Qualitative Theory of Differential Equations

Further Reading for Part 1

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The following books are useful. Each is available from the link provided.

[D] P. Dawkins, *Differential Equations*, Lecture Notes, 2018
`http://tutorial.math.lamar.edu/Classes/DE/DE.aspx` and use the 'Download' button

[S] B. Schroers, *Differential Equations: A Practical Guide*, Cambridge University Press, 2011
`http://www.macs.hw.ac.uk/ bernd/schroers.pdf`

[HSD] M.W. Hirsch, S. Smale and R.L. Devaney, *Differential Equations, Dynamical Systems and an Introduction to Chaos,* 2nd Edition, Elsevier, 2004.
`https://thalis.math.upatras.gr/ bountis/files/def-eq.pdf`

[M] J.D. Murray, *Mathematical Biology I: An Introduction,* 3rd Edition, Springer, 2002.
`http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.725.7032rep=rep1type=pdf`

Sections from these sources that detail and extend the topics covered in the Worksheets are given below.

Worksheet 1: First Order Differential Equations

Chapter 1 of [D] is a useful review of basic ideas. Chapter 2 covers methods for solving first order equations analytically and looks briefly at examples of some applications that are similar to those considered in Worksheet 1. The final section of the chapter touches on the more qualitative ideas that we have been focusing on, but using 'direction fields' instead of the graph of $f(x)$.

First order equations are treated briefly in the first chapters of both [HSD] (see especially sections 1.1 - 1.3)and [M] (sections 1.1 - 1.2) in a manner that is closer to the approach used in Worksheet 1. Each includes a harvesting/predation model that is different from the one on the worksheet and uses this to introduce *bifurcations*. The diagram you are asked to draw in part 2 of Exercise VIII of Worksheet 1 is an example of a *bifurcation diagram.*

Chapter 1 of [S] covers similar ground, with the addition of a section on the existence and uniqueness of solutions of first order equations. Although we didn't use existence and uniqueness results in details they are used explicitly in the proofs of the results on the last page of the Worksheet 1, and uniqueness is needed for part 3 of Exercise I of Worksheet 2. They are also used implicitly throughout the course - for example, phase portraits only make sense when solutions exist and are unique. Existence and uniqueness of solutions for higher dimensional systems are covered in Section 2.2 of [S] and Section 7.2 and Chapter 17 of [HSD].

Worksheet 2: Linear Systems of Differential Equations

The *linear algebra* that underpins solutions of (homogeneous, autonomous) linear systems of equations is reviewed briefly in Sections 5.2 and 5.3 of [D], Section 2.4 of [S] and Sections 2.3 - 2.5 of [HSD] (for two dimensional systems). The discussion in a number of exercises on Worksheet 2 was restricted to n-dimensional systems with n linearly independent eigenvectors (such as those with n distinct eigenvalues). A reasonably full account of the linear algebra needed if this restriction is not imposed is given in Chapter 5 of [HSD]. This is the theory of (real)*Jordan Normal Forms*, although it isn't given that name in [HSD].

The application of linear algebra to the solution of (homogeneous, autonomous) linear systems of differential equations is covered briefly for two dimensional systems in Sections 2.6 and 2.7 of [HSD] and then more generally in Sections 6.1 and 6.3. See also Sections 5.5 and 5.7 - 5.9 of [D] and Section 2.4

of [S]. The latter also contains a brief discussion of the linear algebra of *nonautonomous* linear systems, including the definition of the *Wronskian* of such a system.

Systematic discussions of two dimensional *phase portraits* of homogeneous linear systems among the references listed above is provided by Chapter 3 of [HSD] and Section 4.1.3 of [S]. See also Sections 5.6 - 5.9 of [D]. A summary is contained in Appendix A of [M].

The use of the *trace* and *determinant* of a two dimensional homogeneous linear system to determine the configuration of its eigenvalues (real or complex, and the signs of their real parts) and the type of its phase portrait is described in Section 4.1 of [HSD] and summarised in Appendix A of [M]. Appendix B of [M] gives a very brief description of the *Routh-Hurwitz* conditions for stability which generalise the trace/determinant condition for stability of two dimensional systems and the necessary and sufficient conditions for three dimensional systems discussed in part 5 of Exercise VII on Worksheet 2.

Section 2.5 of [S] provides a brief discussion of *inhomogeneous linear systems* with a particular focus on the method of *variation of parameters* for finding particular solutions. A brief account is also given in Section 5.10 of [D] which in turn refers back to a more detailed discussion of single inhomogeneous n-th order equations in Sections 3.8 - 3.10.

The damped harmonic oscillator is analysed in Section 3.10 of [D] and Section 3.2 of [S].

Worksheet 3: Nonlinear Systems of Differential Equations

The linearization of a nonlinear systems about one of its solutions is discussed in Section 7.4 of [HSD] and applied to equilibrium points in Chapter 8. Linearization at equilibria is also covered briefly in Section 4.2.1 of [S] and is applied to models of interacting populations in Chapter 3 of [M] and chemical reaction systems in Chapter 6. The same chapters also discuss the phase portraits of the models they consider, while a more formal account is given in Section 9.1 of [HSD]. A useful introduction to periodic solutions of systems of differential equations is provided in Chapter 7 of [M].