IMM
Lahore

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Commutative Algebra Fall 2020 (Part II)

Problem Set 1

(1) (Euclidian Algorithm)

- Find polynomials u, v such that $\gcd(f, g) = u\, f + v\, g$ for $f = x^4 - 2x^2 + 1$, $g = (x^2 + 4x + 3)^2 \in \mathbb{R}[x]$
- Find polynomials u, v such that $\gcd(f, g) = u\, f + v\, g$ for $f = x^4 - 2x^2 + 1$, $g = (x^2 + 4x + 3)^2 \in \mathbb{F}_2[x]$ (here $\mathbb{F}_2$ is the field of the integers mod 2, e.g. $4 = 0$ and $3 = 1$).
- Let $f, g \in \mathbb{Q}[x]$. Then one can also consider f, g as polynomials in $\mathbb{R}[x]$.
  Show that $\gcd(f, g)$ is independent of considering f, g in $\mathbb{R}[x]$ or in $\mathbb{Q}[x]$.

(2) Find a polynomial $f \in \mathbb{R}[x]$ such that

$$(f) = \Big(x^4 + 3x^2 - 4, x^3 + 2x^2 - 1, x^4 + 4x^3 + 6x^2 + 4x + 1\Big).$$

(3) Let $m_1, \ldots, m_r$ be monomials in $S = \mathbb{K}[x_1, \ldots, x_n]$, such that as vectorspaces $\dim_{\mathbb{K}} S/(m_1, \ldots, m_r) < \infty$. Show that: For all $1 \leq i \leq n$ there is a $1 \leq j \leq r$ such that mit $m_j = x_i^{a_i}$ for some $a_i \geq 0$.

(4) Let I be a monomial ideal in $\mathbb{K}[x_1, \ldots, x_n]$. Show that:

$$\sqrt{I} = \{f \in k[x_1, \ldots, x_n] \mid \exists\, n \,:\, f^n \in I\}$$

is a monomial ideal generated by squarefree monomials. (The fact that $\sqrt{I}$ is an ideal, holds for a general ideal I in a commutative ring R).