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Probability

Probabilistic model

1) Ω sample space (countable)

   ex. • 1 dice throw $\Omega = \{1, \dots, 6\}$

   • arrival time of the bus

   $\Omega = [0, \infty)$

   • 2 dice throws ?

2) events

   $\mathcal{F} = \mathcal{P}(\Omega)$

   all subsets of Ω

   special events: • $\emptyset$ impossible

   • Ω certain

   examples $\Omega = \{1, \dots, 6\}$

   $A = (\text{"even number"}) = \{2, 4, 6\} \subseteq \Omega$

   $B = (\text{"multiple of 3"}) = \{3, 6\}$

   $C = (\text{"even number } \underset{\text{and}}{\_} \text{ multiple of 3"}) = A \cap B = \{6\}$

statements	set operations
A and B	$A \cap B$
A or B	$A \cup B$
not A	A^c

3) probability

mathematically, $\mathbb{P} : \mathcal{P}(\Omega) \longrightarrow [0,1]$

$$A \longmapsto \mathbb{P}(A) \in [0,1]$$

<u>def</u> (axioms of probability) Ω countable set

$$\mathbb{P} : \mathcal{P}(\Omega) \longrightarrow [0,1]$$

such that

(A1) $\mathbb{P}(\Omega) = 1$

(σ-additivity) (A2) $(A_i)_i \subseteq \mathcal{P}(\Omega)$ such that

$$A_i \cap A_j = \emptyset \qquad i,j = 1,\dots$$

then

$$\mathbb{P}\Big(\dot{\bigcup_{i=1}^{\infty}} A_i\Big) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

<u>example</u> $\Omega = \{1,\dots,6\}$

$\mathbb{P}(i) = \frac{1}{6} \qquad i = 1,\dots,6$

$$\mathbb{P}(\{1,2\}) \underset{\substack{\uparrow \\ \text{must be}}}{=} \mathbb{P}(\{1\}) + \mathbb{P}(\{2\}) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

$(\Omega, \mathcal{F} = \mathcal{P}(\Omega), \mathbb{P})$ is called a (discrete) probability space

<u>properties</u> $(\Omega, \mathcal{F}, \mathbb{P})$. Then

(i) $\mathbb{P}(\emptyset) = 0$

(ii) finite additivity

$(A_i)_{i=1}^{n}$ $\quad A_i \cap A_j = \emptyset \qquad i,j = 1,\dots n$, $i \neq j$

then

$$\mathbb{P}\Big(\dot{\bigcup_{i=1}^{n}} A_i\Big) = \sum_{i=1}^{n} \mathbb{P}(A_i)$$

proof (i) $\mathbb{P}(\emptyset) \underset{(A2)}{=} \sum_{i=1}^{\infty} \mathbb{P}(\emptyset) \quad \Rightarrow \quad \mathbb{P}(\emptyset) = 0$

$\emptyset = \emptyset \cup \emptyset \ldots$

$\left.\begin{array}{l} B_1 = \emptyset, B_2 = \emptyset \ldots \\ B_i \cap B_j = \emptyset \end{array}\right\} \Rightarrow \mathbb{P}\left(\bigcup_{i=1}^{\infty} \emptyset\right) = \sum_{i=1}^{\infty} \mathbb{P}(\emptyset)$

(ii) $A_1, \ldots, A_n, \emptyset, \ldots \emptyset \ldots$

$$\mathbb{P}(A_1 \cup \ldots \cup A_n \cup \emptyset \cup \ldots) \underset{(A2)}{=} \sum_{i=1}^{n} \mathbb{P}(A_i) + \underbrace{\sum_{i=n+1}^{\infty} \mathbb{P}(\emptyset)}_{=0} \quad \checkmark$$

in particular, $A \cap B = \emptyset$

then $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$

def (probability mass function, discrete density)

pmf dd

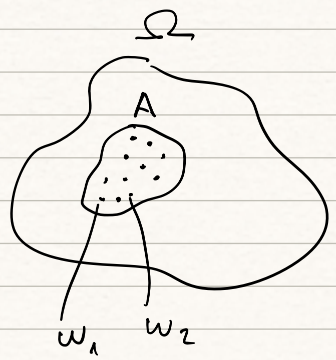
Ω non-empty countable set

$$p : \Omega \to [0,1]$$

s.t. $\sum_{\omega \in \Omega} p(\omega) = 1.$

prop (discrete density $\Rightarrow$ probability)

Let p be a dd. Define for $A \in \mathcal{P}(\Omega)$



$$\mathbb{P}(A) = \sum_{\omega \in A} p(\omega)$$

Then, $\mathbb{P}(.)$ satisfies (A1), (A2)

in other words, $\mathbb{P}$ is a probability.

proof (A2) $\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) \overset{\text{def of } \mathbb{P}}{=} \sum_{\omega \in \bigcup_{i=1}^{\infty} A_i} p(\omega) = \sum_{i=1}^{\infty} \left(\sum_{\omega \in A_i} p(\omega) \right)$

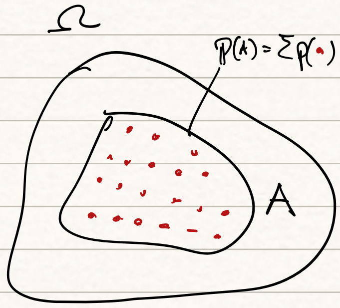
(A1) △

$$= \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

$\mathbb{P}(A_i)$

to summarize, Ω countable set

there is a bijection between probabilities and dd's on Ω.



have

$$\mathbb{P} \longrightarrow p(\omega) := \mathbb{P}(\{\omega\})$$

$\omega \in \Omega$

have

$$\mathbb{P}(A) := \sum_{\omega \in A} p(\omega) \longleftarrow p$$

proof △

<u>example</u> uniform probability

Ω <u>finite</u> set. Define

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|}$$

this is a probability △ verify that it satisfies (A1), (A2)

$$\text{dd} \rightsquigarrow p(\omega) = \frac{1}{|\Omega|}$$

$$\mathbb{P}(\Omega) = 1$$

$$\sum_{\omega \in \Omega} \underline{\underline{p(\omega)}} = 1$$

$$p$$

$$|\Omega| p = 1 \implies p = \frac{1}{|\Omega|}$$

properties $(\Omega, \mathcal{F}, \mathbb{P})$

$A, B \subseteq \Omega$, $\underline{\underline{A \subseteq B}}$

then $\mathbb{P}(B) = \mathbb{P}(B \setminus A) + \mathbb{P}(A)$

$\Rightarrow$ (a) $\mathbb{P}(A) \leq \mathbb{P}(B)$

(b) $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$

proof $\mathbb{P}(B) \underset{A \subseteq B}{=} \mathbb{P}(A \cup (B \setminus A))$

$$\underset{\text{finite additivity}}{=} \mathbb{P}(A) + \mathbb{P}(B \setminus A)$$

prop (inclusion - exclusion formula)

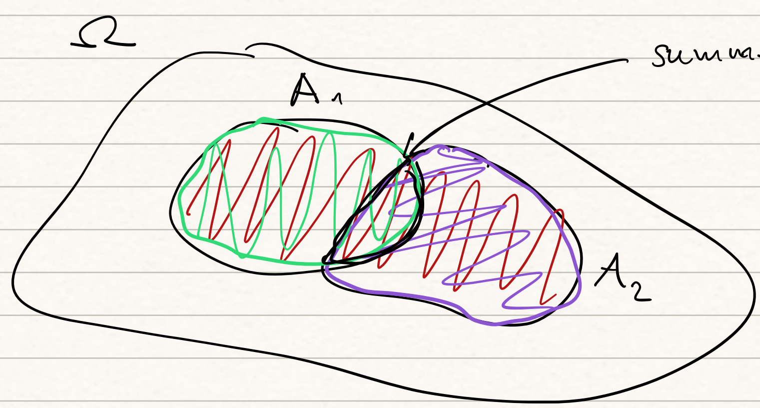
$A_1, \dots A_n \subseteq \Omega$ <u>any</u> sequence

$$\mathbb{P}(A_1 \cup \dots \cup A_n) = \sum_{k=1}^{n} \sum_{\substack{J \subseteq \{1 \dots n\} \\ |J| = k}} (-1)^{k+1} \mathbb{P}\Big(\bigcap_{i \in J} A_i\Big)$$

proof by induction.

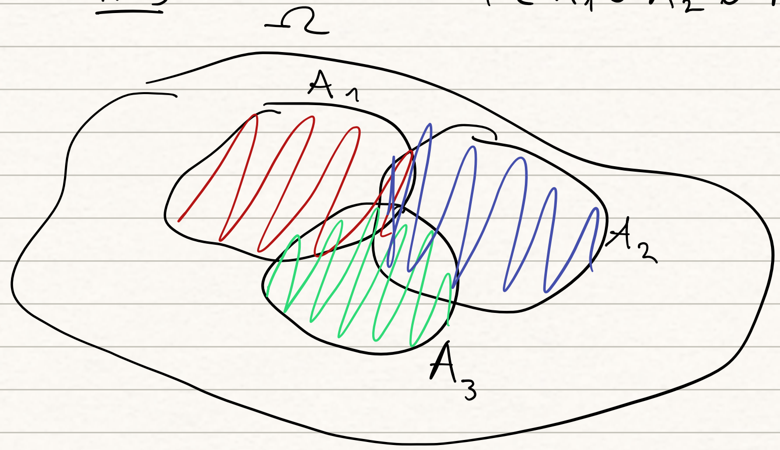
examples: $n = 2$

$$\mathbb{P}(A_1 \cup A_2) = \mathbb{P}(A_1) + \mathbb{P}(A_2) - \mathbb{P}(A \cap B)$$



summed one time too many

<u>n=3</u>



$$\mathbb{P}(A_1 \cup A_2 \cup A_3) = \mathbb{P}(A_1)+\mathbb{P}(A_2)+\mathbb{P}(A_3)$$
$$- \mathbb{P}(A_1 \cap A_2)$$
$$- \mathbb{P}(A_2 \cap A_3)$$
$$- \mathbb{P}(A_1 \cap A_3)$$
$$+ \mathbb{P}(A_1 \cap A_2 \cap A_3)$$

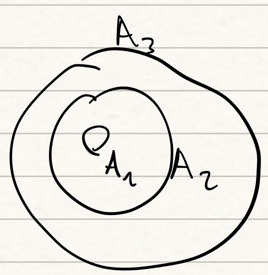
<u>Prop</u> (continuity of probability)

power set of Ω

$\mathbb{P} : \mathcal{P}(\Omega) \to [0,1]$ s.t. it satisfies • (A1)
• finite additivity

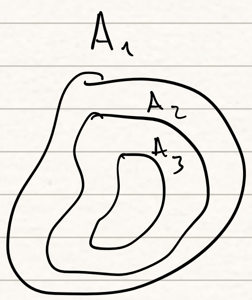
then, the following are equivalent

(i) $\mathbb{P}$ satisfies σ-additivity



(ii) for any increasing sequence of events, i.e. $(A_i)_{i=1}^{\infty}$ s.t. $A_1 \subseteq A_2 \subseteq A_3 \dots$

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{i\to\infty} \mathbb{P}(A_i)$$



(iii) $(A_i)_{i=1}^{\infty}$ decreasing sequence of events

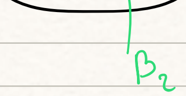
$$\mathbb{P}\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{i\to\infty} \mathbb{P}(A_i)$$

<u>proof</u>

(i) $\Rightarrow$ (ii) $B_1 := A_1$, $B_2 := A_2 \setminus A_1$

$B_n := A_n \setminus A_{n-1}$

notice that, $\bigcup_{i=1}^{n} B_i = A_n$



$\cdot \ \bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i$

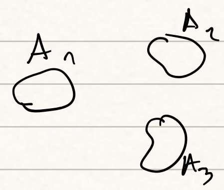
$$\mathbb{P}\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \mathbb{P}\Big(\bigcup_{i=1}^{\infty} B_i\Big) \underset{\uparrow}{=} \sum_{i=1}^{\infty} \mathbb{P}(B_i) = \lim_{n\to\infty} \sum_{i=1}^{n} \mathbb{P}(B_i) =$$

σ-additivity since $B_i \cap B_j = \emptyset$ $i \neq j$

$$\underset{\uparrow}{=} \lim_{n\to\infty} \mathbb{P}(A_n)$$

finite additivity

(ii) $\Longrightarrow$ (i) $\quad (A_i)_{i=1}^{\infty}$ disjoint events



$B_1 = A_1$, $B_2 = A_1 \cup A_2$, ... $B_n = \bigcup_{i=1}^{n} A_i$

$(B_i)_{i=1}^{\infty}$ increasing , $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$

$$\mathbb{P}\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \mathbb{P}\Big(\bigcup_{i=1}^{\infty} B_i\Big) \underset{\uparrow}{=} \lim_{n\to\infty} \mathbb{P}(B_n)$$

continuity (ii)

$$\underset{\uparrow}{=} \lim_{n\to\infty} \sum_{i=1}^{n} \mathbb{P}(A_i) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

finite additivity

<u>corollary</u> $\quad (A_i)_{i=1}^{\infty}$ events

$$\mathbb{P}\Big(\bigcup_{i=1}^{\infty} A_i\Big) \leq \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$