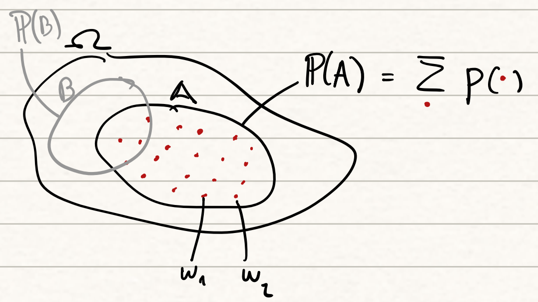
Ω

probabilities $\qquad\qquad$ discrete densities

$\mathbb{P} : \mathcal{P}(\Omega) \to [0,1] \qquad\qquad p : \Omega \to [0,1]$

$$\mathbb{P}(\underset{\subseteq \Omega}{A}) := \sum_{\omega \in A} p(\omega) \longleftarrow p$$



Conditional probability

ex
- what is the probability that a cricket match is cancelled?
- what is the probability that a cricket match is cancelled if today it's raining?

def (cond. probability)

A, B be events s.t. $\mathbb{P}(B) > 0$

$$\mathbb{P}(A \mid B) := \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

prob. on A conditioned on B

prop B event $\mathbb{P}(B) > 0$

then
$$\mathcal{P}(\Omega) \longrightarrow [0,1]$$
$$A \longmapsto \mathbb{P}(A \mid B)$$

is a probability on Ω.

proof $\triangle$

<u>prop</u> (law of total probability)

$(\Omega, \mathcal{P}(\Omega), \mathbb{P})$

$(B_i)_i$ a (finite or countable) partition of Ω.

$$\begin{cases} \bigcup_{i \in I} B_i = \Omega \\ B_i \cap B_j = \emptyset \quad i, j \end{cases}$$

Then

(1) $\mathbb{P}(A) = \sum_i \mathbb{P}(A \cap B_i)$

Moreover, if $\mathbb{P}(B_i) > 0$, then

(2) $\mathbb{P}(A) = \sum_i \mathbb{P}(A \mid B_i)\, \mathbb{P}(B_i)$

<u>proof</u> (1) $A = A \cap \Omega = A \cap \left(\bigcup_{i \in I} B_i \right) = \bigcup_{i \in I} \underbrace{(A \cap B_i)}_{\text{all disjoint}}$

$\rightsquigarrow$ σ-additivity

(2) apply definition of conditional probability

$$\mathbb{P}(A \cap B_i) = \mathbb{P}(A \mid B_i)\, \mathbb{P}(B_i)$$

Theorem (Bayes' formula)

A, B events, $\mathbb{P}(A), \mathbb{P}(B) > 0$

$$\mathbb{P}(B \mid A) = \frac{\mathbb{P}(A \mid B)\, \mathbb{P}(B)}{\mathbb{P}(A)}$$

<u>proof</u> $\mathbb{P}(B \mid A)\mathbb{P}(A) = \mathbb{P}(B \cap A) = \mathbb{P}(A \mid B)\mathbb{P}(B)$

def. of conditional probability

<u>example</u> test to detect a virus

- if virus present: 99% positive
- if virus NOT present: 2% positive

assume that 4 people out of 10000 have the virus.

Question: if one person tests positive, what is the probability that they actually have the virus?

A = "test is positive"

B = "individual is sick"

$$\mathbb{P}(B \mid A) = \frac{\overbrace{\mathbb{P}(A \mid B)}^{0.99}\ \overbrace{\mathbb{P}(B)}^{0.0004}}{\mathbb{P}(A)}$$

$$= \frac{\mathbb{P}(A|B)\mathbb{P}(B)}{\underbrace{\mathbb{P}(A|B)\mathbb{P}(B)}_{} + \underbrace{\mathbb{P}(A|B^c)}_{0.02}\underbrace{\mathbb{P}(B^c)}_{1-0.0004=0.9996}} \simeq 0.02$$

law of total probability

Independence

$$\mathbb{P}(A|B) := \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

if $\mathbb{P}(A \mid B) = \mathbb{P}(A)$, then $\underline{\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)}$

def (independence of two events)

A, B are independent if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$$

example • extract card from a poker deck (52 cards) 4 suits, 13 cards each

A = "card is ♠"

B = "card is a three"

⚠ A, B are independent

• throw two 6-sided dice one at a time

A = "sum is 7", B = "first die is 3"

second \ first	1	2	3	4	5	6
1	(1,1)	(2,1)	(3,1)			
2						
3						
4			$\boxed{(3,4)}$			
5						
6						

$$P(A \cap B) = \boxed{\frac{1}{36}}$$

$$\mathbb{P}(A) = \mathbb{P}((3,4),(4,3),(1,6),(6,1),(2,5),(5,2)) = \frac{6}{36}$$

$$\mathbb{P}(B) = \mathbb{P}((3,1),(3,2)\dots(3,6)) = \frac{6}{36}$$

$$\Rightarrow \mathbb{P}(A)\mathbb{P}(B) = \frac{1}{36}$$

more generally,

__def__ A family of events $(A_i)_{i \in I}$ is said to be independent if

$$\mathbb{P}\Big(\bigcap_{i \in J} A_i\Big) = \prod_{i \in J} \mathbb{P}(A_i)$$

for __every__ subset $J \subseteq I$.

__example__

throw two 6-sided dice

$$\Omega = \{1,2\dots,6\}^2 = \{(1,1),(1,2)\dots,(6,6)\}$$

$\mathbb{P}$ uniform probability

$$\mathbb{P}((1,1)) = \frac{1}{36} = \mathbb{P}((1,2)) = \mathbb{P}((1,3)) = \dots \mathbb{P}((6,6))$$

$A = \{$the second die is 1, 2 or 5$\}$

$B = \{$the second die is 4, 5 or 6$\}$

$C = \{$the sum of the two dice is 9$\}$

write them as subsets of Ω

check if $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C)$?

$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}((4,5)) = \frac{1}{36}$$

$$\mathbb{P}(A) = \frac{1}{2} = \mathbb{P}(B)$$

$$\mathbb{P}(C) = \mathbb{P}((3,6),(6,3),(4,5),(5,4)) = \frac{4}{36} = \frac{1}{9}$$

$$\mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C) = \frac{1}{36}$$

$$\mathbb{P}(A \cap B) = \mathbb{P}(\text{second die is } 5) = \frac{1}{6}$$

$$\mathbb{P}(A)\mathbb{P}(B) = \frac{1}{4} \neq$$

Combinatorics

focus on uniform probability

Ω finite sample space

$$\omega \in \Omega \quad \mathbb{P}(\{\omega\}) = \frac{1}{|\Omega|}$$

$$\underset{\Omega}{\underset{\cap}{\mathbb{P}(A)}} = \frac{|A|}{|\Omega|}$$ need to determine these

common situations

1) arrangements with repetitions

there is a set A and I have to extract k elements from it, and each time I put the element I extracted back.

prop the number of arrangements with repetitions is n^k

2) arrangements **without** repetitions

there is a set A and I have to extract k elements from it, and each time I DON'T put the element I extracted back.

prop number of arrangements without repetitions is

$$D_{n,k} = \frac{n!}{(n-k)!}$$

number of elements in the set

number of elements extracted

why? $x_1 \times x_2 \times \dots \times x_n$

n choices, $n-1$ choices, $n-k+1$ choices $= \frac{n!}{(n-k)!}$

permutation group S_n $\quad n=k \rightsquigarrow |S_n| = n!$

3) **Combinations**

def A finite set, the combinations of k elements extracted from A are **all** the subsets of A of cardinality k.

Prop the number of combinations is

$$C_{n,k} = \binom{n}{k} := \frac{n!}{k!\,(n-k)!}$$

proof $C_{n,k} = \dfrac{D_{n,k}}{k!}$

$(1,2,3)$ $(3,2,1)$

$(2,1,3)$

3

$A \cap B = \emptyset$

$$\mathbb{P}(A \cap B) = \mathbb{P}(\emptyset) = 0 \cancel{\overset{?}{=}} \underset{\substack{\vee \\ 0}}{\mathbb{P}(A)}\,\underset{\substack{\vee \\ 0}}{\mathbb{P}(B)}$$

two 6-sided dice

$A =$ "the first die is 6"

$B =$ "the sum of the two dice is 12"

$\mathbb{P}(A) = \dfrac{1}{6}$

$\mathbb{P}(B) = \mathbb{P}(\,(6,6)\,) = \dfrac{1}{36}$

$\mathbb{P}(A \cap B) = \dfrac{1}{36} = \mathbb{P}(\,(6,6)\,)$