FUNCTIONAL ANALYSIS - IMM

LIST 4: WEAK TOPOLOGIES

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The objective of this list is to prove that when a Banach space $\mathcal{H}$ is separable, then the closed unit ball B_1^* of the dual $\mathcal{H}^*$ is *metrizable* in the w^*-topology.

1. Let T_1 and T_2 be two topologies on a set X such that
   (1) X is Hausdorff with respect to T_1,
   (2) X is compact with respect to T_2, and
   (3) $T_1 \subseteq T_2$.

Prove that $T_1 = T_2$. *Hint: Show that any T_2-closed subset F of X is also T_1-closed.*

2. Let $(x_n)_{n\in\mathbb{N}}$ be a countable dense subset of the unit ball B_1 of $\mathcal{H}$. Let ρ be the weak topology on $\mathcal{H}^*$ induced by the family of evaluations at each x_n

$$\mathcal{F} = \{\hat{x}_n : \mathcal{H}^* \to \mathbb{C} : \hat{x}_n(\phi) = \phi(x_n)\}$$

Show that $\rho \subseteq w^*$-topology and that a base of neighborhoods for the topology ρ at a point $\phi_0 \in \mathcal{H}^*$ is given by sets of the form

$$V_\epsilon(\phi_0; x_1, \ldots, x_N) = \{\phi \in \mathcal{H}^* : |\phi(x_n) - \phi_0(x_n)| < \epsilon \ \ \forall\, n = 1, \ldots, N\}$$

Hint: Remember $\mathcal{F}$ is the topology of pointwise convergence at each x_n.

3. Prove that ρ is *metrizable* by showing that

$$d(\phi_1, \phi_2) := \sum_{n=1}^{\infty} 2^{-n} \min(1, |\phi_1(x_n) - \phi_2(x_n)|) \quad \text{for} \quad \phi_1, \phi_2 \in \mathcal{H}^*$$

defines a metric on $\mathcal{H}^*$ that generates the topology ρ. *Hint: Use question 2.*

4. Show that the toplogy ρ coincides with the w^*-topology on B_1^*. Therefore B_1^* is a compact metric space in the w^*-topology and hence every sequence in B_1^* has a w^*-convergent subsequence.

Hint: Use questions 1, 2 and 3, the Banach-Alaoglu Theorem, and the fact that metric topologies are Hausdorff.