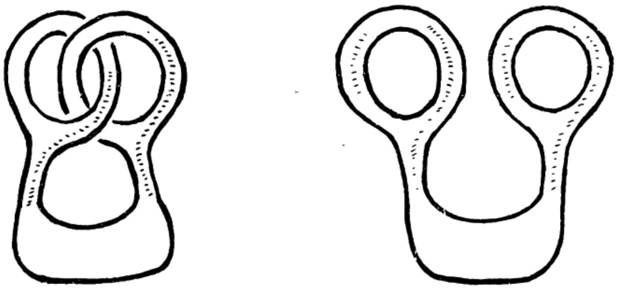
A disk with the center deleted and a circle are homotopic (as exercise, define the homotopy), but they are not isotopic, the disk is a surface and the circle is a curve, and a isotopy is one-to-one and onto.
More generally, two subsets X and Y of the topological space Z are isotopic if they can be distorted one into the other within Z without any cutting (breaking, tearing) or gluing.
Summary

- Two sets are isotopic if they can be distorted into each other without any cutting or tearing.
- Two sets are homeomorphic if there is a bijection from one to the other which is continuous and its inverse is continuous.
- If two sets are isotopic then they are homeomorphic. If two sets are homeomorphic, they are not necessarily isotopic.

7.4.1 Exercises

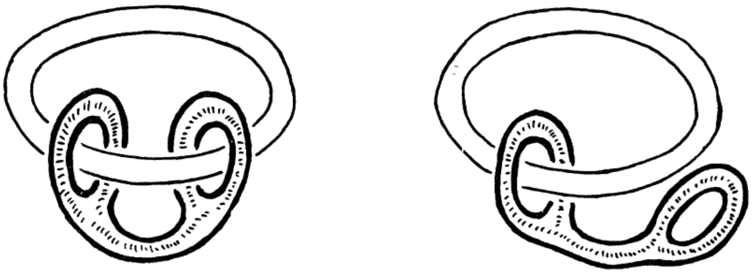
All the spaces are in $\mathbb{R}^3$.

1. Show that the two spaces are isotopic.



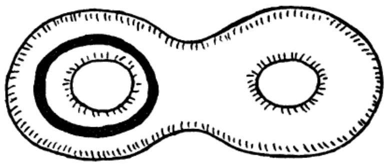
(a) (b)

2. Show that the two spaces are isotopic.

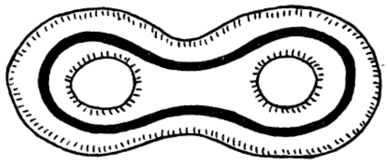


(a) (b)

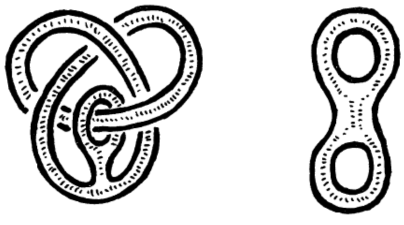
3. Show that the two spaces are isotopic.



(a)

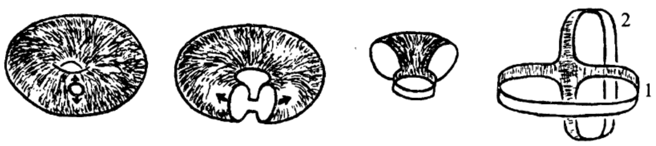


4. Show that the two spaces are isotopic.



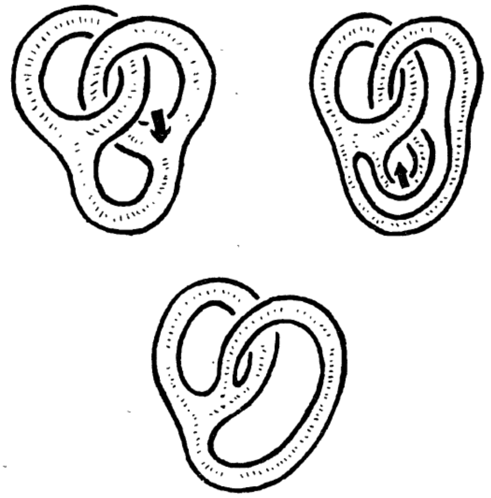
(a) (b)

5. Show that the spaces are isotopic.

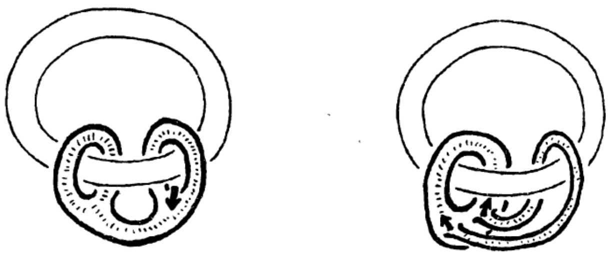


Solutions.

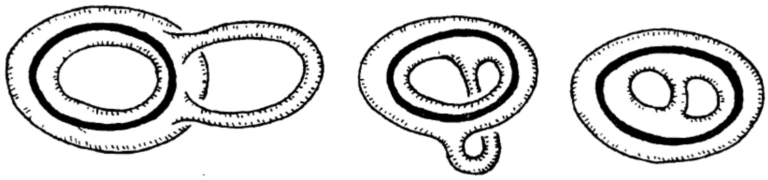
1.



2.



3.



4.

Chapter 8

Covering Spaces

8.1 Introduction

Many functions are "multivalued" and as such they are not functions and they have special properties. Riemann[1] introduced the so called Riemann surfaces on which the multivalued functions become functions.
Let us give some examples. A complex number z is defined by its module and its argument $arg(z)$. The function arg is defined and multivalued for $z \neq 0$. It is $\theta + 2k\pi$ where $0 \leq \theta < 2\pi$ and $k \in \mathbb{Z}$. To $z \in \mathbb{C}^*$, we associate infinitely values $arg(z)$. So, consider $(z, arg(z)) \in \mathbb{C}^* \times \mathbb{R} \subset \mathbb{R}^3$ where we identify $\mathbb{C}$ and $\mathbb{R}^2$. Let define $\widetilde{\mathbb{C}}^* = \{(z, arg(z)) \mid z \in \mathbb{C}^*\}$. $\widetilde{\mathbb{C}}^*$ is an infinite spiral wrapping round the vertical axis in a right-handed fashion.
Let us give some properties of the space $\widetilde{\mathbb{C}}^*$.

1. There is a map $p : \widetilde{\mathbb{C}}^* \longrightarrow \mathbb{C}^*, \ (z, arg(z)) \longmapsto z$.
2. For any $z \in \mathbb{C}^*, p^{-1}(z)$ is a discrete space.
3. For any $z \in \mathbb{C}^*$, there is a neighbourhood U_z such that $p^{-1}(U_z)$ is an infinite number of disjoint open sets $\widetilde{U}_n$ homeomorphic to U_z for any $n \in \mathbb{Z}$.
4. The group $\mathbb{Z}$ acts on $\widetilde{\mathbb{C}}^*$ by $n.(z, \theta) = (z, \theta + 2n\pi)$.
5. The space $\widetilde{\mathbb{C}}^*$ is simply connected.

A second example is given by the "function" $z \longmapsto z^k$ where $k = 1/2$. For any $z = \rho e^{i\theta} \in \mathbb{C}^*$, there are two values $z_1 = \sqrt{\rho} e^{i\frac{\theta}{2}}$ and $z_2 = \sqrt{\rho} e^{i\left(\frac{\theta}{2} + \pi\right)}$ such that $z_1^2 = z_2^2 = z$.
We define a space, called S, like a spiral with two twists such that over each point $z \neq 0$, there are two points in S over z. More precisely, there is a map $p : S \longrightarrow \mathbb{C}^*$ with the following properties:

[1] Georg Friedrich Bernhard Riemann (17 September 1826 - 20 July 1866) was a German mathematician who made contributions to analysis, number theory, and differential geometry. In the field of real analysis, he is mostly known for the first rigorous formulation of the integral, the Riemann integral, and his work on Fourier series. His contributions to complex analysis include most notably the introduction of Riemann surfaces, breaking new ground in a natural, geometric treatment of complex analysis. His famous 1859 paper on the prime-counting function, containing the original statement of the Riemann hypothesis, is regarded, although it is his only paper in the field, as one of the most influential papers in analytic number theory. Through his pioneering contributions to differential geometry, Bernhard Riemann laid the foundations of the mathematics of general relativity.

1. for each point $z \neq 0$ and not on the x-axis, there is a neighbourhood $U_z \subset \mathbb{C}^*$ such that $p^{-1}(U_z)$ is a disjoint union of two copies U_z^+, U_z^- of U_z where

$$U_z^+ = \{(z', u) \in \mathbb{C}^2 \mid z' \in U_z,\ u^2 = z',\ 0 \leq arg(z') < \pi\}$$

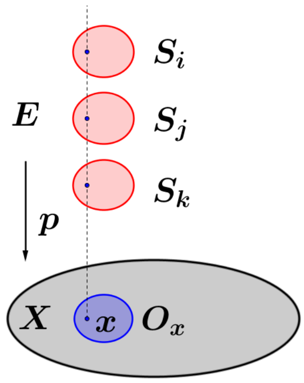
$$U_z^- = \{(z', u) \in \mathbb{C}^2 \mid z' \in U_z,\ u^2 = z',\ \pi \leq arg(z') < 2\pi\}$$

2. There is an action of the quotient group $\pi_1(\mathbb{C}^*)/2\pi_1(\mathbb{C}^*) = \mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ on S where $\bar{0}.(z,u) = (z,u)$ and $\bar{1}.(z,u) = (z,-u)$.

3. The space S is connected, its fundamental group is isomorphic to $\mathbb{Z}$ and the induced map $p_* : \pi_1(S) \longrightarrow \pi_1(\mathbb{C}^*)$ is multiplication by 2.

8.2 Definition & Examples

Definition 8.2.1. *A continuous map $p : E \longrightarrow X$ is a* **covering map** *and E is a* **covering space** *of X, if every $x \in X$ has an open neighbourhood O_x such that $p^{-1}(O_x)$ is a disjoint union of open sets $S_i, i \in I$ in E called* **sheets** *over O_x, each of which is mapped homeomorphically onto O_x by p. Such O_x is said to be* **evenly covered**.



So, a covering of the space X is given by a space E, a continuous map $p : E \longrightarrow X$ with the property of *local trivialization*, i.e. for any $x \in X$, there is a neighbourhood O_x, a discrete space $F \neq \emptyset$ and a homeomorphism $p^{-1}(O_x) \longrightarrow O_x \times F$ such that the following diagram is commutative:

$$\begin{tikzcd} p^{-1}(O_x) \arrow[rr, "\approx"] \arrow[dr, "p"'] & & O_x \times F \arrow[dl, "proj"] \\ & O_x & \end{tikzcd}$$

Definition 8.2.2. *A continuous map $f : X \longrightarrow Y$ is a* **local homeomorphism** *if each point $x \in X$ has an open neighbourhood O_x such that $f(O_x)$ is open in Y and $f_{|O_x}$ is a homeomorphism onto $f(O_x)$.*

Some properties

- For $x \in X$, the **fiber** over x, $p^{-1}(x)$, is discrete and if X is connected, all the fibers have the same cardinality and p is surjective.

- p is a local homeomorphism, but a local homeomorphism is not a covering map. (cf ex 8.2.3.2)
- p is an open map.
- p maps E onto X and X has the quotient topology from E.
- If E is compact (resp. connected, resp. path-connected), then X is compact (resp. connected, resp. path-connected).
- If X is Hausdorff, then E is Hausdorff.

Example 8.2.3. *1. $p : E \longrightarrow X$ a homeomorphism.*

2. $E = X \times Y$ where Y is discrete, and p is the projection onto the first factor.

3. $\exp : \mathbb{R} \longrightarrow \mathbb{S}^1,\ t \longmapsto \exp(2i\pi t)$.

8.2.1 Exercises

1. Prove the properties above.
2. A local homeomorphism such that all fibers are finite with the same cardinality, is a covering map.
3. Let $p : E \longrightarrow X$ be a covering map. If $Y \subset X$, prove that $p_{|p^{-1}(Y)} : p^{-1}(Y) \longrightarrow Y$ is a covering map.
4. Let $p : X \longrightarrow Y$ be a local homeomorphism and s, s' two sections of p. Show that if for some point $y \in Y, s(y) = s'(y)$, then s and s' coincide on a neighbourhood of y.
   If Y is connected, then $s = s'$.
5. Let G be a discrete group acting **totally disconnected** (i.e. $\forall a \in E, \exists O_a$ an open neighbourhood of a such that $g.O_a \cap O_a = \emptyset, \forall g \neq 1$) on the space E. Prove that the quotient map $p : E \longrightarrow E/G$ is a covering map.

8.2.2 Lifting problem

Given a covering map $p : E \longrightarrow X$. Let $f : Y \longrightarrow X$ be a continuous map, the *lifting problem* consists to find a continuous map $F : Y \longrightarrow E$ such that $f = p \circ F$.

$$\begin{tikzcd}
 & E \arrow[d, "p"] \\
Y \arrow[ur, dotted, "F"] \arrow[r, "f"'] & X
\end{tikzcd}$$

Notice that if $f(Y) \subset U$ where U is evenly covered, then we can choose $O \subset E$ homeomorphic under p to U and F such that $F(Y) = O$. So the lifting problem is locally solvable.
In the following, $p : E \longrightarrow X$ is a covering map. Choose $e \in E, x \in X$ and $y \in Y$ such that $f(y) = p(e) = x$.

Theorem 8.2.4 (Unique Lifting theorem). *Assume Y connected. If there exists a map $f : (Y; y) \longrightarrow (X; x)$, i.e. $f(y) = x$, such that $p \circ F = f$, it is unique.*

Proof: Let $G : Y \longrightarrow E$ such that $f = p \circ G$, and $A = \{y' \in Y \mid F(y') = G(y')\}, B = \{y' \in Y \mid F(y') \neq G(y')\}$. Then $A \cup B = Y$ and $y \in A$.
We show that A and B are open and $A \neq \emptyset$. The space Y is supposed to be connected, so $B = \emptyset$ and the result follows.
We show that A is open. Let $y' \in Y$ and U an open neighbourhood of $f(y')$ which is evenly covered and $F(y') = G(y') \in O_i$. Hence $F^{-1}(O_i) \cap G^{-1}(O_i)$ is an open neighbourhood of y' contained in A and A is open.
We show that B is open. Let $y' \in B$. Then $F(y')$ and $G(y')$ lie in two distinct sheets O and O'. Hence $F^{-1}(O) \cap G^{-1}(O')$ is an open neighbourhood of y', contained in B. ■

The notation $f : (X; x) \longrightarrow (Y; y)$ means that the map $f : X \longrightarrow Y$ satisfies the condition $f(x) = y$.

Theorem 8.2.5 (Path Lifting theorem). *Assume Y connected and $p : E \longrightarrow X$ a covering map. If there exists a continuous map $f : ([0,1]; 0) \longrightarrow (X; x)$ be a path, there is a unique path $F : ([0,1]; 0) \longrightarrow (E; e)$ such that $p \circ F = f$.*

Proof:

$$\begin{tikzcd}
 & (E;e) \arrow[d, "p"] \\
([0,1],0) \arrow[ur, dotted, "F"] \arrow[r, "f"'] & (X;x)
\end{tikzcd}$$

- The space X is evenly covered.
  The point e belongs to one sheet O and $q = p_{|O}$ is a homeomorphism onto X. Then $q^{-1} \circ f$ is the suitable lifting.
- The set $f([0,1])$ is a union of open subsets $X_i, i \in I$ which are evenly covered. Then $[0,1] = \bigcup_i f^{-1}(X_i)$, union of open subsets. By compactness of $[0,1]$, there is a finite subcover of $[0,1]$, i.e. finitely many t_j such that $0 < t_1 < \cdots < t_j < \cdots < t_n < 1$ such that $f([t_j, t_{j+1}])$ is evenly covered. We have a lifting on each interval $[t_j, t_{j+1}]$, hence a lifting of the path f.

■

Definition 8.2.6. *A map $p : Y \longrightarrow X$ has the* **path lifting property** *if, given $x \in X, y \in p^{-1}(x)$ and a path $\gamma : [0,1] \longrightarrow X$, $\gamma(0) = x$, then there is a unique lift $\widetilde{\gamma}$ of γ to Y with $\widetilde{\gamma}(0) = y$.*

Theorem 8.2.7 (Covering Homotopy theorem). *Assume Y connected. Suppose the map $f : (Y; y) \longrightarrow (X; x)$ has a lifting $F : (Y; y) \longrightarrow (E; e)$. Then any homotopy $H : Y \times I \longrightarrow X$ with $H(y, 0) = f(y)$ for all $y \in Y$ can be lifted to a homotopy $H' : Y \times I \longrightarrow E$ with $H'(y, 0) = F(y)$ for all $y \in Y$.*

Proof:

$$\begin{tikzcd}
 & (E;e) \arrow[d, "p"] \\
(Y;y) \times 0 \arrow[ur, "F"] \arrow[r, "f"] \arrow[d, hook] & (X;x) \arrow[d, "Id"] \\
(Y;y) \times I \arrow[uur, bend right, "H'"] \arrow[r, "H"] & (X;x)
\end{tikzcd}$$

- If all of X is evenly covered by $S_i, i \in I$, then $p_{|S_i}$ is a homeomorphism and the result is obvious.

- Otherwise, the proof is similar to the previous ones. By compactness of $I = [0,1]$, we can suppose that there is suitable finite partition of I such that for each $y \in Y$, H maps $U_y \times [t_i, t_{i+1}]$, where U_y open neighbourhood of y, into an evenly covered neighbourhood of $H(y, t_i)$. By previous step, we can lift H on $U_y \times I$ to a map $H' : U_y \times I \longrightarrow E$ such that $H'(y', 0) = F(y')$ for all $y' \in U_y$.

- The previous liftings on $U_y \times I$ and $U_{y'} \times I$ agree on $(U_y \cap U_{y'}) \times I$; hence we can piece them together to obtain the liftings of $H_{|y_1 \times I}, y_1 \in U_y \cap U_{y'}$, which agree at the point $(y_1, 0)$. By the unique lifting theorem, $y_1 \times I$ being connected, these two liftings coincide.

■

Corollary 8.2.8. *The map $p_* : \pi_1(E; e) \longrightarrow \pi_1(X; x)$ is a monomorphism (i.e. injective homomorphism).*

Proof: Let γ be a loop based at $e \in E$ such that $[\gamma] \in \ker p_*$, i.e. $p \circ \gamma$ is a loop homotopic to the constant loop at $x = p(e)$. The lifting $(p \circ \gamma)'$ is homotopic to the lifting γ of the constant loop at x. So, γ is homotopic to the constant loop and $[\gamma]$ is trivial in $\pi_1(E; e)$. ■

A topological space X is **locally path-connected** if given any point $x \in X$, and any open subset A containing x, there is a smaller open set containing x, which is path-connected in the subspace A.

Theorem 8.2.9. *Let Y be a path connected and locally path connected space, and $p : E \longrightarrow X$ a covering map. A map $f : (Y; y) \longrightarrow (X; x)$ lifts to a map $F : (Y, y) \longrightarrow (E; e)$ iff $f_*\pi_1(Y; y) \subset p_*\pi_1(E; e)$.*

Proof: $\Longrightarrow$) If f lifts to F, then $f_*\pi_1(Y; y) = p_*F_*\pi_1(Y; y) \subset p_*\pi_1(E; e)$.
$\Longleftarrow$) Let γ be a path from y to $z \in Y$, then $f \circ \gamma$ lifts to a unique path from e to $F(z)$.
Suppose σ is another path from y to z. Then $f(\sigma.\gamma^{-1})$ is a loop based at x and it lifts to a loop based at e. So F is well defined.
F is continuous. ■

Definition 8.2.10. *A map $p : Y \longrightarrow X$ has the* **path lifting property** *if, given $x \in X, y \in p^{-1}(x)$ and a path $\gamma : [0,1] \longrightarrow X$, $\gamma(0) = x$, then there is a unique lift $\widetilde{\gamma}$ of γ to Y with $\widetilde{\gamma}(0) = y$.*

Proposition 8.2.11. *If $p : Y \longrightarrow X$ is an open map and has the path lifting property, and if X is path connected and each point of X has an open simply connected neighbourhood, then p is a covering map.*

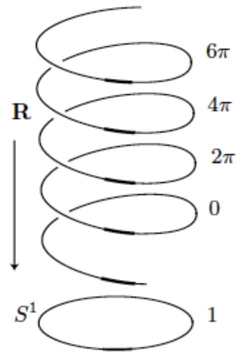
Proof: Let $x \in X$ and O an open simply connected neighbourhood of x. Let $\widetilde{O}$ be a path component of $p^{-1}(O)$ and two points y, y' in the fiber $p^{-1}(x) \cap \widetilde{O}$. Let γ be a path in $\widetilde{O}$ from y to z. Then $p \circ \gamma$ is a loop in X based at x ans its unique lift is γ. The loop $p \circ \gamma$ is homotopic to the constant loop at x. By the homotopy lifting property, $y = z$, so $p_{|\widetilde{O}}$ is injective from $\widetilde{O}$ to O. Since p takes open sets into open sets, $p_{|\widetilde{O}}$ is a homeomorphism and hence p is a covering map. ■

Fundamental Group of the Circle

The unit circle $\mathbb{S}^1$ is the group of complex numbers of absolute value 1. There is a continuous homomorphism of groups

$$\begin{array}{rcl} \phi : \mathbb{R} & \longrightarrow & \mathbb{S}^1 \\ x & \longmapsto & e^{2i\pi x} \end{array}$$

where $\ker(\phi) = \mathbb{Z}$.
ϕ maps $]-\frac{1}{2}, +\frac{1}{2}[$ homeomorphically onto $\mathbb{S}^1 \setminus \{-1\}$ and more generally, any interval $]a, b[$ with $b - a \leq 1$ is homeomorphic to its image. So, ϕ is an open map. Moreover, it is a covering map.
Notice that $\phi(n) = 1$ for any $n \in \mathbb{Z}$, so we choose the point 1 as based point.
Let γ be a path in $\mathbb{S}^1$ with initial point 1, then there is a unique path γ' in $\mathbb{R}$ with initial point 0 such that $\phi \circ \gamma' = \gamma$.
If τ is a path in $\mathbb{S}^1$ with initial point 1 which is homotopic to γ, then there is a unique homotopy from the lifting τ' of τ to γ' and the end point of γ' depends only on the homotopy class of γ.



Theorem 8.2.12. *The fundamental group of the circle is infinite cyclic,*

$$\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$$

Proof: Notice that the fiber $\phi^{-1}(1) = \mathbb{Z}$. Any loop γ in $\mathbb{S}^1$ has a lifting γ' in $\mathbb{R}$ such that $\gamma'(1) \in \mathbb{Z}$. Then we define the map $\chi : \pi_1(\mathbb{S}^1; 1) \longrightarrow \mathbb{Z}$ by $\chi([\gamma]) = \gamma'(1)$.
It is a homomorphism: Given γ and τ two loops based at $1 \in \mathbb{S}^1$ such that $\gamma'(1) = m, \tau'(1) = n$, Define the path σ' in $\mathbb{R}$ from m to $m + n$ given by $\sigma'(s) = \tau'(s) + m$. Then $\phi \circ \sigma' = \tau$, so $\gamma'\tau'$ is the lifting of $\gamma\tau$ with initial point 0; its end point is $m + n$, hence $\chi([\gamma][\tau]) = \chi([\sigma])\chi([\tau])$.
It is onto: Given n, define $\gamma'(s) = ns$. If $\gamma = \phi \circ \gamma'$, then $\chi([\gamma]) = n$.
It is injective: Suppose $\chi([\sigma]) = 0$, so σ' is a loop in $\mathbb{R}$ at 0. $\mathbb{R}$ is contractible, so σ' is homotopic to 0, whence applying ϕ, γ is homotopic to 1 and $[\gamma] = 1$. ∎

Corollary 8.2.13. *The circle is not a retract of the closed disk.*

Proof: Suppose there is a continuous map $r : D_2 \longrightarrow \mathbb{S}^1$ such that $r \circ \iota = Id_{\mathbb{S}^1}$. Then $r_* \circ \iota_* = Id_{\pi_1(\mathbb{S}^1)}$, i.e. the diagram is commutative

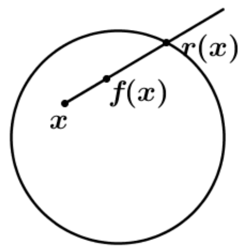
$$\begin{tikzcd} & \pi_1(D_2) = 0 \arrow[dr, "r_*"] & \\ \pi_1(\mathbb{S}^1) = \mathbb{Z} \arrow[ur, "\iota_*"] \arrow[rr, "Id_{\pi_1(\mathbb{S}^1)}"] & & \pi_1(\mathbb{S}^1) = \mathbb{Z} \end{tikzcd}$$

which is not possible. ∎

Corollary 8.2.14. *Any continuous map of a closed disk into itself has a fixed point.*

Proof:

Let $f : D_2 \longrightarrow D_2$ be a continuous map. Suppose $f(x) \neq x$ for any $x \in D_2$. The half line from x to $f(x)$ intersects the boundary of D_2 at the point denoted $r(x)$. Then $r : D_2 \longrightarrow \mathbb{S}^1$ is a retraction, which is a contradiction.



■

This corollary is a particular case to dimension 2 of the Brouwer Fixed Point Theorem.

8.3 Fundamental Groups & Covering Spaces

In this section, we consider a covering space $p : (E; e) \longrightarrow (X; x)$.
The corollary of the covering homotopy theorem is the first result which establishes a relation between the covering maps and the fundamental groups.
Let $p : E \longrightarrow X$ be a covering map.

Theorem 8.3.1 (Covering Homotopy theorem)**.** *Assume Y connected. Let $f : (Y; y) \longrightarrow (X; x)$ be a map which has a lifting $F : (Y; y) \longrightarrow (E; e)$. Then any homotopy $H : Y \times I \longrightarrow X$ with $H(y, 0) = f(y)$ for all $y \in Y$ can be lifted to a homotopy $H' : Y \times I \longrightarrow E$ with $H'(y, 0) = F(y)$ for all $y \in Y$.*

Corollary 8.3.2. *The map $p_* : \pi_1(E; e) \longrightarrow \pi_1(X; x)$ is a monomorphism (i.e. injective homomorphism).*

Proof: Let γ be a loop based at $e \in E$ such that $[\gamma] \in \ker p_*$, i.e. $p \circ \gamma$ is a loop homotopic to the constant loop at $x = p(e)$. The lifting $(p \circ \gamma)'$ is homotopic to the lifting γ of the constant loop at x. So, γ is homotopic to the constant loop and $[\gamma]$ is trivial in $\pi_1(E; e)$. ■

This result shows that the fundamental group $\pi_1(E; e)$ can be viewed as a subgroup of the fundamental group $\pi_1(X; x)$.

Theorem 8.3.3. *Let Y be a path connected and locally path connected space, and let $p : E \longrightarrow X$ a covering map. A map $f : (Y; y) \longrightarrow (X; x)$ lifts to a map $F : (Y, y) \longrightarrow (E; e)$ iff $f_*\pi_1(Y; y) \subset p_*\pi_1(E; e)$.*

Proof: $\Longrightarrow$) If f lifts to F, then $f_*\pi_1(Y; y) = p_*F_*\pi_1(Y; y) \subset p_*\pi_1(E; e)$.
$\Longleftarrow$) Let γ be a path from y to $z \in Y$, then $f \circ \gamma$ lifts to a unique path from e to $F(z)$.
Suppose σ is another path from y to z. Then $f(\sigma.\gamma^{-1})$ is a loop based at x and it lifts to a loop based at e. So F is well defined.
F is continuous. (to be done) ■

Notice that if Y is simply connected, then the lifting problem has a solution (the trivial group is a subgroup of any group).

Proposition 8.3.4. *If $p : Y \longrightarrow X$ is an open map and has the path lifting property, and if X is path connected and each point of X has an open simply connected neighbourhood, then p is a covering map.*

Proof: Let $x \in X$ and O an open simply connected neighbourhood of x. Let $\widetilde{O}$ be a path component of $p^{-1}(O)$ and two points y, y' in the fiber $p^{-1}(x) \cap \widetilde{O}$. Let γ be a path in $\widetilde{O}$ from y to z. Then $p \circ \gamma$ is a loop in X based at x ans its unique lift is γ. The loop $p \circ \gamma$ is homotopic to the constant loop at x. By the homotopy lifting property, $y = z$, so $p_{|\widetilde{O}}$ is injective from $\widetilde{O}$ to O. Since p takes open sets into open sets, $p_{|\widetilde{O}}$ is a homeomorphism and hence p is a covering map. ■

8.3.1 Operation of $\pi_1(X; x)$ on the fiber $p^{-1}(x)$

The previous results on the liftings of paths and homotopies can be viewed as an action of the fundamental group of the base space on the fibers of the covering.

Proposition 8.3.5. *Let $p : (E; e) \longrightarrow (X; x)$ be a locally path-connected covering where X is path connected. Then*

1. *There is a right action of $\pi_1(X; x)$ on the fiber $p^{-1}(x)$.*
2. *The stabilizer of the point x is the subgroup $p_*\pi_1(E; e) \subset \pi_1(X; x)$.*
3. *It is a transitive operation iff E is path connected.*

Proof:

1. Let consider a covering map $p : (E; e) \longrightarrow (X; x)$.
   Let γ be a loop at x in X, its lifting γ' is a path from $a \in p^{-1}$ to $\gamma'_a(1) \in p^{-1}(x)$. This point depends only on the homotopy class of γ, so we define the operation
   $$\begin{array}{rcl} p^{-1}(x) \times \pi_1(X; x) & \longrightarrow & p^{-1}(x) \\ (a, [\gamma]) & \longmapsto & a.[\gamma] = \gamma'_a(1) \end{array}$$
   As exercise, verify that $a.1 = a$ and $a.([\gamma][\tau]) = (a.[\gamma]).[\tau]$ for any $a \in p^{-1}(x)$ and $[\gamma], [\tau] \in \pi_1(X; x)$.
2. The *stabilizer* of the point $e \in p^{-1}(x)$ is the subgroup $p_*\pi_1(E, e)$ of $\pi_1(X; x)$, i.e. the set of all $\sigma \in \pi_1(X; x)$ such that $e.[\sigma] = e$. (Recall that, if G is a group acting on the set S, the stabilizer of $s \in S$ is the subgroup $G_s = \{g \in G \mid s.g = s\}$.)
3. We assume that E is pathwise-connected, then $\pi_1(X; x)$ operates transitively, i.e. for any $a, b \in p^{-1}(x)$ there exists $[\gamma] \in \pi_1(X; x)$ such that $b = a.[\gamma]$.
   The converse is true: if $\pi_1(X; x)$ operates transitively, then E is path connected. (exercise)

■

Thus, we have the following results:

Proposition 8.3.6. *Let $[\gamma] \in \pi_1(X; x)$, and $a \in p^{-1}(x)$, so that $p_*\pi_1(E; a) \subset \pi_1(X; x)$, there is $b \in p^{-1}(x)$ such that $p_*\pi_1(E; b) = [\gamma]^{-1}p_*\pi_1(E; a)[\gamma]$.*

Proof: The loop γ in the base space X lifts to the path γ' from the point $a \in p^{-1}(x)$ to a point $\gamma'(1) = b \in p^{-1}(x)$. Then there are two homomorphisms

$$\begin{array}{rcl} v : \pi_1(X; x) & \longrightarrow & \pi_1(X; x) \\ [\sigma] & \longmapsto & [\gamma^{-1}][\sigma][\gamma] \end{array}$$

and

$$\begin{aligned} u : \pi_1(E;a) &\longrightarrow \pi_1(E;b) \\ [\sigma'] &\longmapsto [\gamma]^{-1}[\sigma'][\gamma] \end{aligned}$$

such that the following diagram commutes.

$$\begin{tikzcd} \pi_1(E;a) \arrow[r, "p_*"] \arrow[d, "u"] & \pi_1(X;x) \arrow[d, "v"] \\ \pi_1(E;b) \arrow[r, "p_*"] & \pi_1(X;x) \end{tikzcd}$$

■

Proposition 8.3.7. *The map $[\gamma] \longmapsto e.[\gamma]$ induces a bijection of the set of all cosets $p^*\pi_1(E;e)[\gamma]$ onto the fiber. If $p^{-1}(x)$ is finite, the number of points in the fiber is equal to the index of the subgroup $p_*\pi_1(E;e)$.*

Proof: Let $\chi : \pi_1(X;x) \longrightarrow \pi_1(X;x)/p^*(\pi_1(E;e))$ be the canonical surjection.
Let $f : \pi_1(X;x) \longrightarrow p^{-1}(x)$ be the map $[\gamma] \longmapsto e.[\gamma]$.
Moreover $e.[\gamma] = e.[\sigma]$ iff $\chi([\gamma]) = \chi([\sigma])$. Then, there is a map

$$\psi : \pi_1(X;x)/p^*(\pi_1(E;e)) \longrightarrow p^{-1}(x)$$

such that the following diagram is commutative

$$\begin{tikzcd} \pi_1(X;x) \arrow[rr, "f"] \arrow[d, "\chi"] & & p^{-1}(x) \\ \pi_1(X;x)/p^*(\pi_1(E;e)) \arrow[urr, dotted, "\psi"] & & \end{tikzcd}$$

(to be finished) ■

8.4 Classification of Coverings

In this section, we suppose that we are able to compute the fundamental groups. So, knowing the fundamental groups, we will classify the covering spaces.

8.4.1 Covering Transformations

We assume that all spaces are path connected and locally path connected.
Let $p_1 : E_1 \longrightarrow X$ and $p_2 : E_2 \longrightarrow X$ be two covering maps of the same space X.

Definition 8.4.1. *A* **covering transformation** *from the covering map p_1 to the covering map p_2 is a continuous map $\phi : E_1 \longrightarrow E_2$ such the diagram commutes:*

$$\begin{tikzcd} E_1 \arrow[rr, "\phi"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

Note that if $a \in p_1^{-1}(x)$, then $\phi(a) \in p_2^{-1}(x)$, i.e. $\phi\left(p_1^{-1}(x)\right) \subseteq p_2^{-1}(x)$ for all $x \in X$.
Note also that the composition of two covering transformations is a covering transformation.

Definition 8.4.2. *A covering transformation ϕ from p_1 to p_2 is a* **isomorphism** *if there is a covering transformation $\psi : E_2 \longrightarrow E_1$ such that $\phi \circ \psi = Id_{E_2}$ and $\psi \circ \phi = Id_{E_1}$.*
An isomorphism is an **automorphism** *if $E_1 = E_2$.*

An isomorphism between two covering spaces is a homeomorphism, but there exist homeomorphisms which are not isomorphisms of covering spaces. For example, consider the two following covering spaces $p_1 : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ where $p_1(z) = z^2$ and $p_2 : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ where $p_2(z) = z^3$. It does not exist some isomorphism from one of these two covering maps to the other one, although the two spaces are clearly homeomorphic under ϕ. If ϕ is an isomorphism of covering spaces, then the fibers $p_1^{-1}(x)$ and $p_2^{-1}(x)$ have the same cardinality, but even if they have same cardinality, ϕ may be not an isomorphism.

Proposition 8.4.3. *Let $(E_i, p_i), i = 1, 2$ be two coverings of the space X such that $p_1(e_1) = p_2(e_2) = x$. Then there exists a covering transformation $\phi : E_1 \longrightarrow E_2$ such that $\phi(e_1) = e_2$ iff $p_{1*}\pi_1(E_1; e_1) \subset p_{2*}\pi_1(E_2; e_2)$.*

Proof: It follows from the theorem 2.2.8 as a particular case. ■

The following corollary is clear.

Corollary 8.4.4. *There exists an isomorphism of coverings $\phi : (E_1; e_1) \longrightarrow (E_2; e_2)$ iff $p_{1*}\pi_1(E_1; e_1) = p_{2*}\pi_1(E_2; e_2)$.*

Notice that the isomorphism respects the base points.
If ϕ is a covering transformation and $p_1 = p_2 \circ \phi$, then one says that p_1 **dominates** p_2.
Let $p_1 : E_1 \longrightarrow X$ and $p_2 : E_2 \longrightarrow X$ be two coverings of the space X, and $e_1 \in p_1^{-1}(x), e_2 \in p_2^{-1}(x)$ for $x \in X$.

Proposition 8.4.5. *The two coverings are isomorphic iff the subgroups $p_{1*}\pi_1(E_1; e_1)$ and $p_{2*}\pi_1(E_2; e_2)$ are conjugate in $\pi_1(X; x)$.*

Proof: Let $\phi : E_1 \longrightarrow E_2$ be some isomorphism, and $\phi(e_1) = b$. Then $p_{1*}\pi_1(E_1; e_1) = p_{2*}\pi_1(E_2; b)$. Let Γ be a path from e_2 to b and γ its projection in X. γ is a loop. Then the group $p_{2*}\pi_1(E_2; b)$ is equal to the conjugate of $\pi_1(E_2; e_2)$ by $[\gamma]$.
Suppose the subgroups $p_{1*}\pi_1(E_1; e_1)$ and $p_{2*}\pi_1(E_2; e_2)$ are conjugate, then the two coverings are isomorphic (exercise). ■

If $E_1 = E_2 = E$, then the set of all covering automorphisms is a group denoted $A(E; p)$.

Proposition 8.4.6. *The automorphism $\phi \in A(E; p)$ is determined by the image $\phi(a)$ for any $a \in E$.*

Proof: See lemma 2.4.1. ■

Proposition 8.4.7. *$A(E; p)$ as a discrete group, acts continuously on E and on the fibers $p^{-1}(x)$.*

Proof: The action of the discrete group $A(E; p)$ on E is

$$\begin{array}{rcl} \varphi : A(E; p) \times E & \longrightarrow & E \\ (g, a) & \longmapsto & g.a \end{array}$$

and $\varphi^{-1}(O) = \bigcup_{g \in A(E;p)} \{g\} \times g^{-1}.O$ which is a union of open sets, so it is open. ■

8.4.2 Galois Coverings

Definition 8.4.8. *A covering map $p : E \longrightarrow X$ is said to be* **Galois** *or* **regular** *if the group $A(E;p)$ acts transitively on the fibers.*

Proposition 8.4.9. *A covering map $p : E \longrightarrow X$ is Galois iff for any $y \in X$ and any $a \in p^{-1}(y)$, $p_*\pi_1(E;a)$ is a normal subgroup of $\pi_1(X;y)$.*
The group $A(E;p)$ is isomorphic to the quotient group $\pi_1(X;y)/p_*\pi_1(E;a)$.

Proof:

■

Corollary 8.4.10. *If E is simply connected, the covering map $p : E \longrightarrow X$ is Galois and $A(E;p)$ is isomorphic to $\pi_1(X;x)$.*

Proof: The space E is simply connected, so its fundamental group is trivial, i.e. $\pi_1(E;a) = 0$ for any $a \in E$ and the result follows. ■

8.4.3 Universal Coverings

We have seen that isomorphic coverings correspond to conjugate subgroups of the fundamental group of the base space of the coverings. Then, we have to construct a covering for each subgroup of the fundamental group of the base space.

Definition 8.4.11. *A covering $p : E \longrightarrow X$ is said to be* **universal** *if the fundamental group $\pi_1(E)$ is trivial.*

Two universal coverings of X are isomorphic (see prop....).

Proposition 8.4.12. *Let $p : E \longrightarrow X$ be a covering of X and $\tilde{p} : \widetilde{E} \longrightarrow X$ the universal covering. Then there is a covering transformation $\phi : \widetilde{E} \longrightarrow E$.*

Proof: Use prop. .... ■
This result explains the terminology "universal".

Note that a universal covering is Galois. Let $p_1 : E_1 \longrightarrow X$ and $p_2 : E_2 \longrightarrow X$ be two regular coverings of X.
Then define $p_1 \leq p_2$ if there is a covering transformation $\phi : E_1 \longrightarrow E_2$. It is an order on the collection of all regular coverings of X, up to isomorphism.
For any regular covering $p : E \longrightarrow X$ of X, there are two covering transformations $E \longrightarrow X$ and $\widetilde{E} \longrightarrow E$, then $p \leq Id$ and $\tilde{p} \leq p$, i.e. the identity covering is a *final* object and the universal covering is an *initial* covering in the category of regular coverings of the space X.

Remark 8.4.13. *Why do we restrict this order relation to regular coverings?*
If we consider any coverings, then it can happen that $p_1 \leq p_2$ and $p_2 \leq p_1$ and the two coverings are not isomorphic. Then it is not an order.

The question is: is there exist a universal covering of any space X?

Proposition 8.4.14. *If the space X has a universal covering, then any point $x \in X$ has a neighbourhood O_x such that any loop in O_x, based at x, is homotopic in X to the constant loop at x.*

Proof: Let s be a section over a neighbourhood O_x, of the universal covering $\tilde{p} : \widetilde{E} \longrightarrow X$. Then the injection $\iota : O_x \hookrightarrow X$ is equal to $\tilde{p} \circ s$ and induces the trivial covering transformation from $\pi_1(O_x; x)$ into $\pi_1(X; x)$. ■

Definition 8.4.15. *A space is* **locally simply connected** *if every point has at least one simply connected open neighbourhood.*

Theorem 8.4.16. *Any path-connected, locally simply connected space has a universal covering.*

Proof: Let X be a path-connected, locally simply connected space and $x \in X$ a base point.
Consider the paths starting at x.Two such paths are equivalent if they have the same end points and they are homotopic. (verify that it is an equivalence relation). Denote $\hat{\gamma}$ the class of the path γ where $\gamma(0) = x$.
Define $\widetilde{E}$ as the set of equivalence classes and define the map $p : \widetilde{E} \longrightarrow X, \hat{\gamma} \mapsto \gamma(1)$.
Define a topology on $\widetilde{E}$ which makes it a simply connected space.
The base space X is locally simply connected, then X is covered by open sets which are connected and such as any loop in the open set is homotopic to the constant loop in X. These open sets are called **fundamental open sets**.
Let O be a fundamental open set, and γ a path from x to $\gamma(1) \in O$, and define $(\widehat{\gamma}, O)$ the subset of $\widetilde{E}$ formed with the paths σ such that there is $\widehat{\gamma'} \in O$ such that $\sigma = \gamma\gamma'$.
These $(\widehat{\gamma}, O)$ are a base for a topology on E, i.e. for any $\sigma \in (\widehat{\gamma_1}, O_1) \cap (\widehat{\gamma_2}, O_2)$, there is a fundamental open set O such that $(\widehat{\sigma}, O) \subset (\widehat{\gamma_1}, O_1) \cap (\widehat{\gamma_2}, O_2)$. It is enough to take any fundamental open set O such that $p(\sigma) \in O \subset O_1 \cap O_2$.

■

8.4.4 Summary of important facts on Coverings

Let $p : E \longrightarrow X$ be a covering, $e \in E$ and $x = p(e) \in X$. Then

- The subgroups $p_*\pi_1(E; e)$ are a class of conjugacy of subgroups of $\pi_1(X, x)$.
- The group of automorphisms of the covering, denoted $A(E, p)$ acts without fixed point on E.
- $A(E, p)$ is isomorphic to the group of automorphisms of $p^{-1}(x)$ considered as right $\pi_1(X; x)$-space.
- If the covering is Galois, then $A(E, p) \simeq \pi_1(X; x)/p_*\pi_1(E, p)$ for any $x \in X$ and $e \in p^{-1}(x)$.
- If the covering is universal,
  - $\pi_1(X)$ is in one-to-one correspondence with the set $p^{-1}(x)$.
  - $\pi_1(X)$ is isomorphic to the group $A(E, p)$.
  - If $e \in E$ such that $p(e) = x \in X$, then for each $e' \in p^{-1}(x)$, there exists a unique covering transformation mapping e to e'.

Appendix A

Group Theory

A.1 Free Groups. Presentation of a Group

Let A be a set considered as an **alphabet**, so that a **word** is a finite juxtaposition of letters of the alphabet. Let $f : A \longrightarrow A'$ be a bijection where $f(a) = a'$, and $W(A \coprod A')$ the set of all words made with the elements of $A \coprod A'$. Let $\mathcal{R}$ be the equivalence relation generated by the relations:

$$(w_1 a a' w_2, w_1 w_2) \in W(A \coprod A')^2 \text{ and } (w_1 a' a w_2, w_1 w_2) \in W(A \coprod A')^2$$

where $w_i \in W(A \coprod A'), i = 1, 2, a \in A, a' = f(a) \in A'$.
The equivalence relation $\mathcal{R}$ is the intersection of all equivalence relations containing the above relations.
We define an associative law $W(A \coprod A') \times W(A \coprod A') \longrightarrow W(A \coprod A')$ by $(w_1, w_2) \longmapsto w_1 w_2$ where the empty word is the neutral element. However, the inverse element of a word w is not defined.
The quotient $W(A \coprod A')/\mathcal{R} := F[A]$ is a group.
The class of the word w is the "reduced" word denoted $\overline{w}$.
The equivalence class of $a' \in A'$ is denoted a^{-1}. We write a^n for the word $\overbrace{a. \cdots .a}^{n \text{ terms}}$, for $n \in \mathbb{N}$ and for the word $\overbrace{a^{-1}. \cdots .a^{-1}}^{-n \text{ terms}}$ for $n \in \mathbb{Z}, n < 0$, and 1 for the word a^0 which correspond to the empty word, a for the word a^1, a^{m+n} for the word $a^m a^n$.
It is easy to show that the law on the quotient induced by the law on $W(A \coprod A')$ is associative and 1 is the neutral element.
Let $w = a_1^{\epsilon_1} a_2^{\epsilon_2} \cdots a_n^{\epsilon_n} \in W(A \coprod A')$ be a word where $a_i \in A$ if $\epsilon_i = 1$ and $a_i \in A'$ if $\epsilon_i = -1$ and $\overline{w}$ its equivalence class. Then the opposite element $\overline{w^{-1}}$ is the equivalence class of $a_n^{-\epsilon_n} \cdots a_2^{-\epsilon_2} a_1^{-\epsilon_1}$.
Let $r_i \in F[A], i \in I$ and R the least normal subgroup of $F[A]$ containing the $r_i, i \in I$.

Definition A.1.1. *An isomorphism of $F[A]/R$ onto a group G is a* **presentation** *of G. The set A is the set of* **generators** *for the presentation and each r_i is a* **relator**. *We denote $G := (A, \{r_i, \})$.*
If $R = \emptyset$, then $F[A]$ is the **free group generated** *by A.*

Universal Mapping Property

Let $f : A \longrightarrow G$ be a map from the set A to a group G. Then, $F[A]$ is a **free group** on the set A iff there is a unique homomorphism $h : F[A] \longrightarrow G$ such that $f = h \circ \iota$ where $\iota : A \longrightarrow F[A]$ is the canonical injection.

$$\begin{tikzcd} A \arrow[r, "\iota"] \arrow[dr, "f"'] & F[A] \arrow[d, dotted, "h"] \\ & G \end{tikzcd}$$

For $a \in A$, we have $\iota(a) \in F[A]$, for simplicity, we denote $\iota(a) = a$. Define $h(a) = f(a)$.
Let $g, g' \in F[A]$, where $g = a_{i_1}^{\alpha_1} \cdots a_{i_n}^{\alpha_n} b_{i_1}^{\beta_1} \cdots b_{i_m}^{\beta_m}$, and $g' = b_{i_m}^{-\beta_m} \cdots b_{i_1}^{-\beta_1} {a'}_{i_1}^{\alpha_1} \cdots {a'}_{i_p}^{\alpha_n}$, with $a_i, a'_i, b_j \in A, \alpha_k = \pm 1, \beta_j = \pm 1$. Then
$gg' = a_{i_1}^{\alpha_1} \cdots a_{i_n}^{\alpha_n} b_{i_1}^{\beta_1} \cdots b_{i_m}^{\beta_m} b_{i_m}^{-\beta_m} \cdots b_{i_1}^{-\beta_1} {a'}_{i_1}^{\alpha_1} \cdots {a'}_{i_p}^{\alpha_n} = a_{i_1}^{\alpha_1} \cdots a_{i_n}^{\alpha_n} {a'}_{i_1}^{\alpha_1} \cdots {a'}_{i_p}^{\alpha_n}$.
We define $h(g) = f(a_{i_1})^{\alpha_1} \cdots f(a_{i_n})^{\alpha_n} f(b_{i_1})^{\beta_1} \cdots f(b_{i_m})^{\beta_m}$,
$h(g') = f(b_{i_m})^{-\beta_m} \cdots f(b_{i_1})^{-\beta_1} f({a'}_{i_1})^{\alpha_1} \cdots f({a'}_{i_p})^{\alpha_n}$.
Then, it is easy to verify that $h(gg') = h(g)h(g')$. Hence h is a homomorphism and is unique.

The **free abelian groups** are defined similarly.
Let $f : A \longrightarrow G$ be a map from the set A to an abelian group G. Then, $F[A]^{ab}$ is a free abelian group on the set A iff there is a unique homomorphism $h : F[A]^{ab} \longrightarrow G$ such that $f = h \circ \iota$ where $\iota : A \hookrightarrow F[A]^{ab}$ is the inclusion.
The free group $F[A]$ is abelian iff $|A| = 1$.
Notice that the free group is defined in the category of groups, and the free abelian group is defined in the (sub)category of abelian groups.
For any group G generated say by a set A, there exists a free group $F[A']$ where $|A'| = |A|$, and an epimorphism $f : F[A'] \longrightarrow G$.

Example A.1.2. *Suppose the group G has generators x and y and relation $(xyx^{-1}y^{-1}, 1)$. Then G is abelian.*

Example A.1.3. *If $A = \{a\}$, then the free group $F[A]$ on one generator a is infinite cyclic.*

A.1.1 Exercises

1. This exercise illustrates the fact that if the group G contains two isomorphic normal subgroups H and K, then G/H need not be isomorphic to G/K.

   (a) Compute $\big(\mathbb{Z}_2 \times \mathbb{Z}_4\big)/ <(1,0)>$. "Compute" means discover to which of the two (up to isomorphism) groups of order 4 this quotient group is isomorphic.

   (b) Repeat with $\big(\mathbb{Z}_2 \times \mathbb{Z}_4\big)/ <(0,2)>$

2. Show that if H and N are subgroups of a group G, and N is normal in G, then $H \cap N$ is normal in H. Show by an example that $H \cap N$ need not be normal in G.

3. Determine the group $G = \big(a, b, c; \{a^3\}, \{b^3\}, \{c^4\}, \{ac = ca^{-1}\}, \{aba^{-1} = bcb^{-1}\}\big)$.

A.2 Product & Coproduct of Groups

In this section, we consider the category of groups $\mathcal{G}rp$.
Let G_1 and G_2 be two groups.

A.2.1 Free Product of Groups

The elements of the groups G_1 and G_2 considered as subgroups of the product $G_1 \times G_2$ commute, i.e. let $g_1 \in G_1, g_2 \in G_2$ and $(g_1, e_2) \in G_1 \times G_2, (e_1, g_2) \in G_1 \times G_2$, then

$$(g_1, e_2) + (e_1, g_2) = (g_1, g_2) = ((e_1, g_2) + (g_1, e_2)$$

We define a nonabelian version of the direct sum.
Let denote $G_1 * G_2 := \{g_1 g_2 \cdots g_m\}$ where $m \in \mathbb{N}$ be the set of words where each letter g_i belongs to G_1 or G_2 and adjacent letters do not belong to the same group. The empty word is allowed. Let define a group operation on $G_1 * G_2$ as follows:

$$(g_1 g_2 \cdots g_m)(h_1 h_2 \cdots h_n) = \begin{cases} g_1 g_2 \cdots g_m h_1 h_2 \cdots h_n & \text{if} \quad g_m \in G_i, \quad h_1 \in G_j \quad 1 \leq i \neq j \leq 2 \\ g_1 \cdots g_{m-1} k h_2 \cdots h_n & \text{if} \quad g_m \in G_i, \quad h_1 \in G_i, \quad\quad k = g_m h_1 \end{cases}$$

The empty word is the neutral element. This operation defines a structure of group on the set $G_1 * G_2$, called the **free product** of the groups G_1 and G_2. Notice that this construction can be generalized to any family of groups.

Example A.2.1. *As an example, $\mathbb{Z} * \mathbb{Z}$ is a free group.*
Another example is $\mathbb{Z}_2 * \mathbb{Z}_2$ where we have $a^2 = e = b^2$, so the words are $a, b, ab, ba, aba, bab, abab, baba, ababa, \ldots$.
The group presented by $\big(a, b; \{a^4\}, \{b^6\}\big)$ is the free product of the cyclic groups $\big(a; \{a^4\}\big)$ and the cyclic group $\big(b; \{b^6\}\big)$.

Universal Mapping Property

Let $k_i : G_i \longrightarrow G_1 * G_2, i = 1, 2$ be the maps defined by sending g_i to the word denoted g_i. The **free product** of two groups is defined by the triple $(G_1 * G_2, k_1, k_2)$ that satisfies the following universal mapping property. Given a group G and two homomorphisms $f_i : G_i \longrightarrow G$, there is a unique homomorphism $h : G_1 * G_2 \longrightarrow G$ such that $f_i = h \circ k_i, i = 1, 2$.

$$\begin{tikzcd} & G & \\ G_1 \arrow[ur, "f_1"] \arrow[r, "k_1"'] & G_1 * G_2 \arrow[u, dotted, "h"] & G_2 \arrow[ul, "f_2"'] \arrow[l, "k_2"] \end{tikzcd}$$

The map h is defined as follows: if $g \in G_1$, $h(g) = f_1(g)$ and if $g \in G_2$, $h(g) = f_2(g)$.
$h(g_1 \ldots g_m) = f_{i_1}(g_1) \cdots f_{i_m}(g_m)$ where $i_j = 1$ if $g_j \in G_1$ and $i_j = 2$ if $g_j \in G_2$, $j = 1, \ldots, m$. It is easy to prove that h is a homomorphism and it is unique (exercise).
Hence, $G_1 * G_2$ satisfies the universal property as the coproduct of G_1 and G_2 in the category of groups $\mathcal{G}rp$.

A.2.2 Direct Product of Groups

Let $(G_1, *)$ and $(G_2, *)$ be two groups. Then the product set $G_1 \times G_2$ where $(g_1, g_2) * (h_1, h_2) = (g_1 * h_1, g_2 * h_2)$ is called the **direct product** of the groups G_1 and G_2 (for simplicity, we denote by the same symbol $*$ the different group operations).

Universal Mapping Property

The **direct product** of two groups G_1, G_2 defined by the triple $(G_1 \times G_1, p_1, p_2)$, satisfies the universal mapping property if for any group G', and any homomorphisms $\phi_i : G' \longrightarrow G_i,\ i = 1, 2$, there exists a unique homomorphism $\phi : G' \longrightarrow G_1 \times G_2$ such that the following diagram is commutative

$$\begin{tikzcd} & G' \arrow[dl, "\phi_1"'] \arrow[d, dotted, "\phi"] \arrow[dr, "\varphi_2"] & \\ G_1 & G_1 \times G_2 \arrow[l, "p_1"] \arrow[r, "p_2"'] & G_2 \end{tikzcd}$$

where $p_i : G_1 \times G_2 \longrightarrow G_i,\ i = 1, 2$ are the canonical projections $(g_1, g_2) \mapsto g_i$. Notice that p_i are homomorphisms.
The map ϕ is defined as $\phi(g') = (\phi_1(g'), \phi_2(g'))$, for $g' \in G'$, as for the case where the groups are Abelian.
As exercise, show that the map h is the unique homomorphism making commutative the above diagram.
A direct product of Abelian groups as Abelian groups, i.e. in the category $\mathcal{A}b$ is the same as their direct product as groups, i.e. in the category $\mathcal{G}rp$.

There is a useful criterion for a group to be a direct product of subgroups:

Proposition A.2.2. *Let G be a group and let H, K be two subgroups such that $H \cap K = \{e\}$, and $\{h * k \mid h \in H, k \in K\} = HK = G$, and such that $h * k = k * h$ for all $h \in H$ and $k \in K$. Then the map*

$$\begin{aligned} H \times K &\longrightarrow G \\ (h, k) &\longmapsto h * k \end{aligned}$$

is an isomorphism.

Proof: It is obviously a homomorphism, which is surjective since $HK = G$. If (h, k) is in its kernel, then $h = k^{-1}$, whence h lies in both H and K, and $h = e$, so that $k = e$ also, and our map is an isomorphism. ■

The free product of two groups can be viewed as the dual of the direct product in the category of the groups $\mathcal{G}rp$.

A.2.3 More on Coproducts of Groups

In the category of Abelian groups, we saw that the direct product and the direct sum of two Abelian groups are equal. The coproduct or direct sum $G_1 \oplus G_2$ of two Abelian groups G_1 and G_2 in the category of Abelian groups $\mathcal{A}b$ is not equal to the coproduct in the category of groups $\mathcal{G}rp$.
Consider two Abelian groups G_1 and G_2, their direct sum $G_1 \oplus G_2$ in the category $\mathcal{A}b$, and their free product $G_1 * G_2$, in the category $\mathcal{G}rp$. They are both coproducts in two different categories. Recall that the triple $(G_1 * G_2, k_1, k_2)$ defines the free product dans la category $\mathcal{G}rp$ and $(G_1 \times G_2, j_1, j_2)$ the direct sum of the two groups G_1 and G_2 dans la category $\mathcal{A}b$.

$$\begin{tikzcd} & G_1 \oplus G_2 & \\ G_1 \arrow[ur, "j_1"] \arrow[dr, "k_1"'] & & G_2 \arrow[ul, "j_2"'] \arrow[dl, "k_2"] \\ & G_1 * G_2 \arrow[uu, dotted, "m"] & \end{tikzcd} \qquad \begin{tikzcd} & (g_1, e_2) & (e_1, g_2) & \\ g_1 \arrow[ur, mapsto] \arrow[dr, mapsto] & & & g_2 \arrow[ul, mapsto] \arrow[dl, mapsto] \\ & \overline{g_1} & \widehat{g_2} & \end{tikzcd}$$

By construction of $G_1 * G_2$, there exists a unique homomorphism $m : G_1 * G_2 \longrightarrow G_1 \oplus G_2$ such that $m(\overline{g_1}) = (g_1, e_2), m(\widehat{g_2}) = (e_1, g_2)$, where $\overline{g_1} = k_1(g_1), \widehat{g_2} = k_2(g_2)$. We have

$$m(\overline{g_1}) + m(\widehat{g_2}) = (g_1, e_2) + (e_1, g_2) = (g_1, g_2) = (e_1, g_2) + (g_1, e_2) = m(\widehat{g_2}) + m(\overline{g_1})$$

Then $m(\overline{g_1}) + m(\widehat{g_2}) = m(\widehat{g_2}) + m(\overline{g_1})$, i.e. $m(\overline{g_1}\widehat{g_2}) = m(\widehat{g_2}\overline{g_1})$, i.e. $m(\overline{g_1}\widehat{g_2}\overline{g_1}^{-1}\widehat{g_2}^{-1}) = (e_1, e_2)$, i.e. $\overline{g_1}\widehat{g_2}\overline{g_1}^{-1}\widehat{g_2}^{-1} = [\overline{g_1}, \widehat{g_2}] = e_{G_1 * G_2}$, hence $[k_1(G_1), k_2(G_2)] = \ker(m)$, so

$$G_1 \oplus G_2 = (G_1 * G_2)^{ab}$$

The direct sum of two Abelian groups in the category of Abelian groups $\mathcal{A}b$ is the abelianization of the free product of the two groups in the category of groups $\mathcal{G}rp$.

A.3 Amalgamated Free Products

Let G_1, G_2 and H be three groups and $f_i : H \longrightarrow G_i, i = 1, 2$ two homomorphisms, N the normal subgroup of $G_1 * G_2$ generated by the elements $f_1(x)f_2(x)^{-1},\ x \in H$. The **amalgamated free product** is denoted $G_1 *_H G_2 := (G_1 * G_2)/N$.
For any $x \in H$, we have the images of $f_1(x)$ and $f_2(x)$ in $G_1 *_H G_2$ are equal.

Universal Mapping Property

Given two homomorphisms $f_i : H \longrightarrow G_i, i = 1, 2$ and the coproduct $(G_1 * G_2, k_1, k_2)$, we get the homomorphisms $\phi_i = can \circ k_i : G_i \longrightarrow G_1 * G_2 \longrightarrow G_1 *_H G_2 = G_1 * G_2/N$. For any homomorphisms $h_i : G_i \longrightarrow G, i = 1, 2$, we have to define the unique homomorphism l making the diagram commutative (exercise).

$$\begin{tikzcd} H \arrow[r, "f_1"] \arrow[d, "f_2"'] & G_1 \arrow[d, "\phi_1"] \arrow[ddr, bend left, "h_1"] & \\ G_2 \arrow[r, "\phi_2"] \arrow[drr, bend right, "h_2"'] & G_1 *_H G_2 \arrow[dr, dotted, "l"] & \\ & & G \end{tikzcd}$$

Notice that if $H = \{e\}$ is the trivial group, then the amalgamated free product $G_1 *_H G_2$ is the free product $G_1 * G_2$.
Let $G_1 = (\alpha_1, \ldots, \alpha_m\ ;\ r_{11} = \cdots = r_{1p} = 1)$ and $G_2 = (\beta_1, \ldots, \beta_n\ ;\ r_{21} = \cdots = r_{2q} = 1)$ be the presentations of the groups G_1, and G_2 and $\{\gamma_1, \ldots, \gamma_s\}$ some generators of H. Then a presentation of the amalgamated free product $G_1 *_H G_2$ is

$$(\phi_1(\alpha_1), \ldots, \phi_1(\alpha_m), \phi_2(\beta_1), \ldots, \phi_2(\beta_n); r_{11} = \cdots = r_{1p} = r_{21} = \cdots = r_{2q} = 1, f_1(\gamma_i) = f_2(\gamma_i), i \leq s)$$

where r_{kl} are the relations in $G_1 *_H G_2$ obtained from the relations r_{kl} in G_i under the homomorphisms ϕ_1 and ϕ_2.

For example, let G be the group presented by $\big(a, b; \{a^4\}, \{b^6\}, \{a^2 = b^3\}\big)$. The homomorphism

$$\begin{aligned} G &\longrightarrow \big(x; \{x^{12}\}\big) \\ a &\longmapsto x^3 \\ b &\longmapsto x^2 \end{aligned}$$

shows that the order of a is four and the order of b is six. Let $G_1 = \big(a; \{a^4\}\big)$ and $G_2 = \big(b; \{b^6\}\big)$ be two groups. Let H_a be the subgroup of G_1 with order two and H_b the subgroup of G_2 with order two. The subgroups H_a and H_b are amalgamated under the mapping $a^2 \longmapsto b^3$ and the group G is the amalgamated free product of the groups H_a and H_b under the mapping $a^2 \longmapsto b^3$.