FUNCTIONAL ANALYSIS - IMM

LIST 2: LINEAR FUNCTIONALS

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1. Show that every linear functional on a Banach space $\mathcal{H}$ is continuous if and only if $\mathcal{H}$ is finite dimensional.

2. Let $\mathcal{H} = \mathcal{C}([0,1])$ with the supremum norm and consider the linear functional $\phi : \mathcal{H} \to \mathbb{C}$ defined by

$$\phi(f) = \int_0^1 f(t)dt.$$

Prove that $\phi \in \mathcal{H}^*$ and $||\phi|| = 1$. *Hint: Consider the functions $f_\alpha(t) = t^\alpha$ for $\alpha > 0$.*

3. Let $M \subset \mathcal{H}$ be a linear subspace of a Banach space $\mathcal{H}$. The *annihilator* $M^\perp$ of M is the subset of $\mathcal{H}^*$ defined by

$$M^\perp := \{\phi \in \mathcal{H}^* : \phi(f) = 0 \ \ \forall \ \ f \in M\}.$$

Prove that $M^\perp$ is a closed linear subspace of $\mathcal{H}^*$.

4. Let $\mathcal{H}$ be a Banach space. For each $f \in \mathcal{H}$, we can define a functional $\hat{f} : \mathcal{H}^* \to \mathbb{C}$ on the dual space $\mathcal{H}^*$ by

$$\hat{f}(\phi) = \phi(f) \qquad \forall \ \ \phi \in \mathcal{H}^*.$$

Prove that $\hat{f}$ belongs to $\mathcal{H}^{**} := (\mathcal{H}^*)^*$ (the *second dual* of $\mathcal{H}$). Show that the map

$$f \longrightarrow \hat{f}$$

is an injective linear map (*linear embedding*) of $\mathcal{H}$ into $\mathcal{H}^{**}$.