Mid-term exam in Algebraic Topology
10 December 2019

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The following exercises are independent.

1. Let $B = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$.
   Show that B is homeomorphic to the quotient space $[0,1] \times [0,1]/\sim$ where the projection is

$$\begin{array}{rcl} [0,1]\times[0,1] & \longrightarrow & B \\ (s,t) & \longmapsto & se^{2i\pi t} \end{array}$$

   Give the equivalence relation $\sim$ and determine the open sets in the quotient space $[0,1] \times [0,1]/\sim$.

2. Let $f, g : X \longrightarrow \mathbb{R}^n$ be two continuous maps where $\mathbb{R}^n$ is equipped with the standard topology.
   Are this maps f and g homotopic?

3. A continuous map $f : X \longrightarrow Y$ is an embedding if the map

$$\tilde{f} : X \longrightarrow f(X)$$

   is a homeomorphism.
   Given a topological space, show that the inclusion of a subspace into the space is an embedding.

4. Is the space $\mathbb{R}^n \setminus \{0\}$ simply connected?

5. Let $(X, \tau_X), (Y, \tau_Y)$ and (Z, τ_Z) be three topological spaces and $f, g : X \longrightarrow Y \times Z$ two continuous maps.
   Show that the maps f and g are homotopic iff the maps $p_Y \circ f$ and $p_Y \circ g$ are homotopic and the maps $p_Z \circ f$ and $p_Z \circ g$

COMSATS

Final exam in Algebraic Topology

2019

The following exercises are independent.

1. Consider the following map

$$\begin{array}{rcll} f:[0,1]\times[0,1] & \longrightarrow & \mathbb{R}^2 & \\ (s,t) & \longmapsto & (-1+2s,2s(2t-1)) & \text{for } 0\leq s\leq \frac{1}{2} \\ (s,t) & \longmapsto & (-1+2s,2(1-s)(2t-1)) & \text{for } \frac{1}{2}\leq s\leq 1 \end{array}$$

   (a) Show that f is a continuous map.
   (b) Determine $f([0,1]\times[0,1])$, i.e. as a subset of $\mathbb{R}^2$.
   (c) Show that $f([0,1]\times[0,1])$ is homeomorphic to the quotient space $[0,1]\times[0,1]/\sim$ where f is the projection.
   (d) Determine the open sets of $f([0,1]\times[0,1])$.

2. Let f and g be two maps of a singleton to the space X where

   (a) $X=\mathbb{R}^2\setminus(\mathbb{Q}\times\{0\})$. Are these two maps homotopic?
   (b) $X=\mathbb{R}^3\setminus\{(x,y,z)\mid z=0\}$. Are these two maps homotopic?
   (c) $X=\mathbb{R}^3\setminus\{(x,y,z)\mid y=0,z=0\}$. Are these two maps homotopic?

3. Given two topological spaces (X,τ_X) and (Y,τ_Y), we consider the homotopy classes of maps of X to Y.

   (a) Let $Y=[0,1]$ as subspace of $\mathbb{R}$ with the standard topology. Show that any two maps of X to Y are homotopic.
   (b) Suppose that X is path-connected. Determine the homotopy classes of X to Y.

4. A space is contractible if the identity map is homotopic to a constant map.

   (a) Show that $[0,1]$ and $\mathbb{R}$ with the respectively induced and standard topologies, are contractible.
   (b) Show that a contractible space is path-connected.
   (c) Show that if Y is contractible, then for any space X, the maps of X to Y are homotopic.
   (d) Show that if X is contractible and Y is path-connected, the maps of X to Y are homotopic.