FUNCTIONAL ANALYSIS - IMM

LIST 1: SPACES OF CONTINUOUS FUNCTIONS

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1. Let $\mathcal{H}$ be a Banach space. Prove that the following functions are continuous:
   - $a : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ defined by $a(f, g) = f + g$,
   - $s : \mathbb{C} \times \mathcal{H} \to \mathcal{H}$ defined by $s(\lambda, f) = \lambda f$, and
   - $n : \mathcal{H} \to \mathbb{R}^+$ defined by $n(f) = ||f||$.

Hint: Use the norms $||(f, g)|| := ||f|| + ||g||$ on $\mathcal{H} \times \mathcal{H}$ and $||(\lambda, f)|| := |\lambda| + ||f||$ on $\mathbb{C} \times \mathcal{H}$ to obtain ϵ-δ proofs for continuity.

2. Let X be a compact Hausdorff space. Prove that $\mathcal{C}(X)$ is finite dimensional if and only if X is finite.

3. Let $(f_n)_{n\in\mathbb{N}} \subset \mathcal{C}(0, 1)$ be the sequence of functions defined by $f_n(x) = x^n$. Prove that f_n converges to 0 pointwise, but does *not* converge in $\mathcal{C}(0, 1)$.

4. Let $\mathcal{F} \subset \mathcal{C}([-\pi, \pi])$ be the collection of all *finite Fourier series*

$$f(x) = \sum_{n=0}^{N} (a_n \sin nx + b_n \cos nx)$$

where $a_n, b_n \in \mathbb{R}$. Show that the derivative map $D : \mathcal{F} \to \mathcal{F}$ defined by $D(f) = f'$ is *not* continuous on $\mathcal{F}$ with the sup-norm metric.

Hint: Find a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{F}$ such that $f_n \to 0$, but $f_n' \nrightarrow 0$ in the sup-norm.

5. Prove that the map $I : \mathcal{C}([a, b]) \to \mathcal{C}([a, b])$ for $a, b \in \mathbb{R}$ defined by

$$I(f)(x) = \int_a^x f(t)dt$$

is continuous (with respect to the sup-norm metric).