DIFFERENTIAL GEOMETRY

PARTIAL EXAM

The allowed time is 2 hours. Please justify carefully every answer.

Exercise 1 Let C and P be the subsets of $\mathbb{R}^3$ given by the equations

$$C := \left\{(x, y, z) \colon x^2 + y^2 = z^2\right\} \quad \text{and} \quad P := \{(x, y, z) \colon z = 2\} .$$

(1) Prove that $C \cap P$ is the support of a closed parametrized regular curve $\sigma : I \to \mathbb{R}^3$, for some interval $I \subset \mathbb{R}$.
(2) Find an explicit parametrization of σ by arclength and compute its length.

Exercise 2 Let $\sigma \colon (0, +\infty) \to \mathbb{R}^3$ be given by

$$\sigma(t) := \left(t, \frac{1+t}{t}, \frac{1-t^2}{t}\right) .$$

(1) Prove that σ is regular.
(2) Compute the curvature of σ.
(3) Compute the torsion of σ.

Exercise 3 Let $\sigma \colon I \to \mathbb{R}^3$ be a regular curve of class C^∞. Let

$$\sigma(t) = (x(t), y(t), z(t))$$

be an explicit parametrization of σ and consider the curve $\gamma \colon I \to \mathbb{R}^3$ parametrized by

$$\gamma(t) = \frac{\sigma'(t)}{\|\sigma'(t)\|} = \frac{1}{\|\left(x'(t), y'(t), z'(t)\right)\|} \left(x'(t), y'(t), z'(t)\right) .$$

(1) [**More difficult**] Prove that γ is regular if and only if σ is biregular (do **not** assume that σ is parametrized by arc length here).
(2) Assume now that σ is parametrized by arclength. Prove that also γ is parametrized by arclength if and only if the curvature κ of σ satisfies $\kappa \equiv 1$.

Date: 21 October 2019.

DIFFERENTIAL GEOMETRY

PARTIAL EXAM

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Exercise 1 Let C and P be the subsets of $\mathbb{R}^3$ given by the equations

$$C := \left\{(x,y,z) \colon x^2+y^2=z^2\right\} \qquad \text{and} \qquad P := \{(x,y,z) \colon z=2\}.$$

(1) Prove that $C\cap P$ is the support of a closed parametrized regular curve $\sigma : I \to \mathbb{R}^3$, for some interval $I \subset \mathbb{R}$.
(2) Find an explicit parametrization of σ by arclength and compute its length.

Solution. (1) The intersection $C \cap P$ is a circle in $\mathbb{R}^3$, of equations

$$\begin{cases} x^2+y^2=4 \\ z=2. \end{cases}$$

In particular, it is a circle of radius 2. A possible way to write a natural parametrization is to consider $I=[0,2\pi]$ and the map $\sigma\colon I \to \mathbb{R}^3$ given by

$$\sigma(t) = (2\cos t, 2\sin t, 2).$$

The curve σ is regular since $\sigma'(t) = (-2\sin t, 2\cos t, 0)$ is non zero for all values of $t \in I$. The curve is closed since $\sigma(0) = (2,0,2) = \sigma(2\pi)$.

(2) We have $\|\sigma'(t)\| = 2$ for every $t \in I$. We thus need to do a change of variables of the form $s = 2t$, with $s \in [0,4\pi]$. We thus get

$$\tilde{\sigma}(s) = (2\cos(s/2), 2\sin(s/2), 2).$$

This is a parametrization of σ by arclength. Indeed, we have $\tilde{\sigma}'(s) = (\cos(s/2), \sin(s/2), 0)$ and thus $\|\tilde{\sigma}'\| = 1$, as required. The length of $\tilde{\sigma}$ (and of σ) is then 4π (which is indeed the length of a circle of radius 2).

□

Exercise 2 Let $\sigma\colon (0,+\infty) \to \mathbb{R}^3$ be given by

$$\sigma(t) := \left(t, \frac{1+t}{t}, \frac{1-t^2}{t}\right).$$

(1) Prove that σ is regular.
(2) Compute the curvature of σ.
(3) Compute the torsion of σ.

Proof. (1) Taking the first derivative of σ we find that

$$\sigma'(t) = \left(1, -\frac{1}{t^2}, -\frac{1}{t^2} - 1\right).$$

Thus, σ is regular as for every $t \in (0,\infty)$ we have that $\sigma'(t) \neq 0$.

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(2) Taking the second derivative of σ we find
$$\sigma''(t) = \left(0, \frac{2}{t^3}, \frac{2}{t^3}\right).$$
Then we compute that
$$\sigma'(t) \wedge \sigma''(t) = \left(\frac{2}{t^3}, -\frac{2}{t^3}, \frac{2}{t^3}\right),$$
and so
$$\|\sigma'(t) \wedge \sigma''(t)\| = \frac{2\sqrt{3}}{t^3}.$$
From part (1) we have
$$\|\sigma'(t)\| = \frac{\sqrt{2(1+t^2+t^4)}}{t^2}.$$
Then using the formula
$$\kappa(t) = \frac{\|\sigma' \wedge \sigma''\|}{\|\sigma'\|^3},$$
we obtain
$$\kappa(t) = \frac{\sqrt{3}\, t^3}{\sqrt{2}(1+t^2+t^4)^{3/2}}.$$

(3) Calculating the third derivative we obtain
$$\sigma'''(t) = \left(0, -\frac{6}{t^4}, -\frac{6}{t^4}\right).$$
To calculate the torsion we employ the formula
$$\tau(t) = \frac{\langle \sigma'(t) \wedge \sigma''(t), \sigma'''(t)\rangle}{\|\sigma'(t) \wedge \sigma''(t)\|^2}.$$
We note that
$$\langle \sigma' \wedge \sigma'', \sigma'''\rangle = \frac{12}{t^7} - \frac{12}{t^7} = 0,$$
and so $\tau \equiv 0$.

It is also possible to directly observe that the support of σ lies in a plane in $\mathbb{R}^3$, and thus the torsion must necessarily be zero. Indeed, writing $\sigma(t) = (x(t), y(t), z(t))$, we can check that
$$x(t) - y(t) + z(t) = -1$$
for every $t \in (0, +\infty)$. This implies the assertion.

□

Exercise 3 Let $\sigma \colon I \to \mathbb{R}^3$ be a regular curve of class C^∞. Let
$$\sigma(t) = (x(t), y(t), z(t))$$
be an explicit parametrization of σ and consider the curve $\gamma \colon I \to \mathbb{R}^3$ parametrized by
$$\gamma(t) = \frac{\sigma'(t)}{\|\sigma'(t)\|} = \frac{1}{\|(x'(t), y'(t), z'(t))\|}\left(x'(t), y'(t), z'(t)\right).$$

(1) **[More difficult]** Prove that γ is regular if and only if σ is biregular (do **not** assume that σ is parametrized by arc length here).

(2) Assume now that σ is parametrized by arclength. Prove that also γ is parametrized by arclength if and only if the curvature κ of σ satisfies $\kappa \equiv 1$.

Solution. (1) Recall that, by definition, a curve is regular if the tangent vector is never zero and biregular if the curvature is never zero. In our case, the curvature κ of σ, is given by

$$\kappa = \frac{\|\sigma' \wedge \sigma''\|}{\|\sigma'\|^3}$$

(since we cannot assume that σ is parametrized by arclength for now). In particular, we have that

$$\sigma \text{ is not regular at } t_0 \text{ if and only if } \exists A \text{ such that } \sigma''(t_0) = A\sigma'(t_0).$$

Let us look for a similar condition for the regularity of γ. We have

$$\gamma'(t_0) = \frac{d}{dt}\frac{\sigma'}{\|\sigma'\|} = \frac{\|\sigma'\|^2\sigma'' - \langle \sigma', \sigma''\rangle\sigma'}{\|\sigma'\|^3}$$

where the right hand side is evaluated at t_0. Thus, γ is not regular if and only if the numerator of the last expression is zero. This happens precisely when σ'' is a multiple of σ'. This gives the desired equivalence.

(2) Assume now that σ is parametrized by arc length. This means that $\|\dot{\sigma}(s)\| = 1$ for every s. In this case, we thus have $\gamma' = \ddot{\sigma}$ and $\kappa = \|\ddot{\sigma}\|$. We thus have

$$\kappa \equiv 1 \quad \Leftrightarrow \quad \|\ddot{\sigma}\| \equiv 1 \quad \Leftrightarrow \quad \|\gamma'\| \equiv 1 \quad \Leftrightarrow \quad \gamma \text{ is parametrized by arc length}$$

The proof is complete.

□

DIFFERENTIAL GEOMETRY

FINAL EXAM

The allowed time is 3 hours. Please justify carefully every answer, and quote the statements for the lectures that you use.

Exercise 1 Let $X \subset \mathbb{R}^3$ be defined as

$$X := \{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 = z\}$$

and let $\phi \colon \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$\phi(u, v) := (u + v, u - v, 2u^2 + 2v^2).$$

(1) Prove that X is a regular surface.
(2) Prove that ϕ is a global parametrization of X.
(3) Compute the Gaussian curvature of X with respect to the parametrization ϕ.
(4) Compute the mean curvature of X with respect to the parametrization ϕ.

Exercise 2 Let $\sigma : \mathbb{R} \to \mathbb{R}^3$ be defined as

$$\sigma(t) = \begin{pmatrix} e^{t \sin\alpha} \cos t \\ e^{t \sin\alpha} \sin t \\ e^{t \sin\alpha} \cot\alpha \end{pmatrix}$$

where $\alpha \in \left(0, \frac{\pi}{2}\right)$ and $\cot\alpha = \frac{\cos\alpha}{\sin\alpha}$.

(1) Prove that σ is a regular curve.
(2) Prove that the support of σ is contained in $C_a \cap \{z > 0\}$ for some positive a, where C_a is the cone of equation $x^2 + y^2 = az^2$. Prove that $\sigma(t) \to (0, 0, 0)$ as $t \to -\infty$
(3) Prove that, for every $T \in \mathbb{R}$, the curve $\sigma_{(-\infty,T]}$ is rectifiable.
(4) Find a parametrization of σ by arclength.

Date: 6 November 2019.

DIFFERENTIAL GEOMETRY

FINAL EXAM

The allowed time is 3 hours. Please justify carefully every answer, and quote the statements for the lectures that you use.

Exercise 1 Let $X \subset \mathbb{R}^3$ be defined as

$$X := \{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 = z\}$$

and let $\phi \colon \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$\phi(u, v) := (u + v, u - v, 2u^2 + 2v^2).$$

(1) Prove that X is a regular surface.
(2) Prove that ϕ is a global parametrization of X.
(3) Compute the Gaussian curvature of X with respect to the parametrization ϕ.
(4) Compute the mean curvature of X with respect to the parametrization ϕ.

Solution. (1) X is the zero level of the function $G \colon \mathbb{R}^3 \to \mathbb{R}$ given by $G(x, y, z) = x^2 + y^2 - z$. Since the preimage of a regular value is a regular surface, it is enough to prove that 0 is a regular value for G. Recall that this means that no point q in the preimage of 0 is critical, i.e., satisfies $\nabla G = 0$. Let us compute the gradient of G. We have

$$\nabla G(x, y, z) = (2x, 2y, -1).$$

We see that ∇G never vanishes. In particular, 0 is a regular value and X is a regular surface.

(2) As X the zero level set of the function G and 0 is regular value for G it suffices to check that $\phi(\mathbb{R}^2) \subset X$, ϕ is injective and $d(\phi)_{(u,v)}$ is injective for all $(u, v) \in \mathbb{R}^2$ in order to conclude that ϕ is a local parametrization of X.

The first check is immediate: for every $(u, v) \in \mathbb{R}^2$ we have

$$(u + v)^2 + (u - v)^2 - (2u^2 + 2v^2) = 0,$$

which proves that $\phi(u, v) \in X$.

For the injectivity of ϕ it is enough to consider the first two coordinates: it is straightforward to check that for every $x, y \in \mathbb{R}^2$ the system

$$\begin{cases} u + v = x \\ u - v = y \end{cases}$$

has one and only one solution.

Finally, let us compute the differential $d\phi$. We have

$$d\phi_{(u,v)} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 4u & 4v \end{pmatrix}.$$

The determinant of the minor given by the first two rows is constantly equal to -2. Thus, the two columns of the matrix representing $d\phi$ are always independent, and this proves that $d\phi$ is injective for all $(u, v) \in \mathbb{R}^2$. This establishes that ϕ is a local parametrization.

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To see that ϕ is a global parametrization it suffices to show that $X = \phi(\mathbb{R}^2)$. Since we have already checked that $\phi(\mathbb{R}^2) \subset X$ it suffices to show the opposite inclusion. Indeed for an arbitrary $(x, y, z) \in X$ we may take $u = \frac{1}{2}(x + y)$, $v = \frac{1}{2}(x - y)$ to see that

$$\phi(u, v) = (x, y, z).$$

This concludes that $X = \phi(\mathbb{R}^2)$ and that ϕ is a global parametrization of X.

(3) In order to compute the Gaussian curvature we compute the metric coefficients E, F, G and the form coefficients e, f, g of the first and second fundamental forms with respect to the parametrization ϕ.

Let us denote by ∂_u and ∂_v the tangent vectors induced by ϕ corresponding to the coordinates (u, v) on $\mathbb{R}^2$. We have

$$\partial_u = (1, 1, 4u)$$

and

$$\partial_v = (1, -1, 4v)$$

Thus, we deduce that at the point $p = \phi(u, v)$ we have

$$E = \langle \partial_u, \partial_u \rangle_p = 2 + 16u^2$$
$$F = \langle \partial_u, \partial_v \rangle_p = 16uv$$
$$G = \langle \partial_v, \partial_v \rangle_p = 2 + 16v^2.$$

In order to compute the form coefficients of the second fundamental form, we need to compute the Gauss map N and the second derivatives of the coordinates of ϕ.

Since X is a regular surface given as level set of a regular value of a smooth function, it is orientable. Thus the Gauss map is well defined. A way to compute it is to compute $\frac{\partial_u \wedge \partial_v}{\|\partial_u \wedge \partial_v\|}$ at the point $p = \phi(u, v)$. From the expression above of ∂_u and ∂_v we get

$$\partial_u \wedge \partial_v = (4u + 4v, 4u - 4v, -2)\,.$$

The norm of the above vector is $2\sqrt{8u^2 + 8v^2 + 1}$. Thus the choice of N induced by this basis is

$$N(p) = N(\phi(u, v)) = \frac{(2u + 2v, 2u - 2v, -1)}{\sqrt{8u^2 + 8v^2 + 1}},$$

and the computation is complete (another possibility to compute N is to consider the versor associated to ∇G at $p = \phi(u, v)$).

The last ingredients we need are the second derivatives of the coordinates of ϕ. As vectors, we get

$$\frac{\partial^2 \phi}{\partial^2 u} = (0, 0, 4)$$
$$\frac{\partial^2 \phi}{\partial u \partial v} = (0, 0, 0)$$
$$\frac{\partial^2 \phi}{\partial^2 v} = (0, 0, 4)$$

and so, with the expression of $N \circ \phi$ found above, we get

$$e = \langle N \circ \phi, \frac{\partial^2 \phi}{\partial^2 u} \rangle = \frac{-4}{\sqrt{8u^2 + 8v^2 + 1}}$$
$$f = \langle N \circ \phi, \frac{\partial^2 \phi}{\partial u \partial v} \rangle = 0$$
$$g = \langle N \circ \phi, \frac{\partial^2 \phi}{\partial^2 v} \rangle = \frac{-4}{\sqrt{8u^2 + 8v^2 + 1}}.$$

We can finally compute the Gaussian curvature. We have

$$K = \frac{eg - f^2}{EG - F^2} = \left(\frac{16}{8u^2 + 8v^2 + 1}\right)\left(\frac{1}{4 + 32u^2 + 32v^2}\right) = \frac{4}{(8u^2 + 8v^2 + 1)^2}.$$

(4) We know that the mean curvature is given by

$$H = \frac{1}{2}\frac{eG - 2fF + gE}{EG - F^2}.$$

Thus, with the values of E, F, G, e, f, g computed above, we get

$$H = \frac{1}{2}\left(\frac{-4(4+16(u^2+v^2))}{\sqrt{8u^2+8v^2+1}}\right)\left(\frac{1}{4+32u^2+32v^2}\right) = -\frac{2+8(u^2+v^2)}{(8u^2+8v^2+1)^{3/2}}.$$

□

Exercise 2 Let $\sigma : \mathbb{R} \to \mathbb{R}^3$ be defined as

$$\sigma(t) = \begin{pmatrix} e^{t\sin\alpha}\cos t \\ e^{t\sin\alpha}\sin t \\ e^{t\sin\alpha}\cot\alpha \end{pmatrix}$$

where $\alpha \in \left(0, \frac{\pi}{2}\right)$ and $\cot\alpha = \frac{\cos\alpha}{\sin\alpha}$.

(1) Prove that σ is a regular curve.
(2) Prove that the support of σ is contained in $C_a \cap \{z > 0\}$ for some positive a, where C_a is the cone of equation $x^2 + y^2 = az^2$. Prove that $\sigma(t) \to (0,0,0)$ as $t \to -\infty$
(3) Prove that, for every $T \in \mathbb{R}$, the curve $\sigma_{(-\infty,T]}$ is rectifiable.
(4) Find a parametrization of σ by arclength.

Solution. (1) In order to check that σ is regular, we need to check that the vector $\sigma'(t)$ is non zero for every $t \in \mathbb{R}$. A direct computation gives

$$\sigma'(t) = e^{t\sin\alpha}\begin{pmatrix} \sin\alpha\cos t - \sin t \\ \sin\alpha\sin t + \cos t \\ \cos\alpha \end{pmatrix}$$

and we see that the last component is non zero for all $t \in \mathbb{R}$ (since the exponential is always positive and $\cos\alpha > 0$).

(2) Let us denote with $(x(t), y(t), z(t))$ the point $\sigma(t)$. For every $t \in \mathbb{R}$ we have

$$x(t)^2 + y(t)^2 = e^{2t\sin\alpha}$$

and

$$z(t)^2 = e^{2t\sin\alpha}(\cot\alpha)^2,$$

Thus, the support of σ is contained in the cone C_a, with $a = \frac{1}{(\cot\alpha)^2}$. whose half-angle at the origin is α.

Since the exponential is positive and $\cot\alpha > 0$ for $\alpha \in (0, \frac{\pi}{2})$, we have $z(t) > 0$ for all $t \in \mathbb{R}$. So, the support is also contained in the half space $\{z > 0\}$.

As $t \to -\infty$, we have $z(t) = e^{t\sin\alpha} \to 0$. Since the components of the vector $(\cos t, \sin t, \cot\alpha)$ are bounded, we have $\sigma(t) \to (0,0,0)$ as $t \to -\infty$, as required.

(3) We need to show that the length L_T of $\sigma_{(-\infty,T]}$ is finite. First of all, we need to compute the norm of $\sigma'(t)$. From the expression found above, we get

$$\|\sigma'(t)\| = \sqrt{2}e^{t\sin\alpha}.$$

So, we deduce that

$$L_T = \int_{-\infty}^{T} \|\sigma'(t)\| dt = \int_{-\infty}^{T} \sqrt{2}e^{t\sin\alpha} dt = \frac{\sqrt{2}}{\sin\alpha}e^{T\sin\alpha},$$

which is finite for every $T \in \mathbb{R}$.

(4) By the computations in the previous item, we can reparametrize σ as $\tilde{\sigma}\colon [0,+\infty) \to \mathbb{R}^3$, with parameter $s = s(t) = L_t$ and $\tilde{\sigma}(0) = (0,0,0)$. We have

$$\frac{\sqrt{2}}{\sin\alpha} e^{t\sin\alpha} = s$$

which gives

$$t = \frac{1}{\sin\alpha} \log\left(\frac{s\sin\alpha}{\sqrt{2}}\right).$$

A parametrization by arclength is then given by

$$\tilde{\sigma}(s) = \sigma(t(s)) = \sigma\left(\frac{1}{\sin\alpha}\log\left(\frac{s\sin\alpha}{\sqrt{2}}\right)\right)$$

which gives

$$\tilde{\sigma}(s) = \frac{s\sin\alpha}{\sqrt{2}} \begin{pmatrix} \cos\left(\frac{1}{\sin\alpha}\log\left(\frac{s\sin\alpha}{\sqrt{2}}\right)\right) \\ \sin\left(\frac{1}{\sin\alpha}\log\left(\frac{s\sin\alpha}{\sqrt{2}}\right)\right) \\ \cot\alpha \end{pmatrix}$$

Let us check that this parametrization is indeed by arclength. We have

$$\tilde{\sigma}'(s) = \frac{\sin\alpha}{\sqrt{2}} \begin{pmatrix} \cos\left(\frac{1}{\sin\alpha}\log\left(\frac{s\sin\alpha}{\sqrt{2}}\right)\right) \\ \sin\left(\frac{1}{\sin\alpha}\log\left(\frac{s\sin\alpha}{\sqrt{2}}\right)\right) \\ \cot\alpha \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} -\sin\left(\frac{1}{\sin\alpha}\log\left(\frac{s\sin\alpha}{\sqrt{2}}\right)\right) \\ \cos\left(\frac{1}{\sin\alpha}\log\left(\frac{s\sin\alpha}{\sqrt{2}}\right)\right) \\ 0 \end{pmatrix}$$

and so

$$\|\tilde{\sigma}'(s)\|^2 = \left(\frac{\sin\alpha}{\sqrt{2}}\right)^2 \left(1 + (\cot\alpha)^2\right) + \frac{1}{2} = \frac{1}{2} + \frac{1}{2} = 1.$$

□