Introduction to the Local Theory of Curves and Surfaces

Notes of a course for the ICTP-CUI Master of Science in Mathematics

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(Much more on this subject can be found in [**1**]*)*

CHAPTER 1

Local theory of curves

Elementary geometry gives a fairly accurate and well-established notion of what is a straight line, whereas is somewhat vague about curves in general. Intuitively, the difference between a straight line and a curve is that the former is, well, straight while the latter is curved. But is it possible to measure how curved a curve is, that is, how far it is from being straight? And what, exactly, is a curve? The main goal of this chapter is to answer these questions. After comparing in the first two sections advantages and disadvantages of several ways of giving a formal definition of a curve, in the third section we shall show how Differential Calculus enables us to accurately measure the curvature of a curve. For curves in space, we shall also measure the torsion of a curve, that is, how far a curve is from being contained in a plane, and we shall show how curvature and torsion completely describe a curve in space.

1.1. How to define a curve

What is a curve (in a plane, in space, in $\mathbb{R}^n$)? Since we are in a mathematical course, rather than in a course about military history of Prussian light cavalry, the only acceptable answer to such a question is a precise definition, identifying exactly the objects that deserve being called curves and those that do not. In order to get there, we start by compiling a list of objects that we consider without a doubt to be curves, and a list of objects that we consider without a doubt not to be curves; then we try to extract properties possessed by the former objects and not by the latter ones.

Example 1.1. Obviously, we have to start from straight lines. A line in a plane can be described in at least three different ways:

- as the graph of a first degree polynomial: $y = mx + q$ or $x = my + q$;
- as the vanishing locus of a first degree polynomial: $ax + by + c = 0$;
- as the image of a map $f\colon \mathbb{R} \to \mathbb{R}^2$ having the form $f(t) = (\alpha t + \beta, \gamma t + \delta)$.

A word of caution: in the last two cases, the coefficients of the polynomial (or of the map) are not uniquely determined by the line; different polynomials (or maps) may well describe the same subset of the plane.

Example 1.2. If $I \subseteq \mathbb{R}$ is an interval and $f\colon I \to \mathbb{R}$ is a (at least) continuous function, then its *graph*

$$\Gamma_f = \bigl\{\bigl(t, f(t)\bigr) \bigm| t \in I\bigr\} \subset \mathbb{R}^2$$

surely corresponds to our intuitive idea of what a curve should be. Note that we have

$$\Gamma_f = \{(x, y) \in I \times \mathbb{R} \mid y - f(x) = 0\} ,$$

that is a graph can always be described as a vanishing locus too. Moreover, it also is the image of the map $\sigma\colon I \to \mathbb{R}^2$ given by $\sigma(t) = \bigl(t, f(t)\bigr)$.

Remark 1.1. To be pedantic, the graph defined in last example is a graph with respect to the *first* coordinate. A graph with respect to the *second* coordinate is a set of the form $\bigl\{\bigl(f(t), t\bigr) \bigm| t \in I\bigr\}$, and has the same right to be considered a curve. Since we obtain one kind of graph from the other just by permuting the coordinates (an operation which geometrically amounts to reflecting with respect to a line), both kinds of graphs are equally suitable, and in what follows dealing with graphs we shall often omit to specify the coordinate we are considering.

Example 1.3. A *circle* (or *circumference*) with center $(x_0, y_0) \in \mathbb{R}^2$ and radius $r > 0$ is the curve having equation

$$(x - x_0)^2 + (y - y_0)^2 = r^2 .$$

Note that it is not a graph with respect to either coordinate. However, it can be represented as the image of the map $\sigma\colon \mathbb{R} \to \mathbb{R}^2$ given by

$$\sigma(t) = (x_0 + r\cos t, y_0 + r\sin t) .$$

Example 1.4. Open sets in the plane, closed disks and, more generally, subsets of the plane with non-empty interior do not correspond to the intuitive idea of curve, so they are to be excluded. The set $[0, 1] \times [0, 1] \setminus \mathbb{Q}^2$, in spite of having an empty interior, does not look like a curve either.

Let us see which clues we can gather from these examples. Confining ourselves to graphs for defining curves is too restrictive, since it would exclude circles, which we certainly want to consider as curves (however, note that circles locally are graphs; we shall come back to this fact later).

The approach via vanishing loci of functions looks more promising. Indeed, all the examples we have seen (lines, graphs, circles) can be described in this way; on the other hand, an open set in the plane or the set $[0, 1] \times [0, 1] \setminus \mathbb{Q}^2$ cannot be the vanishing locus of a continuous function (why?).

So we are led to consider sets of the form

$$C = \{(x, y) \in \Omega \mid f(x, y) = 0\} \subset \mathbb{R}^2$$

for suitable (at least) continuous functions $f\colon \Omega \to \mathbb{R}$, where $\Omega \subseteq \mathbb{R}^2$ is open.

We must however be careful. Sets of this kind are closed in the open set Ω, and this is just fine. But the other implication hold as well:

Proposition 1.1. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set. Then a subset $C \subseteq \Omega$ is closed in Ω if and only if there exists a continuous function $f\colon \Omega \to \mathbb{R}$ such that $C = \{x \in \Omega \mid f(x) = 0\} = f^{-1}(0)$.*

Proof. It is enough to define $f\colon \Omega \to \mathbb{R}$ by setting

$$f(x) = d(x, C) = \inf\{\|x - y\| \mid y \in C\} ,$$

where $\|\cdot\|$ is the usual Euclidean norm in $\mathbb{R}^n$. Indeed, f is obviously continuous, and $x \in C$ if and only if $f(x) = 0$ (why?). $\square$

So, using continuous functions we get sets that clearly cannot be considered curves. However, the problem could be caused by the fact that continuous functions are too many and not regular enough; we might have to confine ourselves to smooth functions.

(Un)fortunately this precaution is not enough. Indeed, it is possible to prove the following

THEOREM 1.1 (Whitney). *Let $\Omega \subseteq \mathbb{R}^n$ be an open set. Then a subset $C \subseteq \Omega$ is closed in Ω if and only if there exists a function $f\colon \Omega \to \mathbb{R}$ of class C^∞ such that $C = f^{-1}(0)$.*

In other words, *any* closed subsets is the vanishing locus of a C^∞ function, not just of a continuous function, and the idea of defining the curves as vanishing loci of arbitrary smooth functions has no chance of working.

Let's take a step back and examine again Examples 1.1, 1.2, and 1.3. In all those cases, it is possible to describe the set as the image of a mapping. This corresponds, in a sense, to a dynamic vision of a curve, thought of as a locus described by a continuously (or differentiably) moving point in a plane or in space or, more in general, in $\mathbb{R}^n$. With some provisos we shall give shortly, this idea turns out to be the right one, and leads to the following definition.

DEFINITION 1.1. Given $k \in \mathbb{N} \cup \{\infty\}$ and $n \geq 2$, a *parametrized curve* of class C^k in $\mathbb{R}^n$ is a map $\sigma\colon I \to \mathbb{R}^n$ of class C^k, where $I \subseteq \mathbb{R}$ is an interval. The image $\sigma(I)$ is often called *support* (or *trace*) of the curve; the variable $t \in I$ is the *parameter* of the curve. If $I = [a, b]$ and $\sigma(a) = \sigma(b)$, we shall say that the curve is *closed*.

REMARK 1.2. If I is not an open interval, and $k \geq 1$, saying that σ is of class C^k in I means that σ can be extended to a C^k function defined in an open interval properly containing I. Moreover, if σ is closed of class C^k, unless stated otherwise we shall always assume that

$$\sigma'(a) = \sigma'(b),\ \sigma''(a) = \sigma''(b),\ \ldots,\ \sigma^{(k)}(a) = \sigma^{(k)}(b)\ .$$

In particular, a closed curve of class C^k can always be extended to a *periodic* map $\hat{\sigma}\colon \mathbb{R} \to \mathbb{R}^n$ of class C^k.

EXAMPLE 1.5. The graph of a map $f\colon I \to \mathbb{R}^{n-1}$ of class C^k is the image of the parametrized curve $\sigma\colon I \to \mathbb{R}^n$ given by $\sigma(t) = \big(t, f(t)\big)$.

EXAMPLE 1.6. For $v_0, v_1 \in \mathbb{R}^n$ with $v_1 \neq O$, the parametrized curve $\sigma\colon \mathbb{R} \to \mathbb{R}^n$ given by $\sigma(t) = v_0 + tv_1$ has as its image the *straight line* through v_0 in the direction v_1.

EXAMPLE 1.7. The two parametrized curves $\sigma_1, \sigma_2\colon \mathbb{R} \to \mathbb{R}^2$ given by

$$\sigma_1(t) = (x_0 + r\cos t, y_0 + r\sin t) \quad \text{and} \quad \sigma_2(t) = (x_0 + r\cos 2t, y_0 + r\sin 2t)$$

both have as their image the *circle* having center $(x_0, y_0) \in \mathbb{R}^2$ and radius $r > 0$.

EXAMPLE 1.8. The parametrized curve $\sigma\colon \mathbb{R} \to \mathbb{R}^3$ given by

$$\sigma(t) = (r\cos t, r\sin t, at)\ ,$$

with $r > 0$ and $a \in \mathbb{R}^*$, has as its image the *circular helix* having *radius* r e *pitch* a; see Fig 1.(a). The image of the circular helix is contained in the right circular cylinder having equation $x^2 + y^2 = r^2$. Moreover, for each $t \in \mathbb{R}$ the points $\sigma(t)$ and $\sigma(t + 2\pi)$ belong to the same line parallel to the cylinder's axis, and have distance $2\pi|a|$.

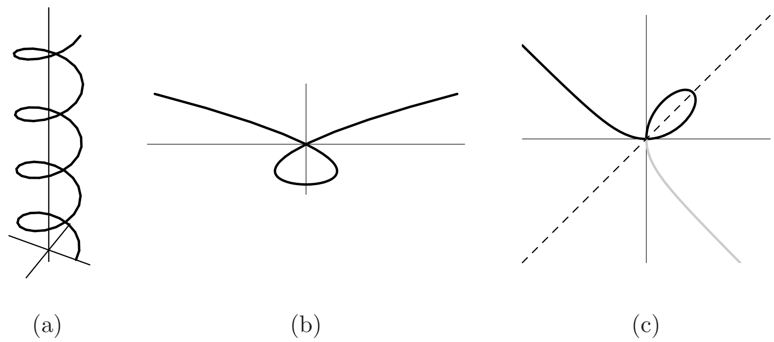


FIGURE 1. (**a**) circular helix; (**b**) non-injective curve; (**c**) folium of Descartes

EXAMPLE 1.9. The curve $\sigma\colon \mathbb{R} \to \mathbb{R}^2$ given by $\sigma(t) = (t, |t|)$ is a continuous parametrized curve which is not of class C^1 (but see Exercise 1.11).

All the parametrized curves we have seen so far (with the exception of the circle; we'll come back to it shortly) provide a homeomorphism between their domain and their image. But it is not always so:

EXAMPLE 1.10. The curve $\sigma\colon \mathbb{R} \to \mathbb{R}^2$ given by $\sigma(t) = (t^3 - 4t, t^2 - 4)$ is a non-injective parametrized curve; see Fig. 1.(b).

EXAMPLE 1.11. The curve $\sigma\colon (-1, +\infty) \to \mathbb{R}^2$ given by

$$\sigma(t) = \left(\frac{3t}{1+t^3}, \frac{3t^2}{1+t^3}\right)$$

is an injective parametrized curve, but it is not a homeomorphism with its image (why?). The set obtained by taking the image of σ, together with its reflection across the line $x = y$, is the *folium of Descartes;* see Fig. 1.(c).

We may also recover some vanishing loci as parametrized curves. Not all of them, by Whitney's Theorem 1.1; but we shall be able to work with vanishing loci of functions f having nonzero *gradient* ∇f, thanks to a classical Calculus theorem, the *implicit function theorem* (you can find its proof, for instance, in [**3**, p. 148]):

THEOREM 1.2 (Implicit function theorem). *Let Ω be an open subset of $\mathbb{R}^m \times \mathbb{R}^n$, and $F\colon \Omega \to \mathbb{R}^n$ a map of class C^k, with $k \in \mathbb{N}^* \cup \{\infty\}$. Denote by (x, y) the coordinates in $\mathbb{R}^{m+n}$, where $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$. Let $p_0 = (x_0, y_0) \in \Omega$ be such that*

$$F(p_0) = O \qquad \textit{and} \qquad \det\left(\frac{\partial F_i}{\partial y_j}(p_0)\right)_{i,j=1,\dots,n} \neq 0\,.$$

Then there exist a neighborhood $U \subset \mathbb{R}^{m+n}$ of p_0, a neighborhood $V \subset \mathbb{R}^m$ of x_0 and a map $g\colon V \to \mathbb{R}^n$ of class C^k such that $U \cap \{p \in \Omega \mid F(p) = O\}$ precisely consists of the points of the form $\big(x, g(x)\big)$ with $x \in V$.

Using this we may prove that the vanishing locus of a function having nonzero gradient is (at least locally) a graph:

PROPOSITION 1.2. *Let $\Omega \subseteq \mathbb{R}^2$ be an open set, and $f\colon \Omega \to \mathbb{R}$ a function of class C^k, with $k \in \mathbb{N}^* \cup \{\infty\}$. Choose $p_0 \in \Omega$ such that $f(p_0) = 0$ but $\nabla f(p_0) \neq O$. Then there exists a neighborhood U of p_0 such that $U \cap \{p \in \Omega \mid f(p) = 0\}$ is the graph of a function of class C^k.*

PROOF. Since the gradient of f in $p_0 = (x_0, y_0)$ is not zero, one of the partial derivatives of f is different from zero in p; up to permuting the coordinates we can assume that $\partial f/\partial y(p_0) \neq 0$. Then the implicit function Theorem 1.2 tells us that there exist a neighborhood U of p_0, an open interval $I \subseteq \mathbb{R}$ including x_0, and a function $g\colon I \to \mathbb{R}$ of class C^k such that $U \cap \{f = 0\}$ is exactly the graph of g. □

REMARK 1.3. If $\partial f/\partial x(p) \neq 0$ then in a neighborhood of p the vanishing locus of f is a graph with respect to the second coordinate.

In other words, the vanishing locus of a function f of class C^1, being locally a graph, is *locally* the support of a parametrized curve near the points where the gradient of f is nonzero.

EXAMPLE 1.12. The gradient of the function $f(x,y) = (x-x_0)^2+(y-y_0)^2-r^2$ is zero only in (x_0, y_0), which does not belong to the vanishing locus of f. Accordingly, each point of the circle with center (x_0, y_0) and radius $r > 0$ has a neighborhood which is a graph with respect to one of the coordinates.

REMARK 1.4. Actually, it can be proved that a subset of $\mathbb{R}^2$ which is locally a graph always is the support of a parametrized curve.

However, the definition of a parametrized curve is not yet completely satisfying. The problem is that it may well happen that two parametrized curves that are different as maps describe what seems to be the same geometric set. An example is given by the two parametrized curves given in Example 1.7, both having as their image a circle; the only difference between them is the speed with which they describe the circle. Another, even clearer example (one you have undoubtedly stumbled upon in previous courses) is the straight line: as recalled in Example 1.1, the same line can be described as the image of infinitely many distinct parametrized curves, just differing in speed and starting point.

On the other hand, considering just the image of a parametrized curve is not correct either. Two different parametrized curves might well describe the same support in geometrically different ways: for instance, one could be injective whereas the other comes back more than once on sections already described before going on. Or, more simply, two different parametrized curves might describe the same image a different number of times, as is the case when restricting the curves in Example 1.7 to intervals of the form $[0, 2k\pi]$.

These considerations suggest to introduce an equivalence relation on the class of parametrized curves, such that two equivalent parametrized curves really describe the same geometric object. The idea of only allowing changes in speed and starting point, but not changes in direction or retracing our steps, is formalized using the notion of diffeomorphism.

DEFINITION 1.2. A *diffeomorphism of class C^k* (with $k \in \mathbb{N}^* \cup \{\infty\}$) between two open sets Ω, $\Omega_1 \subseteq \mathbb{R}^n$ is a homeomorphism $h\colon \Omega \to \Omega_1$ such that both h and its inverse h^{-1} are of class C^k.

More generally, a *diffeomorphism of class* C^k between two sets A, $A_1 \subseteq \mathbb{R}^n$ is the restriction of a diffeomorphism of class C^k of a neighborhood of A with a neighborhood of A_1 and sending A onto A_1.

EXAMPLE 1.13. For instance, $h(x) = 2x$ is a diffeomorphism of class C^∞ of $\mathbb{R}$ with itself, whereas $g(x) = x^3$, even though it is a homeomorphism of $\mathbb{R}$ with itself, is not a diffeomorphism, not even of class C^1, since the inverse function $g^{-1}(x) = x^{1/3}$ is not of class C^1.

DEFINITION 1.3. Two parametrized curves $\sigma\colon I \to \mathbb{R}^n$ and $\tilde{\sigma}\colon \tilde{I} \to \mathbb{R}^n$ of class C^k are *equivalent* if there exists a diffeomorphism $h\colon \tilde{I} \to I$ of class C^k such that $\tilde{\sigma} = \sigma \circ h$; we shall also say that $\tilde{\sigma}$ is a *reparametrization* of σ, and that h is a *parameter change*.

In other words, two equivalent curves only differ in the speed they are traced, while they have the same image, they curve (as we shall see) in the same way, and more generally they have the same geometric properties. So we have finally reached the official definition of what a curve is:

DEFINITION 1.4. A *curve* of class C^k in $\mathbb{R}^n$ is an equivalence class of parametrized curves of class C^k in $\mathbb{R}^n$. Each element of the equivalence class is a *parametrization* of the curve. The *support* of a curve is the support of any parametrization of the curve. A *plane curve* is a curve in $\mathbb{R}^2$.

REMARK 1.5. We shall almost always use the phrase "let $\sigma\colon I \to \mathbb{R}^n$ be a curve" to say that σ is a particular parametrization of the curve under consideration.

Some curves have a parametrization keeping an especially strong connection with its image, and so they deserve a special name.

DEFINITION 1.5. A *Jordan* (or *simple*) *arc* of class C^k in $\mathbb{R}^n$ is a curve admitting a parametrization $\sigma\colon I \to \mathbb{R}^n$ that is a homeomorphism with its image, where $I \subseteq \mathbb{R}$ is an interval. In this case, σ is said to be a *global parametrization* of C. If I is an open (closed) interval, we shall sometimes say that C is an *open* (*closed*) Jordan arc.

DEFINITION 1.6. A *Jordan curve* of class C^k in $\mathbb{R}^n$ is a closed curve C admitting a parametrization $\sigma\colon [a,b] \to \mathbb{R}^n$ of class C^k, injective both on $[a,b)$ and on $(a,b]$. In particular, the image of C is homeomorphic to a circle (why?). The periodic extension $\hat{\sigma}$ of σ mentioned in Remark 1.2 is a *periodic parametrization* of C. Jordan curves are also called *simple curves* (mostly when $n > 2$).

EXAMPLE 1.14. Graphs (Example 1.2), lines (Example 1.6) and circular helices (Example 1.8) are Jordan arcs; the circle (Example 1.3) is a Jordan curve.

EXAMPLE 1.15. The *ellipse* $E \subset \mathbb{R}^2$ with *semiaxes* a, $b > 0$ is the vanishing locus of the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = (x/a)^2 + (y/b)^2 - 1$, that is,

$$E = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right\} .$$

A periodic parametrization of E of class C^∞ is the map $\sigma\colon \mathbb{R} \to \mathbb{R}^2$ given by $\sigma(t) = (a\cos t, b\sin t)$.

EXAMPLE 1.16. The *hyperbola* $I \subset \mathbb{R}^2$ with *semiaxes* a, $b > 0$ is the vanishing locus of the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = (x/a)^2 - (y/b)^2 - 1$, that is,

$$I = \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \right\} .$$

A global parametrization of the component of I contained in the right half-plane is the map $\sigma\colon \mathbb{R} \to \mathbb{R}^2$ given by $\sigma(t) = (a \cosh t, b \sinh t)$.

In the Definition 1.3 of equivalence of parametrized curves we allowed the direction in which the curve is described to be reversed; in other words, we also admitted diffeomorphisms with negative derivative everywhere. As you will see, in some situations it will be important to be able to distinguish the direction in which the curve is traced; so we introduce a slightly finer equivalence relation.

DEFINITION 1.7. Two parametrized curves $\sigma\colon I \to \mathbb{R}^n$ and $\tilde{\sigma}\colon \tilde{I} \to \mathbb{R}^n$ of class C^k are *equivalent with the same orientation* if there exists a parameter change $h\colon \tilde{I} \to I$ from $\tilde{\sigma}$ to σ with positive derivative everywhere; they are *equivalent with opposite orientation* if there exists a parameter change $h\colon \tilde{I} \to I$ from $\tilde{\sigma}$ to σ with negative derivative everywhere (note that the derivative of a diffeomorphism between intervals cannot be zero in any point, so it is either positive everywhere or negative everywhere). An *oriented curve* is then an equivalent class of parametrized curves with the same orientation.

EXAMPLE 1.17. If $\sigma\colon I \to \mathbb{R}^n$ is a parametrized curve, then the parametrized curve $\sigma^-\colon -I \to \mathbb{R}^n$ given by $\sigma^-(t) = \sigma(-t)$, where $-I = \{t \in \mathbb{R} \mid -t \in I\}$, is equivalent to σ but with the opposite orientation.

In general, working with equivalence classes is always a bit tricky; you have to choose a representative element and to check that all obtained results do not depend on that particular representative element. Nevertheless, there is a large class of curves, the *regular curves*, for which it is possible to choose in a canonical way a parametrization that represents the geometry of the curve particularly well: the *arc length parametrization*. The existence of this canonical parametrization permits an effective study of the geometry (and, in particular, of the differential geometry) of curves, confirming a posteriori that this is the right definition.

In the next section we shall introduce this special parametrization.

1.2. Arc length

This is a course about differential geometry; so our basic idea is to study geometric properties of curves (and surfaces) by using techniques borrowed from Mathematical Analysis, and in particular from Differential Calculus. Accordingly,we shall always work with curves of class at least C^1, in order to be able to compute derivatives.

The derivative of a parametrization of a curve tells us the speed at which we are describing the image of the curve. The class of curves for which the speed is nowhere zero (so we always know the direction we are going) is, as we shall see, the right class for differential geometry.

DEFINITION 1.8. Let $\sigma\colon I \to \mathbb{R}^n$ be a parametrized curve of class (at least) C^1. The vector $\sigma'(t)$ is the *tangent vector* to the curve at the point $\sigma(t)$. If $t_0 \in I$ is such that $\sigma'(t_0) \neq O$, then the line through $\sigma(t_0)$ and parallel to $\sigma'(t_0)$ is the *affine*

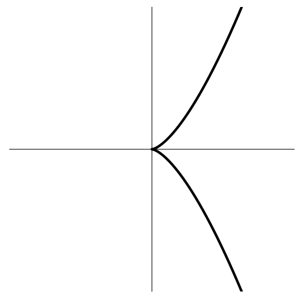


FIGURE 2. A non-regular curve

tangent line to the curve at the point $\sigma(t_0)$. Finally, if $\sigma'(t) \neq O$ for all $t \in I$ we shall say that σ is *regular.*

REMARK 1.6. The notion of a tangent vector depends on the parametrization we have chosen, while the affine tangent line (if any) and the fact of being regular are properties of the curve. Indeed, let $\sigma\colon I \to \mathbb{R}^n$ and $\tilde{\sigma}\colon \tilde{I} \to \mathbb{R}^n$ be two equivalent parametrized curves of class C^1, and $h\colon \tilde{I} \to I$ the parameter change. Then, by computing $\tilde{\sigma} = \sigma \circ h$, we find

$$\tilde{\sigma}'(t) = h'(t)\,\sigma'\big(h(t)\big)\ . \tag{1}$$

Since h' is never zero, we see that the *length* of the tangent vector depends on our particular parametrization, but its *direction* does not; so the affine tangent line in $\tilde{\sigma}(t) = \sigma\big(h(t)\big)$ determined by $\tilde{\sigma}$ is the same as that determined by σ. Moreover, $\tilde{\sigma}'$ is never zero if and only if σ' is never zero; so, being regular is a property of the curve, rather than of a particular representative.

EXAMPLE 1.18. Graphs, lines, circles, circular helices, and the curves in Examples 1.10 and 1.11 are regular curves.

EXAMPLE 1.19. The curve $\sigma\colon \mathbb{R} \to \mathbb{R}^2$ given by $\sigma(t) = (t^2, t^3)$ is a non-regular curve whose image cannot be the image of a regular curve; see Fig 2 and Exercises 1.4 and 1.10.

As anticipated in the previous section, what makes the theory of curves especially simple to deal with is that every regular curve has a canonical parametrization (unique up to its starting point; see Theorem 1.4), strongly related to the geometrical properties common to all parametrizations of the curve. In particular, to study the geometry of a regular curve, we often may confine ourselves to working with the canonical parametrization.

This canonical parametrization basically consists in using as our parameter the length of the curve. So let us start by defining what we mean by length of a curve.

DEFINITION 1.9. Let $I = [a, b]$ be an interval. A *partition* $\mathcal{P}$ of I is a $(k+1)$-tuple $(t_0, \ldots, t_k) \in [a, b]^{k+1}$ with $a = t_0 < t_1 < \cdots < t_k = b$. If $\mathcal{P}$ is partition of I, we set

$$\|\mathcal{P}\| = \max_{1 \leq j \leq k} |t_j - t_{j-1}|\ .$$

Definition 1.10. Given a parametrized curve $\sigma\colon [a,b] \to \mathbb{R}^n$ and a partition $\mathcal{P}$ of $[a,b]$, denote by

$$L(\sigma, \mathcal{P}) = \sum_{j=1}^{k} \|\sigma(t_j) - \sigma(t_{j-1})\|$$

the length of the polygonal closed curve having vertices $\sigma(t_0), \dots, \sigma(t_k)$. We shall say that σ is *rectifiable* if the limit

$$L(\sigma) = \lim_{\|\mathcal{P}\| \to 0} L(\sigma, \mathcal{P})$$

exists and is finite. This limit is the *length* of σ.

Theorem 1.3. *Every parametrized curve $\sigma\colon [a,b] \to \mathbb{R}^n$ of class C^1 is rectifiable, and we have*

$$L(\sigma) = \int_a^b \|\sigma'(t)\| \, \mathrm{d}t \, .$$

Proof. Since σ is of class C^1, the integral is finite. So we have to prove that, for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if $\mathcal{P}$ is a partition of $[a,b]$ with $\|\mathcal{P}\| < \delta$ then

$$\left| \int_a^b \|\sigma'(t)\| \, \mathrm{d}t - L(\sigma, \mathcal{P}) \right| < \varepsilon \, . \tag{2}$$

We begin by remarking that, for each partition $\mathcal{P} = (t_0, \dots, t_k)$ of $[a,b]$ and for each $j = 1, \dots, k$, we have

$$\|\sigma(t_j) - \sigma(t_{j-1})\| = \left\| \int_{t_{j-1}}^{t_j} \sigma'(t) \, \mathrm{d}t \right\| \le \int_{t_{j-1}}^{t_j} \|\sigma'(t)\| \, \mathrm{d}t \, ;$$

so summing over j we find

$$L(\sigma, \mathcal{P}) \le \int_a^b \|\sigma'(t)\| \, \mathrm{d}t \, , \tag{3}$$

independently of the partition $\mathcal{P}$.

Now, fix $\varepsilon > 0$; then the uniform continuity of σ' over the compact interval $[a,b]$ provides us with a $\delta > 0$ such that

$$|t - s| < \delta \Longrightarrow \|\sigma'(t) - \sigma'(s)\| < \frac{\varepsilon}{b - a} \tag{4}$$

for all s, $t \in [a,b]$. Let $\mathcal{P} = (t_0, \dots, t_k)$ be a partition of $[a,b]$ with $\|\mathcal{P}\| < \delta$. For all $j = 1, \dots, k$ and $s \in [t_{j-1}, t_j]$ we have

$$\begin{aligned} \sigma(t_j) - \sigma(t_{j-1}) &= \int_{t_{j-1}}^{t_j} \sigma'(s) \, \mathrm{d}t + \int_{t_{j-1}}^{t_j} \big(\sigma'(t) - \sigma'(s)\big) \, \mathrm{d}t \\ &= (t_j - t_{j-1})\sigma'(s) + \int_{t_{j-1}}^{t_j} \big(\sigma'(t) - \sigma'(s)\big) \, \mathrm{d}t \, . \end{aligned}$$

Hence,

$$\begin{aligned}\|\sigma(t_j)-\sigma(t_{j-1})\| &\geq (t_j-t_{j-1})\|\sigma'(s)\| - \int_{t_{j-1}}^{t_j} \big\|\sigma'(t)-\sigma'(s)\big\|\,\mathrm{d}t \\ &\geq (t_j-t_{j-1})\|\sigma'(s)\| - \frac{\varepsilon}{b-a}(t_j-t_{j-1})\ ,\end{aligned}$$

where the last step follows from the fact that s, $t\in[t_{j-1},t_j]$ implies $|t-s|<\delta$, so we may apply (4). Dividing by t_j-t_{j-1} we get

$$\frac{\|\sigma(t_j)-\sigma(t_{j-1})\|}{t_j-t_{j-1}} \geq \|\sigma'(s)\| - \frac{\varepsilon}{b-a}\ ;$$

then integrating with respect to s over $[t_{j-1},t_j]$ it follows that

$$\|\sigma(t_j)-\sigma(t_{j-1})\| \geq \int_{t_{j-1}}^{t_j} \|\sigma'(s)\|\,\mathrm{d}s - \frac{\varepsilon}{b-a}(t_j-t_{j-1})\ .$$

Summing over $j=1,\dots,k$ we get

$$L(\sigma,\mathcal{P}) \geq \int_a^b \|\sigma'(s)\|\,\mathrm{d}s - \varepsilon\ ,$$

which taken together with (3) gives (2). □

Corollary 1.1. *Length is a geometric property of C^1 curves, and it does not depend on a particular parametrization. In other words, any two equivalent parametrized curves of class C^1 (defined on a compact interval) have the same length.*

Proof. Let $\sigma\colon[a,b]\to\mathbb{R}^n$ and $\tilde\sigma\colon[\tilde a,\tilde b]\to\mathbb{R}^n$ be equivalent parametrized curves, and $h\colon[\tilde a,\tilde b]\to[a,b]$ the parameter change. Then (1) implies

$$L(\tilde\sigma) = \int_{\tilde a}^{\tilde b} \|\tilde\sigma'(t)\|\,\mathrm{d}t = \int_{\tilde a}^{\tilde b} \big\|\sigma'\big(h(t)\big)\big\|\,|h'(t)|\,\mathrm{d}t = \int_a^b \|\sigma'(\tau)\|\,\mathrm{d}\tau = L(\sigma)\ ,$$

thanks to the classical theorem about change of variables in integrals. □

Remark 1.7. Note that the length of a curve does not depend only on its support, since a non-injective parametrization may describe some arc more than once. For instance, the two curves in Example 1.7, restricted to $[0,2\pi]$, have different lengths even though they have the same image.

The time has come for us to define the announced canonical parametrization:

Definition 1.11. Let $\sigma\colon I\to\mathbb{R}^n$ be a curve of class C^k (with $k\geq 1$). Having fixed $t_0\in I$, the *arc length* of σ (measured starting from t_0) is the function $s\colon I\to\mathbb{R}$ of class C^k given by

$$s(t) = \int_{t_0}^{t} \|\sigma'(\tau)\|\,\mathrm{d}\tau\ .$$

We shall say that σ is *parametrized by arc length* if $\|\sigma'\|\equiv 1$. In other words, σ is parametrized by arc length if and only if its arc length is equal to the parameter t up to a translation, that is $s(t)=t-t_0$.

A curve parametrized by arc length is clearly regular. The fundamental result is that the converse implication is true too:

THEOREM 1.4. *Every regular oriented curve admits a unique (up to a translation in the parameter) parametrization by arc length. More precisely, let* $\sigma\colon I \to \mathbb{R}^n$ *be a regular parametrized curve of class* C^k*. Having fixed* $t_0 \in I$*, denote by* $s\colon I \to \mathbb{R}$ *the arc length of* σ *measured starting from* t_0*. Then* $\tilde{\sigma} = \sigma \circ s^{-1}$ *is (up to a translation in the parameter) the unique regular* C^k *curve parametrized by arc length equivalent to* σ *and having the same orientation.*

PROOF. First of all, $s' = \|\sigma'\|$ is positive everywhere, so $s\colon I \to s(I)$ is a monotonically increasing function of class C^k having inverse of class C^k between the intervals I and $\tilde{I} = s(I)$. So $\tilde{\sigma} = \sigma \circ s^{-1}\colon \tilde{I} \to \mathbb{R}^n$ is a parametrized curve equivalent to σ and having the same orientation. Furthermore,

$$\tilde{\sigma}'(t) = \frac{\sigma'\bigl(s^{-1}(t)\bigr)}{\bigl\|\sigma'\bigl(s^{-1}(t)\bigr)\bigr\|} ,$$

so $\|\tilde{\sigma}'\| \equiv 1$, as required.

To prove uniqueness, let σ_1 be another parametrized curve satisfying the hypotheses. Being equivalent to σ (and so to $\tilde{\sigma}$) with the same orientation, there exists a parameter change h with positive derivative everywhere such that $\sigma_1 = \tilde{\sigma} \circ h$. As both $\tilde{\sigma}$ and σ_1 are parametrized by arc length, (1) implies $|h'| \equiv 1$; but $h' > 0$ everywhere, so necessarily $h' \equiv 1$. This means that $h(t) = t + c$ for some $c \in \mathbb{R}$, and thus σ_1 is obtained from $\tilde{\sigma}$ by translating the parameter. □

So, every regular curve admits an essentially unique parametrization by arc length. In some textbooks this parametrization is called the *natural parametrization*.

REMARK 1.8. In what follows, we shall always use the letter s to denote the arc-length parameter, and the letter t to denote an arbitrary parameter. Moreover, the derivatives with respect to the arc-length parameter will be denoted by a dot ($\dot{}$), while the derivatives with respect to an arbitrary parameter by a prime ($'$). For instance, we shall write $\dot{\sigma}$ for $\mathrm{d}\sigma/\mathrm{d}s$, and σ' for $\mathrm{d}\sigma/\mathrm{d}t$. The relation between $\dot{\sigma}$ and σ' easily follows from the chain rule:

$$\sigma'(t) = \frac{\mathrm{d}\sigma}{\mathrm{d}t}(t) = \frac{\mathrm{d}\sigma}{\mathrm{d}s}\bigl(s(t)\bigr)\frac{\mathrm{d}s}{\mathrm{d}t}(t) = \|\sigma'(t)\|\,\dot{\sigma}\bigl(s(t)\bigr) . \tag{5}$$

Analogously we have

$$\dot{\sigma}(s) = \frac{1}{\bigl\|\sigma'\bigl(s^{-1}(s)\bigr)\bigr\|}\,\sigma'\bigl(s^{-1}(s)\bigr) ,$$

where in last formula the letter s denotes both the parameter and the arc length function. As you will see, using the same letter to represent both concepts will not cause, once you get used to it, any confusion.

EXAMPLE 1.20. Let $\sigma\colon \mathbb{R} \to \mathbb{R}^n$ be a line parametrized as in Example 1.6. Then the arc length of σ starting from 0 is $s(t) = \|v_1\|t$, and thus $s^{-1}(s) = s/\|v_1\|$. In particular, a parametrization of the line by arc length is $\tilde{\sigma}(s) = v_0 + sv_1/\|v_1\|$.

EXAMPLE 1.21. Let $\sigma\colon [0, 2\pi] \to \mathbb{R}^2$ be the parametrization of the circle with center $(x_0, y_0) \in \mathbb{R}^2$ and radius $r > 0$ given by $\sigma(t) = (x_0 + r\cos t, y_0 + r\sin t)$. Then the arc length of σ starting from 0 is $s(t) = rt$, so $s^{-1}(s) = s/r$. In particular, a parametrization $\tilde{\sigma}\colon [0, 2\pi r] \to \mathbb{R}^2$ by arc length of the circle is given by $\tilde{\sigma}(s) = \bigl(x_0 + r\cos(s/r), y_0 + r\sin(s/r)\bigr)$.

EXAMPLE 1.22. The circular helix $\sigma\colon \mathbb{R} \to \mathbb{R}^3$ with radius $r > 0$ and pitch $a \in \mathbb{R}^*$ described in Example 1.8 has $\|\sigma'\| \equiv \sqrt{r^2 + a^2}$. So an arc length parametrization is

$$\tilde{\sigma}(s) = \left(r\cos\frac{s}{\sqrt{r^2+a^2}}, r\sin\frac{s}{\sqrt{r^2+a^2}}, \frac{as}{\sqrt{r^2+a^2}} \right) .$$

EXAMPLE 1.23. The *catenary* is the graph of the hyperbolic cosine function; so a parametrization is the curve $\sigma\colon \mathbb{R} \to \mathbb{R}^2$ given by $\sigma(t) = (t, \cosh t)$. It is one of the few curves for which we can explicitly compute the arc length parametrization using elementary functions. Indeed, $\sigma'(t) = (1, \sinh t)$; so

$$s(t) = \int_0^t \sqrt{1 + \sinh^2\tau}\, \mathrm{d}\tau = \int_0^t \cosh\tau\, \mathrm{d}\tau = \sinh t$$

and

$$s^{-1}(s) = \operatorname{arc\,sinh} s = \log\bigl(s + \sqrt{1+s^2}\bigr) .$$

Now, $\cosh\bigl(\log\bigl(s + \sqrt{1+s^2}\bigr)\bigr) = \sqrt{1+s^2}$, and thus the parametrization of the catenary by arc length is

$$\tilde{\sigma}(s) = \Bigl(\log\bigl(s + \sqrt{1+s^2}\bigr), \sqrt{1+s^2}\Bigr) .$$

EXAMPLE 1.24. Let E be an ellipse having semiaxes a, $b > 0$, parametrized as in Example 1.15, and assume $b > a$. Then

$$s(t) = \int_0^t \sqrt{a^2\sin^2\tau + b^2\cos^2\tau}\, \mathrm{d}\tau = b\int_0^t \sqrt{1 - \left(1 - \frac{a^2}{b^2}\right)\sin^2\tau}\, \mathrm{d}\tau$$

is an elliptic integral of the second kind, whose inverse is expressed using Jacobi elliptic functions. So, to compute the arc-length parametrization of the ellipse we have to resort to non-elementary functions.

REMARK 1.9. Theorem 1.4 says that every regular curve can be parametrized by arc length, at least in principle. In practice, finding the parametrization by arc length of a particular curve might well be impossible: as we have seen in the previous examples, in order to do so it is necessary to compute the inverse of a function given by an integral. For this reason, from now on we shall use the parametrization by arc length to introduce the geometric quantities (like curvature, for instance) we are interested in, but we shall always explain how to compute those quantities starting from an arbitrary parametrization too.

1.3. Curvature and torsion

In a sense, a straight line is a curve that never changes direction. More precisely, the image of a regular curve is contained in a line if and only if the direction of its tangent vector σ' is constant (see Exercise 1.22). As a result, it is reasonable to suppose that the variation of the direction of the tangent vector could tell us how far a curve is from being a straight line. To get an effective way of measuring this variation (and so the curve's curvature), we shall use the tangent versor.

DEFINITION 1.12. Let $\sigma\colon I \to \mathbb{R}^n$ be a regular curve of class C^k. The *tangent versor* (also called *unit tangent vector*) to σ is the map $\vec{t}\colon I \to \mathbb{R}^n$ of class C^{k-1} given by

$$\vec{t} = \frac{\sigma'}{\|\sigma'\|} ;$$

we shall also say that the versor $\vec{t}(t)$ is tangent to the curve σ at the point $\sigma(t)$.

Remark 1.10. Equation (1) implies that the tangent vector only depends on the oriented curve, and not on a particular parametrization we might have chosen. In particular, if the curve σ is parametrized by arc length, then

$$\vec{t} = \dot{\sigma} = \frac{\mathrm{d}\sigma}{\mathrm{d}s} .$$

On the other hand, the tangent versor does depend on the orientation of the curve. If $\vec{t}^-$ is the tangent versor to the curve (introduced in Example 1.17) σ^- having opposite orientation, then

$$\vec{t}^-(t) = -\vec{t}(-t) ,$$

that is the tangent versor changes sign when the orientation is reversed.

The variations in the direction of the tangent vector can be measured by the variation of the tangent versor, that is, by the derivative of $\vec{t}$.

Definition 1.13. Let $\sigma\colon I \to \mathbb{R}^n$ be a regular curve of class C^k (with $k \geq 2$) parametrized by arc length. The *curvature* of σ is the function $\kappa\colon I \to \mathbb{R}^+$ of class C^{k-2} given by

$$\kappa(s) = \|\dot{\vec{t}}(s)\| = \|\ddot{\sigma}(s)\| .$$

Clearly, $\kappa(s)$ is the curvature of σ at the point $\sigma(s)$. We shall say that σ is *biregular* if κ is everywhere nonzero. In this case the *radius of curvature* of σ at the point $\sigma(s)$ is $r(s) = 1/\kappa(s)$.

Remark 1.11. If $\sigma\colon I \to \mathbb{R}^n$ is an arbitrary regular parametrized curve, the *curvature* $\kappa(t)$ of σ at the point $\sigma(t)$ is defined by reparametrizing the curve by arc length. If $\sigma_1 = \sigma \circ s^{-1}$ is a parametrization of σ by arc length, and κ_1 is the curvature of σ_1, then we define $\kappa\colon I \to \mathbb{R}^+$ by setting $\kappa(t) = \kappa_1\bigl(s(t)\bigr)$, so the curvature of σ at the point $\sigma(t)$ is equal to the curvature of σ_1 at the point $\sigma_1\bigl(s(t)\bigr) = \sigma(t)$.

Example 1.25. A line parametrized as in Example 1.20 has a constant tangent versor. So the curvature of a straight line is everywhere zero.

Example 1.26. Let $\sigma\colon [0, 2\pi r] \to \mathbb{R}^2$ be the circle with center $(x_0, y_0) \in \mathbb{R}^2$ and radius $r > 0$, parametrized by arc length as in Example 1.21. Then

$$\vec{t}(s) = \dot{\sigma}(s) = \bigl(-\sin(s/r), \cos(s/r)\bigr) \text{ and } \dot{\vec{t}}(s) = \frac{1}{r}\bigl(-\cos(s/r), -\sin(s/r)\bigr) ,$$

so σ has constant curvature $1/r$. This is the reason why the reciprocal of the curvature is called radius of curvature.

Example 1.27. Let $\sigma\colon \mathbb{R} \to \mathbb{R}^3$ be the circular helix with radius $r > 0$ and pitch $a \in \mathbb{R}^*$, parametrized by arc length as in Example 1.22. Then,

$$\vec{t}(s) = \left(-\frac{r}{\sqrt{r^2+a^2}} \sin\frac{s}{\sqrt{r^2+a^2}}, \frac{r}{\sqrt{r^2+a^2}} \cos\frac{s}{\sqrt{r^2+a^2}}, \frac{a}{\sqrt{r^2+a^2}}\right)$$

and

$$\dot{\vec{t}}(s) = -\frac{r}{r^2+a^2}\left(\cos\frac{s}{\sqrt{r^2+a^2}}, \sin\frac{s}{\sqrt{r^2+a^2}}, 0\right) ;$$

so the helix has constant curvature

$$\kappa \equiv \frac{r}{r^2+a^2} .$$

EXAMPLE 1.28. Let $\sigma\colon \mathbb{R} \to \mathbb{R}^2$ be the catenary, parametrized by arc length as in Example 1.23. Then

$$\vec{t}(s) = \left(\frac{1}{\sqrt{1+s^2}}, \frac{s}{\sqrt{1+s^2}} \right)$$

and

$$\dot{\vec{t}}(s) = \left(-\frac{s}{(1+s^2)^{3/2}}, \frac{1}{(1+s^2)^{3/2}} \right) ;$$

so the catenary has curvature

$$\kappa(s) = \frac{1}{1+s^2} .$$

Now, it stands to reason that the direction of the vector $\dot{\vec{t}}$ should also contain significant geometric information about the curve, since it gives the direction the curve is following. Moreover, the vector $\dot{\vec{t}}$ cannot be just any vector. Indeed, since $\vec{t}$ is a versor, we have

$$\langle \vec{t}, \vec{t} \rangle \equiv 1 ,$$

where $\langle \cdot\,, \cdot \rangle$ is the canonical scalar product in $\mathbb{R}^n$; hence, after taking the derivative, we get

$$\langle \dot{\vec{t}}, \vec{t} \rangle \equiv 0 .$$

In other words, *$\dot{\vec{t}}$ is orthogonal to $\vec{t}$ everywhere.*

DEFINITION 1.14. Let $\sigma\colon I \to \mathbb{R}^n$ be a biregular curve of class C^k (with $k \geq 2$) parametrized by arc length. The *normal versor* (also called *unit normal vector*) to σ is the map $\vec{n}\colon I \to \mathbb{R}^n$ of class C^{k-2} given by

$$\vec{n} = \frac{\dot{\vec{t}}}{\|\dot{\vec{t}}\|} = \frac{\dot{\vec{t}}}{\kappa} .$$

The plane through $\sigma(s)$ and parallel to $\mathrm{Span}\big(\vec{t}(s), \vec{n}(s)\big)$ is the *osculating plane* to the curve at $\sigma(s)$. The *affine normal line* of σ at the point $\sigma(s)$ is the line through $\sigma(s)$ parallel to the normal versor $\vec{n}(s)$.

Before going on, we must show how to compute the curvature and the normal versor without resorting to the arc-length, fulfilling the promise we made in Remark 1.9:

PROPOSITION 1.3. *Let $\sigma\colon I \to \mathbb{R}^n$ be any regular parametrized curve. Then the curvature $\kappa\colon I \to \mathbb{R}^+$ of σ is given by*

$$\kappa = \frac{\sqrt{\|\sigma'\|^2 \|\sigma''\|^2 - |\langle \sigma'', \sigma' \rangle|^2}}{\|\sigma'\|^3} . \tag{6}$$

In particular, σ is biregular if and only if σ' and σ'' are linearly independent everywhere; in this case,

$$\vec{n} = \frac{1}{\sqrt{\|\sigma''\|^2 - \frac{|\langle \sigma'', \sigma' \rangle|^2}{\|\sigma'\|^2}}} \left(\sigma'' - \frac{\langle \sigma'', \sigma' \rangle}{\|\sigma'\|^2} \sigma' \right) . \tag{7}$$

PROOF. Let $s\colon I \to \mathbb{R}$ be the arc length of σ measured starting from an arbitrary point. Equation (5) gives

$$\vec{t}\bigl(s(t)\bigr) = \frac{\sigma'(t)}{\|\sigma'(t)\|} \ ;$$

since

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{t}\bigl(s(t)\bigr) = \frac{\mathrm{d}\vec{t}}{\mathrm{d}s}\bigl(s(t)\bigr)\,\frac{\mathrm{d}s}{\mathrm{d}t}(t) = \|\sigma'(t)\|\dot{\vec{t}}\bigl(s(t)\bigr) \ ,$$

we find

$$\begin{aligned} \dot{\vec{t}}\bigl(s(t)\bigr) &= \frac{1}{\|\sigma'(t)\|}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\sigma'(t)}{\|\sigma'(t)\|}\right) \\ &= \frac{1}{\|\sigma'(t)\|^2}\left(\sigma''(t) - \frac{\langle\sigma''(t),\sigma'(t)\rangle}{\|\sigma'(t)\|^2}\sigma'(t)\right) \ ; \qquad (8) \end{aligned}$$

note that $\dot{\vec{t}}\bigl(s(t)\bigr)$ is a multiple of the component of $\sigma''(t)$ orthogonal to $\sigma'(t)$. Finally,

$$\kappa(t) = \bigl\|\dot{\vec{t}}\bigl(s(t)\bigr)\bigr\| = \frac{1}{\|\sigma'(t)\|^2}\sqrt{\|\sigma''(t)\|^2 - \frac{|\langle\sigma''(t),\sigma'(t)\rangle|^2}{\|\sigma'(t)\|^2}} \ ,$$

and the proof is complete, as the last claim follows from the Cauchy-Schwarz inequality, and (7) follows from (8). □

Let us see how to apply this result in several examples.

EXAMPLE 1.29. Let $\sigma\colon \mathbb{R} \to \mathbb{R}^2$ be the ellipse having semiaxes a, $b > 0$, with the parametrization described in Example 1.15. Then $\sigma'(t) = (-a\sin t, b\cos t)$, and hence $\sigma''(t) = (-a\cos t, -b\sin t)$. Therefore

$$\vec{t}(t) = \frac{\sigma'(t)}{\|\sigma'(t)\|} = \frac{1}{\sqrt{a^2\sin^2 t + b^2\cos^2 t}}(-a\sin t, b\cos t)$$

and the curvature of the ellipse is given by

$$\kappa(t) = \frac{ab}{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}} \ .$$

EXAMPLE 1.30. The normal versor of a circle with radius $r > 0$ is

$$\vec{n}(s) = \bigl(-\cos(s/r), -\sin(s/r)\bigr) \ ;$$

that of a circular helix with radius $r > 0$ and pitch $a \in \mathbb{R}^*$ is

$$\vec{n}(s) = \left(-\cos\frac{s}{\sqrt{r^2+a^2}}, -\sin\frac{s}{\sqrt{r^2+a^2}}, 0\right) \ ;$$

that of the catenary is

$$\vec{n}(s) = \left(-\frac{s}{\sqrt{1+s^2}}, \frac{1}{\sqrt{1+s^2}}\right) \ ;$$

and that of the ellipse with semiaxes a, $b > 0$ is

$$\vec{n}(t) = \frac{1}{\sqrt{a^2\sin^2 t + b^2\cos^2 t}}(-b\cos t, -a\sin t) \ .$$

EXAMPLE 1.31. Let $\sigma\colon I \to \mathbb{R}^n$, given by $\sigma(t) = \big(t, f(t)\big)$, be the graph of a map $f\colon I \to \mathbb{R}^{n-1}$ of class (at least) C^2. Then

$$\vec{t} = \frac{1}{\sqrt{1+\|f'\|^2}}(1, f')\ ,$$

$$\vec{n}(t) = \frac{1}{\sqrt{\|f''\|^2 - |\langle f'', f'\rangle|^2/(1+\|f'\|^2)}}\left(-\frac{\langle f'', f'\rangle}{1+\|f'\|^2}, f'' - \frac{\langle f'', f'\rangle}{1+\|f'\|^2} f'\right)\ .$$

and

$$\kappa = \frac{\sqrt{(1+\|f'\|^2)\|f''\|^2 - |\langle f'', f'\rangle|^2}}{(1+\|f'\|^2)^{3/2}}\ .$$

In particular, σ is biregular if and only if f'' is never zero (why?).

REMARK 1.12. To define the normal versor we had to assume the biregularity of the curve. However, if the curve is *plane*, to define a normal versor regularity is enough.

Indeed, if $\sigma\colon I \to \mathbb{R}^2$ is a plane curve of class C^k parametrized by arc length, for all $s \in I$ there exists a unique versor $\tilde{\vec{n}}(s)$ that is orthogonal to $\vec{t}(s)$ and such that the pair $\{\vec{t}(s), \tilde{\vec{n}}(s)\}$ has the same orientation as the canonical basis. In coordinates,

$$\vec{t}(s) = (a_1, a_2) \quad\Longrightarrow\quad \tilde{\vec{n}}(s) = (-a_2, a_1)\ ;$$

in particular, the map $\tilde{\vec{n}}\colon I \to \mathbb{R}^2$ is of class C^{k-1}, just like $\vec{t}$. Moreover, since $\dot{\vec{t}}(s)$ is orthogonal to $\vec{t}(s)$, it has to be a multiple of $\tilde{\vec{n}}(s)$; so there exists a function $\tilde{\kappa}\colon I \to \mathbb{R}$ of class C^{k-2} such that we have

$$\dot{\vec{t}} = \tilde{\kappa}\tilde{\vec{n}}\ . \tag{9}$$

DEFINITION 1.15. If $\sigma\colon I \to \mathbb{R}^2$ is a regular plane curve of class C^k (with $k \ge 2$) parametrized by arc length, the map $\tilde{\vec{n}}\colon I \to \mathbb{R}^2$ of class C^{k-1} just defined is the *oriented normal versor* of σ, while the function $\tilde{\kappa}\colon I \to \mathbb{R}$ of class C^{k-2} is the *oriented curvature* of σ.

REMARK 1.13. Since, by construction, we have $\det(\vec{t}, \tilde{\vec{n}}) \equiv 1$, the oriented curvature of a plane curve is given by the formula

$$\tilde{\kappa} = \det(\vec{t}, \dot{\vec{t}})\ . \tag{10}$$

To put it simply, this means that, if $\tilde{\kappa} > 0$, then the curve is bending in a counterclockwise direction, while if $\tilde{\kappa} < 0$ then the curve is bending in a clockwise direction. Finally, if $\sigma\colon I \to \mathbb{R}^2$ is an arbitrary parametrized plane curve, then the oriented curvature of σ in the point $\sigma(t)$ is given by (see Problem 1.1)

$$\tilde{\kappa}(t) = \frac{1}{\|\sigma'(t)\|^3}\det\big(\sigma'(t), \sigma''(t)\big)\ . \tag{11}$$

REMARK 1.14. The oriented curvature $\tilde{\kappa}$ of a plane curve is related to the usual curvature κ by the identity $\kappa = |\tilde{\kappa}|$. In particular, the normal versor introduced in Definition 1.14 coincides with the oriented normal versor $\tilde{\vec{n}}$ when the oriented curvature is positive, and with its opposite when the oriented curvature is negative.

EXAMPLE 1.32. Example 1.26 show that the oriented curvature of the circle with center $(x_0, y_0) \in \mathbb{R}^2$ and radius $r > 0$, parametrized by arc length as in Example 1.21, is constant, equal to $1/r$. On the other hand, let $\sigma = (\sigma_1, \sigma_2)\colon I \to \mathbb{R}^2$ be a regular curve parametrized by arc length, with constant oriented curvature equal to $1/r \neq 0$. Then the coordinates of σ satisfy the linear system of ordinary differential equations

$$\begin{cases} \ddot{\sigma}_1 = -\frac{1}{r}\dot{\sigma}_2 \ , \\ \ddot{\sigma}_2 = \frac{1}{r}\dot{\sigma}_1 \ . \end{cases}$$

Keeping in mind that $\dot{\sigma}_1^2 + \dot{\sigma}_2^2 \equiv 1$, we find that there exists a $s_0 \in \mathbb{R}$ such that

$$\dot{\sigma}(s) = \left(-\sin\frac{s+s_0}{r}, \cos\frac{s+s_0}{r} \right) ,$$

so the support of σ is contained (why?) in a circle with radius $|r|$. In other words, circles are characterized by having a constant nonzero oriented curvature.

As we shall shortly see (and as the previous example suggests), the oriented curvature completely determines a plane curve in a very precise sense: two plane curves parametrized by arc length having the same oriented curvature only differ by a rigid plane motion (Theorem 1.6 and Exercise 1.48).

Space curves, on the other hand, are not completely determined by their curvature. This is to be expected: in space, a curve may bend and also twist, that is leave any given plane. And, if $n > 3$, a curve in $\mathbb{R}^n$ may hypertwist in even more dimensions. For the sake of clarity, *in the rest of this section we shall* (almost) *uniquely consider curves in the space* $\mathbb{R}^3$.

If the support of a regular curve is contained in a plane, it is clear (why? see the proof of Proposition 1.4) that the osculating plane of the curve is constant. This suggests that it is possible to measure how far a space curve is from being plane by studying the variation of its osculating plane. Since a plane (through the origin of $\mathbb{R}^3$) is completely determined by the direction orthogonal to it, we are led to the following

Definition 1.16. Let $\sigma\colon I \to \mathbb{R}^3$ be a biregular curve of class C^k. The *binormal versor* (also called *unit binormal vector*) to the curve is the map $\vec{b}\colon I \to \mathbb{R}^3$ of class C^{k-2} given by $\vec{b} = \vec{t} \wedge \vec{n}$, where $\wedge$ denotes the vector product in $\mathbb{R}^3$. The *affine binormal line* of σ at the point $\sigma(s)$ is the line through $\sigma(s)$ parallel to the binormal versor $\vec{b}(s)$.

Finally, the triple $\{\vec{t}, \vec{n}, \vec{b}\}$ of $\mathbb{R}^3$-valued functions is the *Frenet frame* of the curve. Sometimes, the maps $\vec{t}$, $\vec{n}$, $\vec{b}\colon I \to \mathbb{R}^3$ are also called *spherical indicatrices* because their image is contained in the unit sphere of $\mathbb{R}^3$.

So we have associated to each point $\sigma(s)$ of a biregular space curve σ an orthonormal basis $\{\vec{t}(s), \vec{n}(s), \vec{b}(s)\}$ of $\mathbb{R}^3$ having the same orientation as the canonical basis, and varying along the curve (see Fig. 3).

Remark 1.15. The Frenet frame depends on the orientation of the curve. Indeed, if we denote by $\{\vec{t}^-, \vec{n}^-, \vec{b}^-\}$ the Frenet frame associated with the curve $\sigma^-(s) = \sigma(-s)$ equivalent to σ having opposite orientation, we have

$$\vec{t}^-(s) = -\vec{t}(-s) \ , \qquad \vec{n}^-(s) = \vec{n}(-s) \ , \qquad \vec{b}^-(s) = -\vec{b}(-s) \ .$$

On the other hand, since it was defined using a parametrization by arc length, the Frenet frame only depends on the oriented curve, and not on the specific parametrization chosen to compute it.

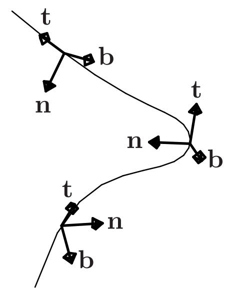


FIGURE 3. The Frenet frame

EXAMPLE 1.33. Let $\sigma\colon \mathbb{R} \to \mathbb{R}^3$ be the circular helix with radius $r > 0$ and pitch $a \in \mathbb{R}^*$, parametrized by arc length as in Example 1.22. Then

$$\vec{b}(s) = \left(\frac{a}{\sqrt{r^2+a^2}} \sin \frac{s}{\sqrt{r^2+a^2}}, -\frac{a}{\sqrt{r^2+a^2}} \cos \frac{s}{\sqrt{r^2+a^2}}, \frac{r}{\sqrt{r^2+a^2}} \right) .$$

EXAMPLE 1.34. If $\sigma\colon I \to \mathbb{R}^3$ is the graph of a map $f = (f_1, f_2)\colon I \to \mathbb{R}^2$ such that f'' is nowhere zero, then

$$\vec{b} = \frac{1}{\sqrt{\|f''\|^2 + |\det(f', f'')|^2}} \big(\det(f', f''), -f_2'', f_1''\big) .$$

EXAMPLE 1.35. If we identify $\mathbb{R}^2$ with the plane $\{z = 0\}$ in $\mathbb{R}^3$, we may consider every plane curve as a space curve. With this convention, it is straightforward (why?) to see that the binormal versor of a biregular curve $\sigma\colon I \to \mathbb{R}^2$ is everywhere equal to $(0, 0, 1)$ if the oriented curvature of σ is positive, and everywhere equal to $(0, 0, -1)$ if the oriented curvature of σ is negative.

REMARK 1.16. Keeping in mind Proposition 1.3, we immediately find that the binormal versor of an arbitrary biregular parametrized curve $\sigma\colon I \to \mathbb{R}^3$ is given by

$$\vec{b} = \frac{\sigma' \wedge \sigma''}{\|\sigma' \wedge \sigma''\|} . \tag{12}$$

In particular, we obtain another formula for the computation of the normal versor of curves in $\mathbb{R}^3$:

$$\vec{n} = \vec{b} \wedge \vec{t} = \frac{(\sigma' \wedge \sigma'') \wedge \sigma'}{\|\sigma' \wedge \sigma''\| \, \|\sigma'\|} .$$

Moreover, formula (6) for the computation of the curvature becomes

$$\kappa = \frac{\|\sigma' \wedge \sigma''\|}{\|\sigma'\|^3} . \tag{13}$$

The next proposition confirms the correctness of our idea that the variation of the binormal versor measures how far a curve is from being plane:

PROPOSITION 1.4. *Let $\sigma\colon I \to \mathbb{R}^3$ be a biregular curve of class C^k (with $k \geq 2$). Then the image of σ is contained in a plane if and only if the binormal versor is constant.*

Proof. Without loss of generality, we may assume that the curve σ is parametrized by arc length.

If the image of σ is contained in a plane, then there is a plane $H \subset \mathbb{R}^3$ containing the origin such that $\sigma(s) - \sigma(s') \in H$ for all s, $s' \in I$. Dividing by $s - s'$ and taking the limit as $s' \to s$ we immediately find that $\vec{t}(s) \in H$ for all $s \in I$. In the same way, it can be shown that $\dot{\vec{t}}(s) \in H$ for all $s \in I$, so $\vec{n}(s) \in H$ for all $s \in I$. Hence $\vec{b}(s)$ must always be one of the two versors orthogonal to H; since it changes continuously, it is constant.

On the other hand, assume the binormal versor is a constant vector $\vec{b}_0$; we want to prove that the support of σ is contained in a plane. Now, a plane is determined by one of its points and an orthogonal versor: a point $p \in \mathbb{R}^3$ is in the plane passing through $p_0 \in \mathbb{R}^3$ and orthogonal to the vector $v \in \mathbb{R}^3$ if and only if $\langle p - p_0, v\rangle = 0$. Take $s_0 \in I$; we want to show that the support of σ is contained in the plane through $\sigma(s_0)$ and orthogonal to $\vec{b}_0$. This is the same as showing that $\langle \sigma(s), \vec{b}_0\rangle \equiv \langle \sigma(s_0), \vec{b}_0\rangle$, or that the function $s \mapsto \langle \sigma(s), \vec{b}_0\rangle$ is constant. And indeed we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\langle \sigma, \vec{b}_0\rangle = \langle \vec{t}, \vec{b}_0\rangle \equiv 0 ,$$

as $\vec{t}$ is always orthogonal to the binormal versor, so the support of σ really is contained in the plane of equation $\langle p - \sigma(s_0), \vec{b}_0\rangle = 0$. $\square$

This result suggests that the derivative of the binormal versor might measure how far a biregular curve is from being plane. Now, $\vec{b}$ is a versor; so, taking the derivative of $\langle \vec{b}, \vec{b}\rangle \equiv 1$ we get $\langle \dot{\vec{b}}, \vec{b}\rangle \equiv 0$, that is $\dot{\vec{b}}$ is always orthogonal to $\vec{b}$. On the other hand,

$$\dot{\vec{b}} = \dot{\vec{t}} \wedge \vec{n} + \vec{t} \wedge \dot{\vec{n}} = \vec{t} \wedge \dot{\vec{n}} ,$$

so $\dot{\vec{b}}$ is perpendicular to $\vec{t}$ too; hence, $\dot{\vec{b}}$ has to be a multiple of $\vec{n}$.

Definition 1.17. Let $\sigma\colon I \to \mathbb{R}^3$ be a biregular curve of class C^k (with $k \geq 3$) parametrized by arc length. The *torsion* of σ is the function $\tau\colon I \to \mathbb{R}$ of class C^{k-3} such that $\dot{\vec{b}} = -\tau\vec{n}$. (*Warning*: In some texts the torsion is defined to be the opposite of the function we have chosen.)

Remark 1.17. Proposition 1.4 may then be rephrased by saying that *the image of a biregular curve σ is contained in a plane if and only if the torsion of σ is everywhere zero.*

Remark 1.18. Curvature and torsion do not depend on the orientation of the curve. More precisely, if $\sigma\colon I \to \mathbb{R}^3$ is a biregular curve parametrized by arc length, and σ^- is the usual curve parametrized by arc length equivalent to σ but with the opposite orientation given by $\sigma^-(s) = \sigma(-s)$, then the curvature κ^- and the torsion τ^- of σ^- are such that

$$\kappa^-(s) = \kappa(-s) \quad \text{and} \quad \tau^-(s) = \tau(-s) .$$

Remark 1.19. On the other hand, the oriented curvature and the oriented normal versor of a plane curve depend on the orientation of the curve. Indeed, with the notation of the previous remark applied to a plane curve σ, we find

$$\vec{t}^-(s) = -\vec{t}(-s) , \quad \tilde{\kappa}^-(s) = -\tilde{\kappa}(-s) , \quad \tilde{\vec{n}}^-(s) = -\tilde{\vec{n}}(-s) .$$

REMARK 1.20. To find the torsion of a biregular curve $\sigma\colon I \to \mathbb{R}^3$ with an arbitrary parametrization, first of all note that $\tau = -\langle \dot{\vec{b}}, \vec{n}\rangle$. Taking the derivative of (12), we get

$$\dot{\vec{b}} = \frac{\mathrm{d}\vec{b}}{\mathrm{d}s} = \frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}\vec{b}}{\mathrm{d}t} = \frac{1}{\|\sigma'\|}\left[\frac{\sigma' \wedge \sigma'''}{\|\sigma' \wedge \sigma''\|} - \frac{\langle \sigma' \wedge \sigma'', \sigma' \wedge \sigma'''\rangle}{\|\sigma' \wedge \sigma''\|^3}\sigma' \wedge \sigma''\right] .$$

Therefore, keeping in mind Equation (7), we obtain

$$\tau = -\frac{\langle \sigma' \wedge \sigma''', \sigma''\rangle}{\|\sigma' \wedge \sigma''\|^2} = \frac{\langle \sigma' \wedge \sigma'', \sigma'''\rangle}{\|\sigma' \wedge \sigma''\|^2} .$$

EXAMPLE 1.36. If $\sigma\colon I \to \mathbb{R}^3$ is the usual parametrization $\sigma(t) = \big(t, f(t)\big)$ of the graph of a function $f\colon I \to \mathbb{R}^2$ with f'' nowhere zero, then

$$\tau = \frac{\det(f'', f''')}{\|f''\|^2 + |\det(f', f'')|^2} .$$

EXAMPLE 1.37. Let $\sigma\colon \mathbb{R} \to \mathbb{R}^3$ be the circular helix with radius $r > 0$ and pitch $a \in \mathbb{R}^*$, parametrized by arc length as in Example 1.22. Then, taking the derivative of the binormal versor found in Example 1.33 and keeping in mind Example 1.30, we find

$$\tau(s) \equiv \frac{a}{r^2 + a^2} .$$

Thus both the curvature and the torsion of the circular helix are constant.

We have computed the derivative of the tangent versor and of the binormal versor; for the sake of completeness, let us compute the derivative of the normal versor too. We get

$$\dot{\vec{n}} = \dot{\vec{b}} \wedge \vec{t} + \vec{b} \wedge \dot{\vec{t}} = -\tau\vec{n} \wedge \vec{t} + \vec{b} \wedge \kappa\vec{n} = -\kappa\vec{t} + \tau\vec{b} .$$

DEFINITION 1.18. Let $\sigma\colon I \to \mathbb{R}^3$ be a biregular space curve. The three equations

$$\begin{cases} \dot{\vec{t}} = \kappa\vec{n} , \\ \dot{\vec{n}} = -\kappa\vec{t} + \tau\vec{b} , \\ \dot{\vec{b}} = -\tau\vec{n} , \end{cases} \tag{14}$$

are the *Frenet-Serret formulas* of σ.

REMARK 1.21. There are Frenet-Serret formulas for plane curves too. Since $\dot{\tilde{\vec{n}}}$ is, for the usual reasons, orthogonal to $\tilde{\vec{n}}$, it has to be a multiple of $\vec{t}$. Taking the derivative of $\langle \vec{t}, \tilde{\vec{n}}\rangle \equiv 0$, we find $\langle \vec{t}, \dot{\tilde{\vec{n}}}\rangle = -\tilde{\kappa}$. So the *Frenet-Serret formulas for plane curves* are

$$\begin{cases} \dot{\vec{t}} = \tilde{\kappa}\tilde{\vec{n}} , \\ \dot{\tilde{\vec{n}}} = -\tilde{\kappa}\vec{t} , \end{cases}$$

The basic idea of the local theory of space curves is that the curvature and the torsion completely determine a curve (compare Example 1.37 and Problem 1.7). To convey in precise terms what we mean, we need a definition.

DEFINITION 1.19. A *rigid motion* of $\mathbb{R}^n$ is an affine map $\rho\colon \mathbb{R}^n \to \mathbb{R}^n$ of the form $\rho(x) = Ax + b$, where $b \in \mathbb{R}^n$ and

$$A \in SO(n) = \{A \in GL(n, \mathbb{R}) \mid A^T A = I \text{ and } \det A = 1\} .$$

In particular, when $n = 3$ every rigid motion is a rotation about the origin followed by a translation.

If a curve is obtained from another through a rigid motion, both curves have the same curvature and torsion (Exercise 1.26); conversely, the *fundamental theorem of the local theory of curves* states that any two curves with equal curvature and torsion can always be obtained from one another through a rigid motion.

Frenet-Serret formulas are exactly the tool that will enable us to prove this result, using the classical Analysis theorem about the existence and uniqueness of the solutions of a linear system of ordinary differential equations (see [**5**, p. 162]):

THEOREM 1.5. *Given an interval $I \subseteq \mathbb{R}$, a point $t_0 \in I$, a vector $u_0 \in \mathbb{R}^n$, and two functions $f\colon I \to \mathbb{R}^n$ and $A\colon I \to M_{n,n}(\mathbb{R})$ of class C^k, with $k \in \mathbb{N}^* \cup \{\infty\}$, where $M_{p,q}(\mathbb{R})$ denotes the space of $p\times q$ real matrices, there exists a unique solution $u\colon I \to \mathbb{R}^n$ of class C^{k+1} to the Cauchy problem*

$$\begin{cases} u' = Au + f , \\ u(t_0) = u_0 . \end{cases}$$

In particular, the solution of the Cauchy problem for *linear* systems of *ordinary* differential equations exists over the whole domain of definition of the coefficients.

This is what we need to prove the *fundamental theorem of the local theory of curves:*

THEOREM 1.6 (Fundamental theorem of the local theory of curves). *Given two functions $\kappa\colon I \to \mathbb{R}^+$ and $\tau\colon I \to \mathbb{R}$, with κ always positive and of class C^{k+1} and τ of class C^k (with $k \in \mathbb{N}^* \cup \{\infty\}$), there exists a unique (up to a rigid motion) biregular curve $\sigma\colon I \to \mathbb{R}^3$ of class C^{k+3} parametrized by arc length with curvature κ and torsion τ.*

PROOF. We prove existence first. Frenet-Serret formulas (14) form a linear system of ordinary differential equations in 9 unknowns (the components of $\vec{t}$, $\vec{n}$, and $\vec{b}$); so we can apply Theorem 1.5.

Fix a point $s_0 \in I$ and an orthonormal basis $\{\vec{t}_0, \vec{n}_0, \vec{b}_0\}$ with the same orientation as the canonical basis. Theorem 1.5 provides us with a unique triple of functions $\vec{t}$, $\vec{n}$, $\vec{b}\colon I \to \mathbb{R}^3$, with $\vec{t}$ of class C^{k+2} and $\vec{n}$ and $\vec{b}$ of class C^{k+1}, satisfying (14) and such that $\vec{t}(s_0) = \vec{t}_0$, $\vec{n}(s_0) = \vec{n}_0$, and $\vec{b}(s_0) = \vec{b}_0$.

We want to prove that the triple $\{\vec{t}, \vec{n}, \vec{b}\}$ we have just found is the Frenet frame of some curve. We show first that being an orthonormal basis in s_0 forces it to be so in every point. From (14) we deduce that the functions $\langle \vec{t}, \vec{t}\rangle$, $\langle \vec{t}, \vec{n}\rangle$, $\langle \vec{t}, \vec{b}\rangle$, $\langle \vec{n}, \vec{n}\rangle$, $\langle \vec{n}, \vec{b}\rangle$, and $\langle \vec{b}, \vec{b}\rangle$ satisfy the following system of six linear ordinary differential equations in 6 unknowns

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}s}\langle \vec{t}, \vec{t}\rangle = 2\kappa\langle \vec{t}, \vec{n}\rangle \;, \\ \frac{\mathrm{d}}{\mathrm{d}s}\langle \vec{t}, \vec{n}\rangle = -\kappa\langle \vec{t}, \vec{t}\rangle + \tau\langle \vec{t}, \vec{b}\rangle + \kappa\langle \vec{n}, \vec{n}\rangle \;, \\ \frac{\mathrm{d}}{\mathrm{d}s}\langle \vec{t}, \vec{b}\rangle = -\tau\langle \vec{t}, \vec{n}\rangle + \kappa\langle \vec{n}, \vec{b}\rangle \;, \\ \frac{\mathrm{d}}{\mathrm{d}s}\langle \vec{n}, \vec{n}\rangle = -2\kappa\langle \vec{t}, \vec{n}\rangle + 2\tau\langle \vec{n}, \vec{b}\rangle \;, \\ \frac{\mathrm{d}}{\mathrm{d}s}\langle \vec{n}, \vec{b}\rangle = -\kappa\langle \vec{t}, \vec{b}\rangle - \tau\langle \vec{n}, \vec{n}\rangle + \tau\langle \vec{b}, \vec{b}\rangle \;, \\ \frac{\mathrm{d}}{\mathrm{d}s}\langle \vec{b}, \vec{b}\rangle = -2\tau\langle \vec{n}, \vec{b}\rangle \;, \end{cases}$$

with initial conditions

$$\langle \vec{t}, \vec{t}\rangle(s_0) = 1 \;, \quad \langle \vec{t}, \vec{n}\rangle(s_0) = 0 \;, \quad \langle \vec{t}, \vec{b}\rangle(s_0) = 0 \;,$$
$$\langle \vec{n}, \vec{n}\rangle(s_0) = 1 \;, \quad \langle \vec{n}, \vec{b}\rangle(s_0) = 0 \;, \quad \langle \vec{b}, \vec{b}\rangle(s_0) = 1 \;.$$

But it is straightforward to verify that

$$\langle \vec{t}, \vec{t}\rangle \equiv \langle \vec{n}, \vec{n}\rangle \equiv \langle \vec{b}, \vec{b}\rangle \equiv 1 \;, \quad \langle \vec{t}, \vec{n}\rangle \equiv \langle \vec{t}, \vec{b}\rangle \equiv \langle \vec{n}, \vec{b}\rangle \equiv 0 \tag{15}$$

is a solution of the same system of differential equations, satisfying the same initial conditions in s_0. So the functions $\vec{t}$, $\vec{n}$ and $\vec{b}$ have to satisfy equalities (15), and the triple $\{\vec{t}(s), \vec{n}(s), \vec{b}(s)\}$ is orthonormal for all $s \in I$. Moreover, it has the same orientation of the canonical basis of $\mathbb{R}^3$ everywhere: indeed, $\langle \vec{t}\wedge\vec{n}, \vec{b}\rangle$ is a continuous function on I with values in $\{+1, -1\}$, whose value is $+1$ in s_0; hence, necessarily, $\langle \vec{t}\wedge\vec{n}, \vec{b}\rangle \equiv +1$, which implies (why?) that $\{\vec{t}(s), \vec{n}(s), \vec{b}(s)\}$ has the same orientation as the canonical basis everywhere.

Finally, define the curve $\sigma\colon I \to \mathbb{R}^3$ by setting

$$\sigma(s) = \int_{s_0}^{s} \vec{t}(t)\,\mathrm{d}t \;.$$

The curve σ is of class C^{k+3} with derivative $\vec{t}(s)$, so it is regular, parametrized by arc length, and with tangent versor $\vec{t}$. Since the equations (14) give $\ddot{\sigma} = \kappa\vec{n}$ with $\kappa > 0$ everywhere, we deduce that κ is the curvature and $\vec{n}$ the normal versor of σ (in particular, σ is biregular). It follows that $\vec{b}$ is the binormal versor and, thanks to (14) once more, that τ is the torsion of σ, as required.

Let us now prove uniqueness. Let $\sigma_1\colon I \to \mathbb{R}^3$ be another biregular curve of class C^{k+3}, parametrized by arc length, with curvature κ and torsion τ. Fix $s_0 \in I$; up to a rigid motion, we may assume that $\sigma(s_0) = \sigma_1(s_0)$, and that σ and σ_1 have the same Frenet frame at s_0. By the uniqueness of the solution of (14), it follows that σ and σ_1 have the same Frenet frame at all points of I; in particular, $\dot{\sigma} \equiv \dot{\sigma}_1$. But this implies

$$\sigma(s) = \sigma(s_0) + \int_{s_0}^{s} \dot{\sigma}(t)\,\mathrm{d}t = \sigma_1(s_0) + \int_{s_0}^{s} \dot{\sigma}_1(t)\,\mathrm{d}t = \sigma_1(s) \;,$$

and $\sigma_1 \equiv \sigma$. □

Therefore curvature and torsion are all we need to completely describe a curve in space. For this reason, curvature and torsion are sometimes called *intrinsic* or *natural equations* of the curve.

REMARK 1.22. Exactly in the same way (Exercise 1.48) it is possible to prove the following result: *Given a function* $\tilde{\kappa}: I \to \mathbb{R}$ *of class* C^k, *with* $k \in \mathbb{N}^* \cup \{\infty\}$, *there exists a unique (up to a rigid motion in the plane) regular curve* $\sigma: I \to \mathbb{R}^2$ *of class* C^{k+2} *parametrized by arc length having oriented curvature* $\tilde{\kappa}$.

Guided problems

For convenience, we repeat here the Frenet-Serret formulas, and the formulas (given in Remarks 1.16 and 1.20, and useful to solve the exercises) for the computation of curvature, torsion and Frenet frame of an arbitrarily parametrized biregular space curve:

$$\boxed{\begin{gathered}\vec{t} = \frac{\sigma'}{\|\sigma'\|}\ , \quad \vec{b} = \frac{\sigma' \wedge \sigma''}{\|\sigma' \wedge \sigma''\|}\ , \quad \vec{n} = \frac{(\sigma' \wedge \sigma'') \wedge \sigma'}{\|\sigma' \wedge \sigma''\|\,\|\sigma'\|}\ , \\ \kappa = \frac{\|\sigma' \wedge \sigma''\|}{\|\sigma'\|^3}\ , \quad \tau = \frac{\langle \sigma' \wedge \sigma'', \sigma''' \rangle}{\|\sigma' \wedge \sigma''\|^2}\ ; \\ \begin{cases} \dot{\vec{t}} = \kappa \vec{n}\ , \\ \dot{\vec{n}} = -\kappa \vec{t} + \tau \vec{b}\ , \\ \dot{\vec{b}} = -\tau \vec{n}\ . \end{cases}\end{gathered}} \tag{15}$$

Problem 1.1. Let $\sigma\colon I \to \mathbb{R}^2$ be a biregular plane curve, parametrized by an arbitrary parameter t. Show that the oriented curvature of σ is given by

$$\tilde{\kappa} = \frac{1}{\|\sigma'\|^3} \det(\sigma', \sigma'') = \frac{x'y'' - x''y'}{\big((x')^2 + (y')^2\big)^{3/2}}\ ,$$

where x, $y\colon I \to \mathbb{R}$ are defined by $\sigma(t) = \big(x(t), y(t)\big)$.

Solution. By formula (10), the oriented curvature is given by $\tilde{\kappa} = \det(\vec{t}, \dot{\vec{t}})$. To complete the proof, it is sufficient to substitute $\vec{t} = \sigma'/||\sigma'||$ and

$$\begin{aligned} \dot{\vec{t}}\big(s(t)\big) &= \frac{1}{\|\sigma'(t)\|} \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\sigma'(t)}{\|\sigma'(t)\|} \right) \\ &= \frac{1}{\|\sigma'(t)\|^2} \left(\sigma''(t) - \frac{\langle \sigma''(t), \sigma'(t) \rangle}{\|\sigma'(t)\|^2} \sigma'(t) \right) \end{aligned}$$

in (10). Since the determinant is linear and alternating with respect to columns, we get $\tilde{\kappa} = \det(\sigma', \sigma'')/\|\sigma'(t)\|^3$, as desired. □

Problem 1.2. Let $\sigma\colon I \to \mathbb{R}^n$ be a regular curve of class C^2 parametrized by arc length. Denote by $\theta(\varepsilon)$ the angle between the versors $\vec{t}(s_0)$ and $\vec{t}(s_0 + \varepsilon)$, tangent to σ respectively in $\sigma(s_0)$ and in a nearby point $\sigma(s_0 + \varepsilon)$, for $\varepsilon > 0$ small. Show that the curvature $\kappa(s_0)$ of σ in $\sigma(s_0)$ satisfies the equality

$$\kappa(s_0) = \lim_{\varepsilon \to 0} \left| \frac{\theta(\varepsilon)}{\varepsilon} \right|\ .$$

Deduce that the curvature κ measures the rate of variation of the direction of the tangent line, with respect to the arc length.

Solution. Consider the versors $\vec{t}(s_0)$ and $\vec{t}(s_0 + \varepsilon)$, having as their initial point the origin O; the triangle they determine is isosceles, and the length of the third side is given by $\|\vec{t}(s_0 + \varepsilon) - \vec{t}(s_0)\|$. The Taylor expansion of the sine function yields

$$\|\vec{t}(s_0 + \varepsilon) - \vec{t}(s_0)\| = 2\left|\sin\big(\theta(\varepsilon)/2\big)\right| = \left|\theta(\varepsilon) + o\big(\theta(\varepsilon)\big)\right|\ .$$

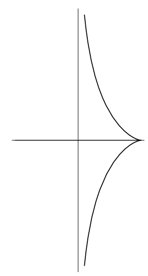


FIGURE 4. The tractrix

Keeping in mind the definition of curvature, we conclude that

$$\begin{aligned}\kappa(s_0) = \|\dot{\vec{t}}(s_0)\| &= \lim_{\varepsilon\to 0}\left\|\frac{\vec{t}(s_0+\varepsilon)-\vec{t}(s_0)}{\varepsilon}\right\| \\ &= \lim_{\varepsilon\to 0}\left|\frac{\theta(\varepsilon)+o\big(\theta(\varepsilon)\big)}{\varepsilon}\right| .\end{aligned}$$

As $\lim_{\varepsilon\to 0}\theta(\varepsilon) = 0$, the assertion follows. □

PROBLEM 1.3. **The tractrix.** Let $\sigma\colon (0,\pi)\to\mathbb{R}^2$ be the plane curve defined by

$$\sigma(t) = \left(\sin t, \cos t + \log\tan\frac{t}{2}\right) ;$$

the image of σ is called *tractrix* (Fig. 4). [Note: This curve will be used in later chapters to define surfaces with important properties; see Example 3.37.]

(i) Prove that σ is a parametrization of class C^∞, regular everywhere except in $t=\pi/2$.
(ii) Check that the length of the segment of the tangent line to the tractrix from the point of tangency to the y-axis is always 1.
(iii) Determine the arc length of σ starting from $t_0=\pi/2$.
(iv) Compute the curvature of σ where it is defined.

SOLUTION. (i) Since $\tan(t/2)>0$ for all $t\in(0,\pi)$, the curve σ is of class C^∞. Moreover,

$$\sigma'(t) = \left(\cos t, \frac{\cos^2 t}{\sin t}\right) \qquad \text{and} \qquad \|\sigma'(t)\| = \frac{|\cos t|}{\sin t} ,$$

so $\sigma'(t)$ is zero only for $t=\pi/2$, as desired.

(ii) If $t_0\neq\pi/2$, the affine tangent line $\eta\colon\mathbb{R}\to\mathbb{R}^2$ to σ at the point $\sigma(t_0)$ is given by

$$\eta(x) = \sigma(t_0)+x\sigma'(t_0) = \left(\sin t_0 + x\cos t_0, \cos t_0 + \log\tan\frac{t_0}{2} + x\frac{\cos^2 t_0}{\sin t_0}\right) .$$

The tangent line intersects the y-axis in the point where the first coordinate of η is zero, that is, for $x=-\tan t_0$. So the length we are looking for is

$$\|\eta(-\tan t_0)-\eta(0)\| = \|(-\sin t_0, -\cos t_0)\| = 1 ,$$

as stated.

In a sense, this result is true for $t_0 = \pi/2$ too. Indeed, even if the tangent vector to σ tends to O for $t \to \pi/2$, the tangent line to σ at $\sigma(t)$ tends to the x-axis for $t \to \pi/2$, since

$$\lim_{t\to\pi/2^-} \frac{\sigma'(t)}{\|\sigma'(t)\|} = (1,0) = -(-1,0) = -\lim_{t\to\pi/2^+} \frac{\sigma'(t)}{\|\sigma'(t)\|}\ .$$

So, if we consider the x-axis as the tangent line to the support of the tractrix at the point $\sigma(\pi/2) = (1,0)$, in this case too the segment of the tangent line from the point of the curve to the y-axis has length 1.

(iii) If $t > \pi/2$ we have

$$s(t) = \int_{\pi/2}^{t} \|\sigma'(\tau)\|\,\mathrm{d}\tau = -\int_{\pi/2}^{t} \frac{\cos\tau}{\sin\tau}\,\mathrm{d}\tau = -\log\sin t\ .$$

Analogously, if $t < \pi/2$ we have

$$s(t) = \int_{\pi/2}^{t} \|\sigma'(\tau)\|\,\mathrm{d}\tau = -\int_{t}^{\pi/2} \frac{\cos\tau}{\sin\tau}\,\mathrm{d}\tau = \log\sin t\ .$$

In particular,

$$s^{-1}(s) = \begin{cases} \pi - \arcsin \mathrm{e}^{-s} \in [\pi/2,\pi) & \text{if } s \in [0,+\infty)\ , \\ \arcsin \mathrm{e}^{s} \in (0,\pi/2] & \text{if } s \in (-\infty,0]\ , \end{cases}$$

and using the formula $\tan\frac{x}{2} = \frac{\sin x}{1+\cos x}$ we see that the reparametrization of σ by arc length is given by

$$\sigma\bigl(s^{-1}(s)\bigr) = \begin{cases} \left(\mathrm{e}^{-s}, -s - \sqrt{1-\mathrm{e}^{-2s}} - \log\left(1-\sqrt{1-\mathrm{e}^{-2s}}\right)\right) & \text{if } s > 0\ , \\ \left(\mathrm{e}^{s}, s + \sqrt{1-\mathrm{e}^{2s}} - \log\left(1+\sqrt{1-\mathrm{e}^{2s}}\right)\right) & \text{if } s < 0\ . \end{cases}$$

(iv) Using the reparametrization $\sigma_1 = \sigma \circ s^{-1}$ of σ by arc length we have just computed, we find

$$\dot\sigma_1(s) = \begin{cases} \left(-\mathrm{e}^{-s}, -\dfrac{1-\mathrm{e}^{-2s}-\sqrt{1-\mathrm{e}^{-2s}}}{1-\sqrt{1-\mathrm{e}^{-2s}}}\right) & \text{if } s > 0\ , \\ \left(\mathrm{e}^{s}, \dfrac{1-\mathrm{e}^{2s}+\sqrt{1-\mathrm{e}^{2s}}}{1+\sqrt{1-\mathrm{e}^{2s}}}\right) & \text{if } s < 0\ , \end{cases}$$

and

$$\ddot\sigma_1(s) = \begin{cases} \left(\mathrm{e}^{-s}, \dfrac{\mathrm{e}^{-2s}}{\sqrt{1-\mathrm{e}^{-2s}}}\right) & \text{if } s > 0\ , \\ \left(\mathrm{e}^{s}, -\dfrac{\mathrm{e}^{2s}}{\sqrt{1-\mathrm{e}^{2s}}}\right) & \text{if } s < 0\ . \end{cases}$$

So the curvature κ_1 of σ_1 for $s \neq 0$ is given by

$$\kappa(s) = \|\ddot\sigma(s)\| = \frac{1}{\sqrt{\mathrm{e}^{2|s|}-1}}\ ,$$

and (keeping in mind Remark 1.11) the curvature κ of σ for $t \neq \pi/2$ is

$$\kappa(t) = \kappa_1\bigl(s(t)\bigr) = |\tan t|\ .$$

As an alternative, we could have computed the curvature of σ by using the formula for curves with an arbitrary parametrization (see next problem and Problem 1.1). □

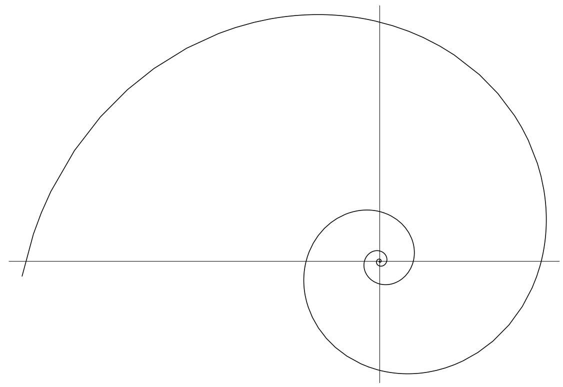


FIGURE 5. Logarithmic spiral

PROBLEM 1.4. **Logarithmic spiral.** Fix two real numbers $a > 0$ and $b < 0$. The *logarithmic spiral* (Fig. 5) is the plane curve $\sigma\colon \mathbb{R} \to \mathbb{R}^2$ given by

$$\sigma(t) = (a\mathrm{e}^{bt} \cos t, a\mathrm{e}^{bt} \sin t) .$$

(i) Show that the support of the spiral satisfies the equation $r = a\mathrm{e}^{b\theta}$, expressed in the polar coordinates (r, θ).
(ii) Show that $\sigma(t)$ winds around the origin O tending to it as $t \to \infty$.
(iii) Determine the arc length of σ, starting from $t = 0$. Find the arc length in the case $a = 1/2$ and $b = -1$.
(iv) Determine the curvature and the torsion of σ, and remark that the curvature is never zero.

SOLUTION. (i) We have $r^2 = x^2 + y^2 = a^2\mathrm{e}^{2bt}(\cos^2 t + \sin^2 t) = a^2\mathrm{e}^{2bt}$, and the assertion follows because r is always positive.

(ii) First of all, by (i) we have $\|\sigma(t)\| = a\mathrm{e}^{bt}$, and thus $\sigma(t) \to O$ as $t \to \infty$, because $b < 0$.

Moreover, t coincides with the argument θ of $\sigma(t)$ up to a multiple of 2π; so the argument of $\sigma(t)$ is periodic of period 2π and assumes all possible values, that is σ winds around the origin.

(iii) Note that the parametrization of σ is of class C^∞. Differentiating, we find

$$\sigma'(t) = a\mathrm{e}^{bt}(b \cos t - \sin t, b \sin t + \cos t)$$

and so

$$\|\sigma'(t)\| = a\mathrm{e}^{bt}\sqrt{b^2 + 1} .$$

We deduce from this that the arc length of σ starting from $t = 0$ is given by:

$$s(t) = \int_0^t \|\sigma'(\tau)\| \,\mathrm{d}\tau = a\sqrt{b^2+1} \int_0^t \mathrm{e}^{b\tau} \,\mathrm{d}\tau = a\sqrt{b^2+1} \left[\frac{\mathrm{e}^{bt} - 1}{b} \right] .$$

In the case $a = 1/2$, $b = -1$, the arc length is $s(t) = (1 - \mathrm{e}^{-t})/\sqrt{2}$.

(iv) By applying the usual formulas we find

$$\begin{aligned}\kappa(t) &= \frac{\|\sigma' \wedge \sigma''\|}{\|\sigma'\|^3}\\ &= \frac{-(b^2-1)\sin^2 t + 2b^2\cos^2 t - (b^2-1)\cos^2 t + 2b^2\sin^2 t}{a\,\mathrm{e}^{bt}\,(b^2+1)^{3/2}}\\ &= \frac{2b^2-(b^2-1)}{a\,\mathrm{e}^{bt}\,(b^2+1)^{3/2}} = \frac{b^2+1}{a\,\mathrm{e}^{bt}\,(b^2+1)^{3/2}} = \frac{1}{a\,\mathrm{e}^{bt}\,(b^2+1)^{1/2}}\ .\end{aligned}$$

In particular, the curvature is never zero, and the curve is biregular.

Finally, since the curve σ is plane and biregular, its torsion is defined and is zero everywhere. □

Problem 1.5. **Twisted cubic.** Determine the curvature, the torsion, and the Frenet frame of the curve $\sigma\colon \mathbb{R} \to \mathbb{R}^3$ defined by $\sigma(t) = (t, t^2, t^3)$.

Solution. Differentiating the expression of σ we find

$$\sigma'(t) = (1, 2t, 3t^2)\ , \quad \sigma''(t) = (0, 2, 6t) \quad \text{and} \quad \sigma'''(t) = (0, 0, 6)\ .$$

In particular, σ' is nowhere zero; thus σ is regular and

$$\vec{t}(t) = \frac{1}{\sqrt{1+4t^2+9t^4}}(1, 2t, 3t^2)\ .$$

Next,

$$\sigma'(t) \wedge \sigma''(t) = (6t^2, -6t, 2)$$

is never zero, so σ' and σ'' are always linearly independent, and σ is biregular. Using the formulas recalled at the beginning of this section we get

$$\vec{b}(t) = \frac{\sigma'(t) \wedge \sigma''(t)}{\|\sigma'(t) \wedge \sigma''(t)\|} = \frac{1}{2\sqrt{1+9t^2+9t^4}}(6t^2, -6t, 2)\ ,$$

$$\vec{n}(t) = \vec{b}(t) \wedge \vec{t}(t) = \frac{(-9t^3-2t, 1-9t^4, 6t^3+3t)}{\sqrt{(1+4t^2+9t^4)(1+9t^2+9t^4)}}\ ,$$

$$\kappa(t) = \frac{\|\sigma'(t) \wedge \sigma''(t)\|}{\|\sigma'(t)\|^3} = \frac{2\sqrt{1+9t^2+9t^4}}{(1+4t^2+9t^4)^{3/2}}\ ,$$

and

$$\tau(t) = \left\langle \vec{b}(t), \frac{\sigma'''(t)}{\|\sigma'(t) \wedge \sigma''(t)\|} \right\rangle = \frac{3}{1+9t^2+9t^4}\ .$$

□

Problem 1.6. Prove that the curve $\sigma\colon (0, +\infty) \to \mathbb{R}^3$ defined by

$$\sigma(t) = \left(t, \frac{1+t}{t}, \frac{1-t^2}{t}\right)$$

is contained in a plane.

Solution. By noting that

$$\sigma'(t) = \left(1, -\frac{1}{t^2}, -\frac{1}{t^2} - 1\right) \qquad \text{and} \qquad \sigma''(t) = \left(0, \frac{2}{t^3}, \frac{2}{t^3}\right)\ ,$$

we find that the vector product

$$\sigma'(t) \wedge \sigma''(t) = \frac{2}{t^3}\,(1, -1, 1)$$

is nowhere zero, so the curvature $\kappa = \|\sigma' \wedge \sigma''\|/\|\sigma'\|^3$ is nowhere zero. By Remark 1.17, we may conclude that σ is a plane curve if and only if the torsion $\tau = \langle \sigma' \wedge \sigma'', \sigma''' \rangle / \|\sigma' \wedge \sigma''\|^2$ is zero everywhere, that is, if and only if $\langle \sigma' \wedge \sigma'', \sigma''' \rangle$ is zero everywhere. But

$$\langle \sigma' \wedge \sigma'', \sigma''' \rangle = \det \begin{pmatrix} 1 & -\frac{1}{t^2} & -\frac{1}{t^2} - 1 \\ 0 & \frac{2}{t^3} & \frac{2}{t^3} \\ 0 & -\frac{6}{t^4} & -\frac{6}{t^4} \end{pmatrix} \equiv 0 \,,$$

and the assertion follows. □

PROBLEM 1.7. Let $\sigma \colon I \to \mathbb{R}^3$ be a biregular curve parametrized by arc length, having constant curvature $\kappa_0 > 0$ and constant torsion $\tau_0 \in \mathbb{R}$. Prove that, up to rotations and translations of $\mathbb{R}^3$, σ is an arc of a circular helix.

SOLUTION. If $\tau_0 = 0$, then Proposition 1.4 and Example 1.32 tell us that σ is an arc of a circle, so it can be considered an arc of the degenerate circular helix with pitch 0.

Assume, on the other hand, $\tau_0 \neq 0$. Then

$$\frac{\mathrm{d}}{\mathrm{d}s}(\tau_0 \vec{t} + \kappa_0 \vec{b}) = \tau_0 \kappa_0 \vec{n} - \kappa_0 \tau_0 \vec{n} \equiv O \,;$$

so $\tau_0 \vec{t} + \kappa_0 \vec{b}$ has to be everywhere equal to a constant vector $\vec{v}_0$ having length $\sqrt{\kappa_0^2 + \tau_0^2}$. Up to rotations in $\mathbb{R}^3$ (which do not change the curvature nor the torsion; see Exercise 1.26), we may assume

$$\vec{v}_0 = \sqrt{\kappa_0^2 + \tau_0^2}\, \vec{e}_3 \quad \Longrightarrow \quad \vec{e}_3 \equiv \frac{\tau_0}{\sqrt{\kappa_0^2 + \tau_0^2}} \vec{t} + \frac{\kappa_0}{\sqrt{\kappa_0^2 + \tau_0^2}} \vec{b} \,,$$

where $\vec{e_3} = (0, 0, 1)$ is the third vector of the canonical basis of $\mathbb{R}^3$. Let then $\sigma_1 \colon I \to \mathbb{R}^3$ be defined by

$$\sigma_1(s) = \sigma(s) - \frac{\tau_0 s}{\sqrt{\kappa_0^2 + \tau_0^2}}\, \vec{e}_3$$

(beware: as we shall see shortly, s is *not* the arc length parameter of σ_1). We want to show that σ_1 is the parametrization of an arc of a circle contained in a plane orthogonal to $\vec{e}_3$. First of all,

$$\frac{\mathrm{d}}{\mathrm{d}s} \langle \sigma_1, \vec{e}_3 \rangle = \langle \sigma_1', \vec{e}_3 \rangle = \langle \vec{t}, \vec{e}_3 \rangle - \frac{\tau_0}{\sqrt{\kappa_0^2 + \tau_0^2}} \equiv 0 \,;$$

hence $\langle \sigma_1, \vec{e}_3 \rangle$ is constant, and so the support of σ_1 is contained in a plane orthogonal to $\vec{e}_3$, as claimed. Moreover,

$$\sigma_1' = \vec{t} - \frac{\tau_0}{\sqrt{\kappa_0^2 + \tau_0^2}}\, \vec{e}_3 = \frac{\kappa_0^2}{\kappa_0^2 + \tau_0^2}\, \vec{t} - \frac{\kappa_0 \tau_0}{\kappa_0^2 + \tau_0^2}\, \vec{b} \quad \text{and} \quad \sigma_1'' = \kappa_0 \vec{n} \,;$$

hence

$$\|\sigma_1'\| \equiv \frac{\kappa_0}{\sqrt{\kappa_0^2 + \tau_0^2}} \quad \text{and} \quad \sigma_1' \wedge \sigma_1'' = \frac{\kappa_0^3}{\kappa_0^2 + \tau_0^2}\, \vec{b} + \frac{\kappa_0^2 \tau_0}{\kappa_0^2 + \tau_0^2}\, \vec{t} \,.$$

So, using (13) we find that the curvature κ_1 of σ_1 is

$$\kappa_1 = \frac{\|\sigma_1' \wedge \sigma_1''\|}{\|\sigma_1'\|^3} \equiv \frac{\kappa_0^2 + \tau_0^2}{\kappa_0} \,.$$

Thus σ_1, being a plane curve with constant curvature, by Example 1.32 parametrizes an arc of a circle with radius $r = \kappa_0/(\kappa_0^2+\tau_0^2)$ and contained in a plane orthogonal to $\vec{e}_3$. Up to a translation in $\mathbb{R}^3$, we may assume that this circle is centered at the origin, and hence σ indeed is a circular helix with radius r and pitch $a = \tau_0/(\kappa_0^2+\tau_0^2)$, as stated. $\square$

PROBLEM 1.8. **Curves on a sphere.** Let $\sigma\colon I \to \mathbb{R}^3$ be a biregular curve parametrized by arc length.

(i) Prove that if the support of σ is contained in a sphere with radius $R > 0$ then

$$\tau^2 + (\dot{\kappa}/\kappa)^2 \equiv R^2\kappa^2\tau^2 \; . \tag{16}$$

(ii) Prove that if $\dot{\kappa}$ is nowhere zero and σ satisfies (16) then the support of σ is contained in a sphere with radius $R > 0$. [Note: You can find further information about curves contained in a sphere in Exercise 1.55.]

SOLUTION. (i) Up to a translation in $\mathbb{R}^3$ (which does not change curvatures and torsions; see Exercise 1.26), we may assume the center of the sphere to be in the origin. So $\langle \sigma, \sigma \rangle \equiv R^2$; differentiating three times and applying Frenet-Serret formulas we find

$$\langle \vec{t}, \sigma \rangle \equiv 0, \quad \kappa\langle \vec{n}, \sigma \rangle + 1 \equiv 0 \quad \text{and} \quad \dot{\kappa}\langle \vec{n}, \sigma \rangle + \kappa\tau\langle \vec{b}, \sigma \rangle \equiv 0 \; . \tag{17}$$

Now, $\{\vec{t}, \vec{n}, \vec{b}\}$ is an orthonormal basis; in particular, we may write

$$\sigma = \langle \sigma, \vec{t} \rangle \vec{t} + \langle \sigma, \vec{n} \rangle \vec{n} + \langle \sigma, \vec{b} \rangle \vec{b} \; ,$$

so $|\langle \sigma, \vec{t} \rangle|^2 + |\langle \sigma, \vec{n} \rangle|^2 + |\langle \sigma, \vec{b} \rangle|^2 \equiv R^2$, and (17) implies (16).

(ii) Since $\dot{\kappa}$ is nowhere zero, by equation (16) so is τ; hence we may divide (16) by $\tau^2\kappa^2$, obtaining

$$\frac{1}{\kappa^2} + \left(\frac{1}{\tau} \frac{\mathrm{d}}{\mathrm{d}s} \left(\frac{1}{\kappa} \right) \right)^2 \equiv R^2 \; .$$

Differentiating and recalling that $\dot{\kappa} \neq 0$, we find

$$\frac{\tau}{\kappa} + \frac{\mathrm{d}}{\mathrm{d}s} \left(\frac{1}{\tau} \frac{\mathrm{d}}{\mathrm{d}s} \left(\frac{1}{\kappa} \right) \right) \equiv 0 \; .$$

Define now $\eta\colon I \to \mathbb{R}^3$ by setting

$$\eta = \sigma + \frac{1}{\kappa} \vec{n} + \frac{1}{\tau} \frac{\mathrm{d}}{\mathrm{d}s} \left(\frac{1}{\kappa} \right) \vec{b} \; .$$

Then

$$\frac{\mathrm{d}\eta}{\mathrm{d}s} = \vec{t} + \frac{\mathrm{d}}{\mathrm{d}s} \left(\frac{1}{\kappa} \right) \vec{n} - \vec{t} + \frac{\tau}{\kappa} \vec{b} + \frac{\mathrm{d}}{\mathrm{d}s} \left(\frac{1}{\tau} \frac{\mathrm{d}}{\mathrm{d}s} \left(\frac{1}{\kappa} \right) \right) \vec{b} - \frac{\mathrm{d}}{\mathrm{d}s} \left(\frac{1}{\kappa} \right) \vec{n} \equiv O \; ,$$

that is the curve η is constant. This means that there exists a point $p \in \mathbb{R}^3$ such that

$$\|\sigma - p\|^2 = \frac{1}{\kappa^2} + \left(\frac{1}{\tau} \frac{\mathrm{d}}{\mathrm{d}s} \left(\frac{1}{\kappa} \right) \right)^2 \equiv R^2 \; ;$$

hence, the support of σ is contained in the sphere with radius R and center p. $\square$

PROBLEM 1.9. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a C^∞ function; in this problem we shall write $f_x = \partial f/\partial x$, $f_y = \partial f/\partial y$, $f_{xx} = \partial^2 f/\partial x^2$, and so on. Choose a point $p \in f^{-1}(0) = C$, with $f_y(p) \neq 0$, and let $g\colon I \to \mathbb{R}$, with $I \subseteq \mathbb{R}$, a C^∞ function such that $f^{-1}(0)$ is given, in a neighborhood of p, by the graph of g, as in Proposition 1.2. Finally, choose $t_0 \in I$ so that $p = \big(t_0, g(t_0)\big)$.

(i) Show that the tangent vector to C in p is parallel to $\big(-f_y(p), f_x(p)\big)$, and thus the vector $\nabla f(p) = \big(f_x(p), f_y(p)\big)$ is orthogonal to the tangent vector.

(ii) Show that the oriented curvature at p of C is given by

$$\tilde{\kappa} = \frac{f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2}{\|\nabla f\|^3} ,$$

where we have oriented C in such a way that the tangent versor at p is a positive multiple of $\big(-f_y(p), f_x(p)\big)$.

(iii) If $f(x,y) = x^4 + y^4 - xy - 1$ and $p = (1,0)$, compute the oriented curvature of C at p.

SOLUTION. (i) Consider the parametrization $\sigma(t) = \big(t, g(t)\big)$. The tangent vector is parallel to $\sigma'(t_0) = \big(1, g'(t_0)\big)$. Since $f\big(t, g(t)\big) \equiv 0$, differentiating with respect to t we find that

$$f_x\big(t, g(t)\big) + f_y\big(t, g(t)\big)\, g'(t) \equiv 0 \; ; \tag{18}$$

so

$$g'(t_0) = -\frac{f_x(p)}{f_y(p)} ,$$

and the assertion immediately follows.

(ii) By differentiating again we find $\sigma''(t_0) = \big(0, g''(t_0)\big)$; so using the formula $\tilde{\kappa} = \|\sigma'\|^{-3} \det(\sigma', \sigma'')$ proved in Problem 1.1 we get

$$\tilde{\kappa} = \frac{|f_y(p)|^3 g''(t_0)}{\|\nabla f(p)\|^3} . \tag{19}$$

Take now one more derivative of (18) and evaluate it at t_0: we find that

$$f_{xx}(p) + f_{xy}(p)\, g'(t_0) + [f_{yx}(p) + f_{yy}(p)\, g'(t_0)]\, g'(t_0) + f_y(p)\, g''(t_0) \equiv 0 .$$

The parametrization σ is oriented as required if and only if $f_y(p) < 0$. So, extracting $g''(t_0)$ from the previous expression and inserting it in (19), we find the formula we were seeking.

(iii) In this case,

$$f_x(p) = 4 , \quad f_y(p) = -1 , \quad f_{xx}(p) = 12 , \quad f_{xy}(p) = -1 \quad f_{yy}(p) = 0 ,$$

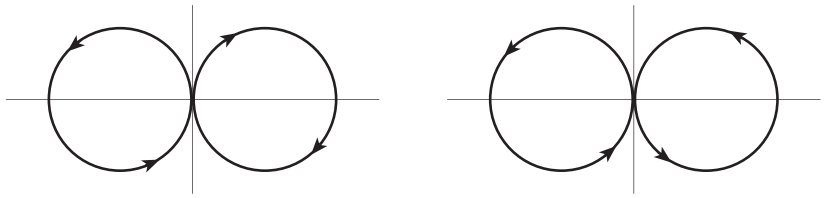
and so $\tilde{\kappa} = 4/17^{3/2}$. □

Exercises

PARAMETRIZATIONS AND CURVES

1.1. Prove that the curve $\sigma\colon \mathbb{R} \to \mathbb{R}^2$ given by $\sigma(t) = \big(t/(1+t^4), t/(1+t^2)\big)$ is an injective regular parametrization, but not a homeomorphism with its image.

1.2. Draw the support of the curve parametrized, in polar coordinates (r, θ), by $\sigma_1(\theta) = (a\cos\theta, \theta)$, for $\theta \in [0, 2\pi]$. Note that the image is contained in a circle, and that it is defined by the equation $r = a\cos\theta$.



(a) (b)

Figure 6

1.3. Prove that the relation introduced in Definition 1.3 actually is an equivalence relation on the set of parametrizations of class C^k.

1.4. Prove that the curve $\sigma\colon \mathbb{R} \to \mathbb{R}^2$ given by $\sigma(t) = (t^2, t^3)$ is not regular and that no parametrization equivalent to it can be regular.

1.5. Prove that every open interval $I \subseteq \mathbb{R}$ is C^∞-diffeomorphic to $\mathbb{R}$.

1.6. Prove that every interval $I \subseteq \mathbb{R}$ is C^∞-diffeomorphic to one of the following: $[0,1)$, $(0,1)$, or $[0,1]$. In particular, every regular curve admits a parametrization defined in one of these intervals.

1.7. Determine the parametrization $\sigma_1\colon (-\pi,\pi) \to \mathbb{R}^3$ equivalent to the parametrization $\sigma\colon \mathbb{R} \to \mathbb{R}^2$ given by $\sigma(t) = (r\cos t, r\sin t)$ of the circle, obtained by the parameter change $s = \arctan(t/4)$.

1.8. Prove that the two parametrizations $\sigma, \sigma_1\colon [0,2\pi] \to \mathbb{R}^2$ of class C^∞ of the circle defined by $\sigma(t) = (\cos t, \sin t)$ and $\sigma_1(t) = (\cos 2t, \sin 2t)$ are not equivalent (see Example 1.7 and Remark 1.7).

1.9. Let $\sigma_1\colon [0,2\pi] \to \mathbb{R}^2$ be defined by

$$\sigma_1(t) = \begin{cases} (\cos t, \sin t) & \text{for } t \in [0,\pi]\ , \\ (-1, 0) & \text{for } t \in [\pi, 2\pi]\ . \end{cases}$$

(i) Show that σ_1 is continuous but not of class C^1.

(ii) Prove that σ_1 is not equivalent to the usual parametrization of the circle $\sigma\colon [0,2\pi] \to \mathbb{R}^2$ given by $\sigma(t) = (\cos t, \sin t)$.

1.10. Prove that the support of the curve of the Example 1.19 cannot be the image of a regular curve.

1.11. For all $k \in \mathbb{N}^* \cup \{\infty\}$, find a parametrized curve $\sigma\colon \mathbb{R} \to \mathbb{R}^2$ of class C^k having as its support the graph of the absolute value function. Show next that no such curve can be regular.

1.12. Let $\sigma\colon [0,1] \to \mathbb{R}^2$ given by

$$\sigma(t) = \begin{cases} \bigl(-1 + \cos(4\pi t), \sin(4\pi t)\bigr) & \text{for } t \in [0,1/2]\ , \\ \bigl(1 + \cos(-4\pi t - \pi), \sin(-4\pi t - \pi)\bigr) & \text{for } t \in [1/2,1]\ ; \end{cases}$$

see Fig. 6.(a).

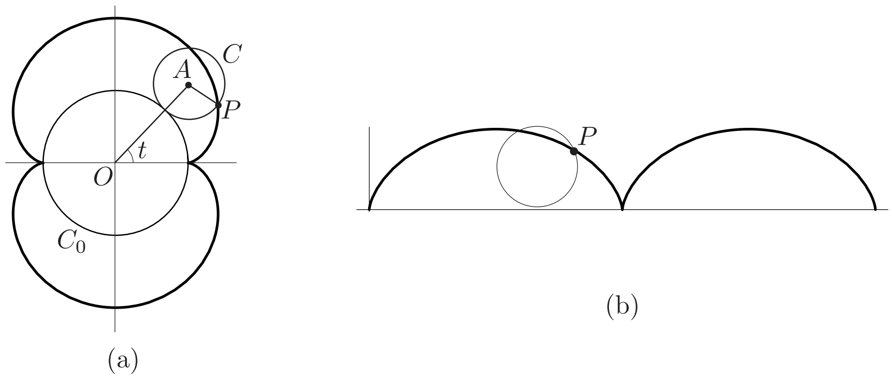


FIGURE 7. (**a**) epicycloid; (**b**) cycloid

(i) Show that σ defines a parametrization of class C^1 but not C^2.
(ii) Consider $\sigma_1 \colon [0,1] \to \mathbb{R}^2$ given by

$$\sigma_1(t) = \begin{cases} \sigma(t) & \text{for } t \in [0,1/2] \;, \\ \bigl(1+\cos(4\pi t+\pi), \sin(4\pi t+\pi)\bigr) & \text{for } t \in [1/2,1] \;; \end{cases}$$

see Fig. 6.(b). Show that σ and σ_1 are not equivalent, not even as continuous parametrizations.

1.13. The *conchoid of Nicomedes* is the plane curve described, in polar coordinates, by the equation $r = b + a/\cos\theta$, with fixed a, $b \neq 0$, and $\theta \in [-\pi,\pi]$. Draw the support of the conchoid and determine a parametrization in Cartesian coordinates.

1.14. Show, using the parameter change $v = \tan(t/2)$, that the parametrizations $\sigma_1 \colon [0,\infty) \to \mathbb{R}^3$ and $\sigma_2 \colon [0,\pi) \to \mathbb{R}^3$ of the circular helix, given by

$$\sigma_1(v) = \left(r\,\frac{1-v^2}{1+v^2}, \frac{2rv}{1+v^2}, 2a \arctan v \right) \quad \text{and} \quad \sigma(t) = (r\cos t, r\sin t, at)$$

respectively, are equivalent.

1.15. **Epicycloid.** An *epicycloid* is the plane curve described by a point P of a circle C with radius r that rolls without slipping on the outside of a circle C_0 with radius R. Assume that the center of C_0 is the origin, and that the point P starts in $(R,0)$ and moves counterclockwise. Finally, denote by t the angle between the positive x-axis and the vector OA, joining the origin and the center A of C; see Fig. 7.(a).

(i) Show that the center A of C has coordinates $\bigl((r+R)\cos t, (r+R)\sin t\bigr)$.
(ii) Having computed the coordinates of the vector AP, determine a parametrization of the epicycloid.

LENGTH AND RECTIFIABLE CURVES

1.16. Let $\sigma \colon [a,b] \to \mathbb{R}^n$ be a rectifiable curve. Show that

$$L(\sigma) \geq \|\sigma(b) - \sigma(a)\| \;,$$

and deduce that a line segment is the shortest curve between two points.

1.17. Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function given by

$$f(t) = \begin{cases} t\sin(\pi/t) & \text{if } t \neq 0 \,, \\ 0 & \text{if } t = 0 \,. \end{cases}$$

Show that the curve $\sigma\colon [0,1] \to \mathbb{R}^2$ given by $\sigma(t) = \bigl(t, f(t)\bigr)$ is an injective, non-rectifiable, continuous curve.

1.18. **Cycloid.** In the xy-plane consider a circle with radius 1 rolling without slipping on the x-axis, as in Fig. 7.(b). The path described by a point of the circle is called *cycloid*. Following the motion of such a point P, starting from the origin and up to the moment when it arrives back to the x-axis, we get a regular curve $\sigma\colon [0, 2\pi] \to \mathbb{R}^2$ with the cycloid as its support. Show that σ is defined by $\sigma(t) = (t - \sin t, 1 - \cos t)$, and determine its length.

1.19. Let $\sigma\colon [a,b] \to \mathbb{R}^3$ be the usual parametrization $\sigma(t) = \bigl(t, f(t)\bigr)$ of the graph of a function $f\colon [a,b] \to \mathbb{R}$ of class C^1. Prove that the length of σ is

$$L(\sigma) = \int_a^b \sqrt{1 + |f'(t)|^2}\,\mathrm{d}t \,.$$

1.20. Prove that if $\sigma\colon [0, +\infty) \to \mathbb{R}^2$ is the logarithmic spiral parametrized as in Problem 1.4, then the limit $\lim\limits_{t\to+\infty} \int_0^t \|\sigma'(\lambda)\|\,\mathrm{d}\lambda$ exists and is finite. In a sense, we may say that the logarithmic spiral has a finite length.

1.21. Determine a parametrization by arc length for the parabola $\sigma\colon \mathbb{R} \to \mathbb{R}^2$ given by $\sigma(t) = (t, a\,t^2)$ with a fixed $a > 0$.

REGULAR AND BIREGULAR CURVES

1.22. Prove that the support of a regular curve $\sigma\colon I \to \mathbb{R}^n$ is contained in a line if and only if the tangent versor $\vec{t}\colon I \to \mathbb{R}^n$ of σ is constant.

1.23. Let $\sigma\colon I \to \mathbb{R}^3$ be a regular curve. Show that $\sigma(t)$ and $\sigma'(t)$ are orthogonal for every value of $t \in I$ if and only if $\|\sigma\|$ is a constant non-zero function.

1.24. Determine which of the following maps $\sigma_i\colon \mathbb{R} \to \mathbb{R}^3$ are regular and/or biregular curves:

(i) $\sigma_1(t) = (\mathrm{e}^{-t}, 2t, t - 1)$;
(ii) $\sigma_2(t) = \bigl(2t, (t^2 - 1)^2, 3t^3\bigr)$;
(iii) $\sigma_3(t) = (t, 2t, t^3)$.

1.25. Let $\sigma\colon [-2\pi, 2\pi] \to \mathbb{R}^3$ be the curve given by

$$\sigma(t) = \bigl(1 + \cos t, \sin t, 2\sin(t/2)\bigr) \,.$$

Prove that it is a regular curve having as its support the intersection of the sphere of radius 2 centered at the origin with the cylinder having equation $(x-1)^2 + y^2 = 1$.

CURVATURE AND TORSION

1.26. Let $\sigma\colon I \to \mathbb{R}^3$ be a biregular curve parametrized by arc length, and $\rho\colon \mathbb{R}^3 \to \mathbb{R}^3$ a rigid motion. Prove that $\rho \circ \sigma$ is a biregular curve parametrized by arc length with the same curvature and the same torsion as σ.

1.27. Let $\sigma\colon \mathbb{R} \to \mathbb{R}^3$ be the curve given by $\sigma(t) = (1 + \cos t, 1 - \sin t, \cos 2t)$. Prove that σ is a regular curve, and compute its curvature and its torsion without reparametrizing it by arc length.

1.28. Let $f\colon U \to \mathbb{R}$ be a function of class C^∞ defined on an open subset U of the plane $\mathbb{R}^2$, and $\sigma\colon I \to U$ a regular curve such that $f \circ \sigma \equiv 0$. Prove that for every $t \in I$ the tangent vector $\sigma'(t)$ is orthogonal to the gradient of f computed in $\sigma(t)$, and determine the oriented curvature of σ as a function of the derivatives of f.

1.29. Let $\sigma\colon I \to \mathbb{R}^2$ be a regular plane curve, given in polar coordinates by the equation $r = \rho(\theta)$, that is,

$$\sigma(\theta) = \big(\rho(\theta)\cos\theta, \rho(\theta)\sin\theta\big)$$

for some function $\rho\colon I \to \mathbb{R}^+$ of class C^∞ nowhere zero. Prove that the arc length of σ is given by

$$s(\theta) = \int_{\theta_0}^{\theta} \sqrt{\rho^2 + (\rho')^2}\,\mathrm{d}\theta\ ,$$

and that the oriented curvature of σ is

$$\tilde{\kappa} = \frac{2(\rho')^2 - \rho\rho'' + \rho^2}{\big(\rho^2 + (\rho')^2\big)^{3/2}}\ .$$

1.30. Let $\sigma\colon (0, +\infty) \to \mathbb{R}^3$ be the curve given by $\sigma(t) = (t, 2t, t^4)$. Prove that σ is a regular curve whose support is contained in a plane, and compute the curvature of σ at each point.

1.31. Determine the arc length, the curvature and the torsion of the curve $\sigma\colon \mathbb{R} \to \mathbb{R}^3$ defined by $\sigma(t) = (a \cosh t, b \sinh t, a\,t)$. Prove that, if $a = b = 1$, then the curvature is equal to the torsion for every value of the parameter.

1.32. Let $\sigma\colon \mathbb{R} \to \mathbb{R}^3$ be the mapping defined by

$$\sigma(t) = (2\sqrt{2}\,t - \sin t, 2\sqrt{2}\sin t + t, 3\cos t)\ .$$

Prove that the curve defined by σ is a circular helix (up to a rigid motion of $\mathbb{R}^3$).

1.33. Consider a plane curve $\sigma\colon I \to \mathbb{R}^2$ parametrized by arc length. Prove that if the vector $O\sigma(s)$ forms a constant angle θ with the tangent versor $\vec{t}(s)$ then σ is a logarithmic spiral (see Problem 1.4).

1.34. Let $\sigma\colon [a, b] \to \mathbb{R}^3$ be a curve of class at least C^2.

(i) Show that if the support of σ is contained in a plane through the origin then the vectors σ, σ', and σ'' are linearly dependent.

(ii) Show that if the vectors σ, σ', and σ'' are linearly dependent but the vectors σ, σ' are linearly independent then the support of σ is contained in a plane through the origin.

(iii) Find an example in which σ and σ' are linearly dependent but the support of σ is not contained in a plane through the origin.

FRENET FRAME AND OSCULATING PLANE

1.35. Let $\sigma\colon \mathbb{R} \to \mathbb{R}^3$ be the curve $\sigma(t) = (\mathrm{e}^t, \mathrm{e}^{2t}, \mathrm{e}^{3t})$. Find the values of $t \in \mathbb{R}$ for which the tangent vector $\sigma'(t)$ is orthogonal to the vector $\vec{v} = (1, 2, 3)$.

1.36. Let $\sigma\colon \mathbb{R} \to \mathbb{R}^3$ be the curve $\sigma(t) = \big((4/5)\cos t, 1 - \sin t, -(3/5)\cos t\big)$. Determine the Frenet frame of σ.

1.37. Let $\sigma\colon \mathbb{R} \to \mathbb{R}^2$ be the plane curve parametrized by $\sigma(t) = (t, \frac{1}{3}t^3)$. Determine the curvature of σ and study the values of the parameter for which it is zero. Determine the normal versor and the oriented normal versor, wherever they are defined.

1.38. Let $\sigma\colon I \to \mathbb{R}^3$ be a curve of class at least C^2. Show that the vector σ'' is parallel to the osculating plane and that its components along the vectors $\mathbf{t}$ and $\mathbf{n}$ are $\|\sigma'\|'$ and $\kappa\|\sigma'\|^2$, respectively.
If σ is biregular, show that the osculating plane in $\sigma(t)$ is the plane through $\sigma(t)$ and parallel to σ' and σ''. So, the equation of the osculating plane is given by $\langle \sigma'(t_0) \wedge \sigma''(t_0), \vec{x} - \sigma(t_0)\rangle = 0$.

1.39. Let σ be a curve such that all its affine tangent lines pass through a given point. Show that if σ is regular then its support is contained in a straight line, and find a counterexample with a non-regular σ (considering the tangent lines only in the points of the curve where they are defined).

1.40. Let $\sigma\colon \mathbb{R} \to \mathbb{R}^3$ be a Jordan arc of class C^2, not necessarily regular, such that all its affine tangent lines pass through a given point P. Show that the support of σ is contained in a straight line, and find a counterexample with a Jordan arc not of class C^2 (considering the tangent lines only in the points of the curve where they are defined).

1.41. Let $\sigma\colon [a, b] \to \mathbb{R}^3$ be a biregular curve such that all its affine normal line pass through a given point. Show that the support of σ is contained in a circle.

1.42. Consider the curve $\sigma\colon \mathbb{R} \to \mathbb{R}^3$ given by $\sigma(t) = (t, (1/2)t^2, (1/3)t^3)$. Show that the osculating planes of σ in three different points $\sigma(t_1)$, $\sigma(t_2)$, $\sigma(t_3)$ intersect in a point belonging to the plane generated by the points $\sigma(t_1)$, $\sigma(t_2)$ and $\sigma(t_3)$.

1.43. Show that the binormal vector to a circular helix parametrized as in Example 1.8 forms a constant angle with the axis of the cylinder containing the helix.

1.44. Show that the curve $\sigma\colon \mathbb{R} \to \mathbb{R}^3$ parametrized by

$$\sigma(t) = (t + \sqrt{3}\,\sin t, 2\cos t, \sqrt{3}\,t - \sin t),$$

is a circular helix (up to a rigid motion of $\mathbb{R}^3$).

1.45. Let $\gamma\colon \mathbb{R} \to \mathbb{R}^3$ be the curve parametrized by $t \mapsto \gamma(t) = (t, At^2, Bt^n)$, where $A, B > 0$ are real numbers and $n \geq 1$ is an integer.

(i) Determine a map $\eta\colon \mathbb{R} \to \mathbb{R}^3$ such that $\eta(t)$ is the intersection point between the affine tangent line to γ in $\gamma(t)$ and the plane $z = 0$.
(ii) Find conditions on A, B, n in order for η to be a regular curve.

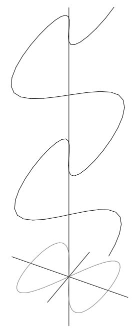


FIGURE 8. Generalized helix $\sigma(t) = (\cos t, \sin(2t), t)$

FRENET-SERRET FORMULAS

1.46. Determine the curvature and the torsion at each point of the curve $\sigma\colon \mathbb{R} \to \mathbb{R}^3$ parametrized by $\sigma(t) = (3t - t^3, 3t^2, 3t + t^3)$.

1.47. Determine the curvature, the torsion, and the Frenet frame of the curve $\sigma\colon \mathbb{R} \to \mathbb{R}^3$ given by $\sigma(t) = \bigl(a(t - \sin\, t), a(1 - \cos\, t), bt\bigr)$.

1.48. Given a function $\tilde{\kappa}\colon I \to \mathbb{R}$ of class C^k, prove that there exists a unique (up to plane rigid motions) regular curve $\sigma\colon I \to \mathbb{R}^2$ of class C^{k+2} parametrized by arc length having oriented curvature $\tilde{\kappa}$.

1.49. **Generalized helices.** Let $\sigma\colon I \to \mathbb{R}^3$ be a biregular curve parametrized by arc length. Prove that the following assertions are equivalent:

(i) there exist two constants a, $b \in \mathbb{R}$, not both zero, such that $a\kappa + b\tau \equiv 0$;
(ii) there exists a nonzero versor $\vec{v}_0$ such that $\langle \vec{t}, \vec{v}_0 \rangle$ is constant;
(iii) there exists a plane π such that $\vec{n}(s) \in \pi$ for all $s \in I$;
(iv) there exists a nonzero versor $\vec{v}_0$ such that $\langle \vec{b}, \vec{v}_0 \rangle$ is constant;
(v) there exist $\theta \in (0, \pi) \setminus \{\pi/2\}$ and a biregular plane curve parametrized by arc length $\eta\colon J_\theta \to \mathbb{R}^3$, where $J_\theta = (\sin\theta)I$, such that
$$\sigma(s) = \eta(s \sin\theta) + s\cos\theta \vec{b}_\eta$$
for all $s \in I$, where $\vec{b}_\eta$ is the (constant!) binormal versor of η;
(vi) the curve σ has a parametrization of the form $\sigma(s) = \eta(s) + (s - s_0)\vec{v}$, where η is a plane curve parametrized by arc length, and v is a vector orthogonal to the plane containing the support of η.

A curve σ satisfying any of these equivalent conditions is called *(generalized) helix*; see Fig. 8. Finally, write the curvature, the torsion and the Frenet frame of σ as functions of the curvature and of the Frenet frame of η.
[*Hint*: if $\tau/\kappa = c$ is constant, set $c = \frac{\cos\alpha}{\sin\alpha}$ and $\vec{v}_0(s) = \cos\alpha\vec{t}(s) + \sin\alpha\vec{n}(s)$ and prove that $\vec{v}_0$ is constant.]

1.50. Check for which values of the constants a, $b \in \mathbb{R}$ the curve $\sigma\colon \mathbb{R} \to \mathbb{R}^3$ parametrized by $\sigma(t) = (at, bt^2, t^3)$ is a generalized helix.

1.51. Let $\sigma\colon I \to R^3$ be a biregular curve parametrized by arc length. Prove that $\kappa \equiv \pm\tau$ if and only if there exists a nonzero versor $\vec{v}$ such that $\langle \vec{t}, \vec{v} \rangle \equiv \langle \vec{b}, \vec{v} \rangle$. Prove furthermore that, in this case, $\langle \vec{t}, \vec{v} \rangle$ is constant.

1.52. Let $\sigma\colon \mathbb{R} \to \mathbb{R}^3$ be the curve $\sigma(t) = \big(1+\cos t, \sin t, 2\sin(t/2)\big)$. Prove that σ is not a plane curve and that its support is contained in the sphere with radius 2 and center at the origin.

1.53. Let $\sigma\colon [a,b] \to \mathbb{R}^3$ be a biregular curve parametrized by arc length. Show that

$$\frac{\mathrm{d}^3\sigma}{\mathrm{d}s^3} = -\kappa^2\vec{t} + \dot{\kappa}\vec{n} + \kappa\tau\vec{b}\,.$$

1.54. Let $\sigma\colon I \to \mathbb{R}^3$ be a biregular curve parametrized by arc length having tangent versor $\mathbf{t}$, and for every $\varepsilon \neq 0$ set $\sigma_\varepsilon = \sigma + \varepsilon\mathbf{t}$. Prove that σ_ε always is a regular curve, and that the normal versor of σ_ε is always orthogonal to the normal versor of σ if the curvature κ of σ is of the form

$$\kappa(s) = c(\mathrm{e}^{2s/\varepsilon} - c^2\varepsilon^2)^{-1/2}$$

for some constant $0 < c < 1/\varepsilon$.

1.55. Let $\sigma\colon I \to \mathbb{R}^3$ be a biregular curve parametrized by arc length, having constant curvature $\kappa_0 > 0$. Prove that the support of σ is contained in a sphere with radius $R > 0$ if and only if $\kappa_0 > 1/R$ and $\tau \equiv 0$.

1.56. Let $\sigma\colon I \to \mathbb{R}^3$ and $\alpha\colon I \to \mathbb{R}^3$ be two different biregular curves parametrized by arc length, having equal affine binormal lines in the points corresponding to the same parameter. Prove that the curves σ and α are plane.

1.57. Determine curvature and torsion of the biregular curve $\sigma\colon (0, +\infty) \to \mathbb{R}^3$ defined by $\sigma(t) = (2t, \frac{1+2t}{2t}, \frac{1-4t^2}{2t})$.

1.58. Determine curvature and torsion of the regular curve $\sigma\colon \mathbb{R} \to \mathbb{R}^3$ defined by $\sigma(t) = (\cos\, t, \sin\, t, 2\sin\frac{t}{2})$.

1.59. **The Darboux vector** Let $\sigma\colon I \to \mathbb{R}$ be a biregular curve parametrized by arc length. The vector $\vec{d}(s) = \tau(s)\vec{t}(s) + \kappa(s)\vec{b}$ is called the Darboux vector at $\sigma(s)$. Show that $\vec{d}$ satisfies $\dot{\vec{t}} = \vec{d} \wedge \vec{t}$, $\dot{\vec{n}} = \vec{d} \wedge \vec{n}$, $\dot{\vec{b}} = \vec{d} \wedge \vec{b}$.

1.60. Let $\sigma\colon I \to \mathbb{R}^3$ be a regular curve parametrized by arc length, having nowhere zero curvature κ and torsion τ; let $s_0 \in I$ be fixed. For every $\varepsilon \in \mathbb{R}$ let $\gamma^\varepsilon\colon I \to \mathbb{R}^3$ be the curve given by $\gamma^\varepsilon(t) = \sigma(t) + \varepsilon\,\mathbf{b}(t)$, where $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ is the Frenet frame of σ. Denote by $\mathbf{t}^\varepsilon$, $\mathbf{n}^\varepsilon$ e $\mathbf{b}^\varepsilon$ the tangent, normal and binormal versors of γ^ε, and by κ^ε, τ^ε the curvature and the torsion of γ^ε. Prove that:

(i) γ^ε is always a biregular curve;
(ii) σ is a plane curve if and only if $\mathbf{b}^\varepsilon \equiv \pm\mathbf{b}$;
(iii) σ is a plane curve if and only if $\mathbf{t}^\varepsilon \equiv \mathbf{t}$ if and only if $\mathbf{n}^\varepsilon \equiv \mathbf{n}$.

1.61. **Bertrand curves.** Two biregular curves σ, $\sigma_1\colon I \to \mathbb{R}^3$, having normal versors $\vec{n}$ and $\vec{n}_1$ respectively, are called *Bertrand curves* if they have the same affine normal line at every point. In particular, possibly modifying the orientation, it is always possible to assume that $\vec{n} \equiv \vec{n}_1$, that is, that the curves have the same normal versors.

(i) Show that if σ and σ_1 are the parametrizations by arc length of two Bertrand curves then there exists a real-valued function $\alpha\colon I \to \mathbb{R}$ such that $\sigma_1 \equiv \sigma + \alpha\vec{n}$.
(ii) Show that the distance between points corresponding to the same parameter of two Bertrand curves is constant, that is, the function α in (i) is constant.
(iii) Show that the angle between the tangent lines in two corresponding points of two Bertrand curves is constant.
(iv) Show that if σ and σ_1 are biregular Bertrand curves with never vanishing torsion then there exist constants $a \in \mathbb{R}$, $b \in \mathbb{R}^*$ such that $\kappa + a\tau \equiv b$, where κ and τ are the curvature and the torsion of σ.
(v) Prove the converse of the previous statement: if σ is a curve having curvature κ and torsion τ, both nowhere zero, such that $\kappa + a\tau \equiv b$ for suitable constants $a \in \mathbb{R}$ and $b \in \mathbb{R}^*$, then there exists another curve σ_1 such that σ and σ_1 are Bertrand curves.
(vi) Show that if σ is a biregular curve with nowhere zero torsion τthen σ is a circular helix if and only if there exist at least two curves σ_1 and σ_2 such that σ and σ_i are Bertrand curves, for $i = 1, 2$. Show that, in this case, there exist infinitely many curves $\tilde{\sigma}$ such that σ and $\tilde{\sigma}$ are Bertrand curves.
(vii) Prove that if two Bertrand curves σ and σ_1 have the same binormal versor then there exists a constant $a > 0$ such that $a(\kappa^2 + \tau^2) = \kappa$.

THE FUNDAMENTAL THEOREM OF THE LOCAL THEORY OF CURVES

1.62. Find a plane curve, parametrized by arc length $s > 0$, with curvature $\kappa(s) = 1/s$. Do the same with the oriented curvature $\tilde{\kappa}(s) = 1/s$ instead of the usual curvature.

1.63. Compute the curvature of the catenary $\sigma\colon \mathbb{R} \to \mathbb{R}^2$ parametrized by $\sigma(t) = \big(a\cosh(t/a), t\big)$, where a is a real constant.

1.64. Given $a > 0$, determine a curve having curvature and torsion given respectively by $\kappa(s) = \sqrt{1/2as}$ and $\tau(s) = 0$ for $s > 0$.

1.65. Compute curvature and torsion of the curve $\sigma\colon \mathbb{R} \to \mathbb{R}^3$ parametrized by $\sigma(t) = \mathrm{e}^t(\cos t, \sin t, 3)$.

1.66. **Regular curves with nonzero constant torsion.** We know from Example 1.37 and Problem 1.7 that the circular helix is characterized by having both curvature and torsion constant (and both nonzero, except for the degenerate case with support contained in a plane circle). The aim of this exercise is to study biregular curves having nonzero constant torsion in $\mathbb{R}^3$.

(i) Show that if σ is a biregular curve having constant torsion $\tau \equiv a$ then

$$\sigma(t) = a^{-1}\int_{t_0}^{t} \vec{b}(s) \wedge \dot{\vec{b}}\,\mathrm{d}s.$$

Moreover, prove that the vectors $\vec{b}$, $\dot{\vec{b}}$, and $\ddot{\vec{b}}$ are linearly independent for all values of the parameter.

(ii) Consider, on the other hand, a function $f: I \to \mathbb{R}^3$ of class at least C^2, having values in the unitary sphere (that is, $\|f\| \equiv 1$), and such that the vectors $f(\lambda)$, $f'(\lambda)$, and $f''(\lambda)$ are linearly independent for all $\lambda \in I$. Consider the curve $\sigma: I \to \mathbb{R}^3$ given by

$$\sigma(t) = a \int_{t_0}^{t} f(\lambda) \wedge f'(\lambda)\, \mathrm{d}\lambda \,,$$

for some nonzero constant a and a fixed value $t_0 \in I$. Show that σ is regular and that it has constant torsion $\tau \equiv a^{-1}$.

EVOLVENT, EVOLUTE, INVOLUTE

Definition 1.E.1. Let $\sigma: I \to \mathbb{R}^2$ be a biregular plane curve, parametrized by arc length. The plane curve $\beta: I \to \mathbb{R}^2$ parametrized by

$$\beta(s) = \sigma(s) + \frac{1}{\kappa(s)}\vec{n}(s)$$

is the *evolute* of σ. Now let $\sigma: I \to \mathbb{R}^3$ be a regular plane curve, parametrized by arc length. A curve $\tilde{\sigma}: I \to \mathbb{R}^3$ (not necessarily parametrized by arc length) is an *involute* (or *evolvent*) of σ if $\dot{\sigma}(s)$ is parallel to $\tilde{\sigma}(s) - \sigma(s)$ and orthogonal to $\tilde{\sigma}'(s)$ for all $s \in I$.

1.67. Show that the affine normal line to a biregular plane curve $\sigma: I \to \mathbb{R}^2$ at the point $\sigma(s)$ is equal to the affine tangent line of its evolute β at the point $\beta(s)$. In particular, the affine tangent line to the evolute at $\beta(s)$ is orthogonal to the affine tangent line to the original curve at $\sigma(s)$.

1.68. Show that the evolute of the catenary $\sigma(t) = (t, \cosh t)$ is parametrized by $\beta(t) = (t - \sinh t \cosh t, 2 \cosh t)$.

1.69. Find the evolute β of the curve $\sigma(t) = (\cos^3 t, \sin^3 t)$.

1.70. Given $a > 0$ and $b < 0$, find the evolute of the logarithmic spiral parametrized by $\sigma(t) = (a\mathrm{e}^{bt} \cos t, a\mathrm{e}^{bt} \sin t)$.

1.71. Prove that any biregular plane curve σ is an involute of its evolute. Moreover, prove that any couple of involutes of σ are Bertrand curves (see Exercise 1.61).

1.72. Let $\sigma: I \to \mathbb{R}^3$ be a regular curve parametrized by arc length. For $c \in \mathbb{R}$, define $\sigma_c: I \to \mathbb{R}^3$ by $\sigma_c(s) = \sigma(s) + (c - s)\dot{\sigma}(s)$ for all $s \in I$.

(i) Prove that a curve $\tilde{\sigma}: I \to \mathbb{R}^3$ is an involute of σ if and only if there exists $c \in \mathbb{R}$ such that $\tilde{\sigma}(s) = \sigma_c(s)$ for all $s \in I$.

(ii) Assume that σ is biregular and prove that the involute σ_c is biregular for all values of $s \neq c$.Prove, moreover, that the tangent versor of σ_c in $\sigma_c(s)$ is parallel to the normal versor of σ in $\sigma(s)$ and, in general, σ_c is not parametrized by arc length.

(iii) Assume that σ is biregular and prove that the curvature of an involute σ_c is given by $\frac{\sqrt{\kappa^2+\tau^2}}{|(c-s)\kappa|}$, in terms of the curvature κ and the torsion τ of σ.

(iv) Let σ be the circular helix, as in the Example 1.22. Prove that each involute σ_c of σ is a plane curve.

(v) Determine the involute of the catenary $\sigma(t) = (t, \cosh t)$ and of the circle $\sigma_1(t) = (r\cos t, r\sin t)$.

1.73. Let $\sigma\colon I \to \mathbb{R}^3$ be a biregular curve parametrized by arc length and let $\hat{\sigma} = \sigma - \kappa^{-1}\mathbf{n}$, where $\mathbf{n}$ is the normal versor of σ. Prove that if σ is an involute of $\hat{\sigma}$, then σ is a plane curve.

SPHERICAL INDICATRICES

1.74. Let $\sigma\colon I \to \mathbb{R}^3$ be a regular curve of curvature κ. Prove that $\vec{t}\colon I \to \mathbb{R}^3$ is regular if and only if σ is biregular, and that the arc length of σ is an arc length for $\vec{t}$ as well if and only if $\kappa \equiv 1$.

1.75. Let $\sigma\colon I \to \mathbb{R}^3$ be a biregular curve of curvature κ and torsion τ. Denote by s the arc length of σ, and by s_1 the arc length of the normal curve $\vec{n}\colon I \to \mathbb{R}^3$. Prove that $\mathrm{d}s_1/\mathrm{d}s = \sqrt{\kappa^2 + \tau^2}$.

1.76. Let $\sigma\colon \mathbb{R} \to \mathbb{R}^3$ be the circular helix given by $\sigma(t) = (r\cos t, r\sin t, at)$. Prove that its tangent versor $\vec{t}\colon \mathbb{R} \to \mathbb{R}^3$ is a circle with center on the z-axis, and compute its radius of curvature.

1.77. Let $\sigma\colon I \to \mathbb{R}^3$ be a biregular curve. Prove that if the support of its tangent versor $\vec{t}\colon I \to \mathbb{R}^3$ is a circle then σ is (up to a rigid motion of $\mathbb{R}^3$) a circular helix.

1.78. Let $\sigma\colon I \to \mathbb{R}^3$ be a biregular curve. Show that the tangent vector at a point of the tangent curve $\vec{t}\colon I \to \mathbb{R}^3$ to σ is parallel to the affine normal line at the corresponding point of σ.

1.79. Let $\sigma\colon I \to \mathbb{R}^3$ be a biregular curve having curvature κ and torsion τ. Prove that the curvature κ_1 of the tangent curve $\vec{t}\colon I \to \mathbb{R}^3$ of σ is given by

$$\kappa_1 = \sqrt{1 + \frac{\tau^2}{\kappa^2}}\;.$$

CHAPTER 2

Local theory of surfaces

The rest of this notes is devoted to the study of surfaces in space. As we did for the curves, we shall begin by trying to understand how best define a surface; but, unlike what happened for curves, for surfaces it will turn out to be more useful to work with subsets of $\mathbb{R}^3$ that locally look like an open subset of the plane, instead of working with maps from an open subset of $\mathbb{R}^2$ to $\mathbb{R}^3$ having an injective differential.

When we say that a surface locally looks like an open subset of the plane, we are not (only) talking about its topological structure, but (above all) about its differential structure. In other words, it must be possible to differentiate functions on a surface exactly as we do on open subsets of the plane: computing a partial derivative is a purely local operation, so it is has to be possible to perform similar operation in every object that locally looks (from a differential viewpoint) like an open subset of the plane.

To carry out this program, after presenting in Section 2.1 the official definition of what a surface is, in Section 2.2 we shall define precisely the family of functions that are smooth on a surface, that is, the functions we shall be able to differentiate; in Section 2.4 we shall show how to differentiate them, and we shall define the notion of differential of a smooth map between surfaces. Furthermore, in Sections 2.3 and 2.4, we shall introduce the tangent vectors to a surface and we shall explain why they are an embodiment of partial derivatives.

2.1. How to define a surface

As we did for curves, we begin by discussing the question of the correct definition of what a surface is. Our experience from the one-dimensional case suggests two possible approaches: we might define surfaces as subsets of the space with some properties, or we can define them as maps from an open subset of the plane to the space, satisfying suitable regularity properties.

Working with curves we preferred this second approach, since the existence of parameterizations by arc length allowed us to directly relate the geometric properties of the support of the curve with the differential properties of the curve itself.

As we shall see, in the case of surfaces the situation is significantly more complex. The approach that emphasizes maps will be useful to study local questions; but from a global viewpoint it will be more effective to privilege the other approach.

But let us not disclose too much too soon. Let us instead start by introducing the obvious generalization of the notion of a regular curve:

Definition 2.1. An *immersed* (or *parametrized*) *surface* in space is a map $\varphi\colon U \to \mathbb{R}^3$ of class C^∞, where $U \subseteq \mathbb{R}^2$ is an open set, such that the differential $d\varphi_x\colon \mathbb{R}^2 \to \mathbb{R}^3$ is injective (that is, has rank 2) in every point $x \in U$. The image $\varphi(U)$ of φ is the *support* of the immersed surface.

REMARK 2.1. For reasons that will become clear later (see Remark 2.22), when studying surfaces we shall only use C^∞ maps, and we shall not discuss lower regularity issues.

REMARK 2.2. The differential $\mathrm{d}\varphi_x$ of $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ in $x \in U$ is represented by the Jacobian matrix

$$\operatorname{Jac}\varphi\,(x) = \begin{vmatrix} \frac{\partial\varphi_1}{\partial x_1}(x) & \frac{\partial\varphi_1}{\partial x_2}(x) \\ \frac{\partial\varphi_2}{\partial x_1}(x) & \frac{\partial\varphi_2}{\partial x_2}(x) \\ \frac{\partial\varphi_3}{\partial x_1}(x) & \frac{\partial\varphi_3}{\partial x_2}(x) \end{vmatrix} \in M_{3,2}(\mathbb{R}) \; .$$

As for curves, in this definition the emphasis is on the map rather than on its image. Moreover, we are not asking for the immersed surfaces to be a homeomorphism with their images or to be injective (see Example 2.1); both these properties are nevertheless locally true. To prove this, we need a lemma, somewhat technical but extremely useful. In turn, the proof of the lemma will depend on a classical Differential Calculus theorem (see [**3**, p. 140]):

THEOREM 2.1 (Inverse function theorem). *Let $F\colon \Omega \to \mathbb{R}^n$ be a map of class C^k, with $k \in \mathbb{N}^* \cup \{\infty\}$, where Ω is an open subset of $\mathbb{R}^n$. Let $p_0 \in \Omega$ be such that $\det \operatorname{Jac} F(p_0) \neq 0$, where $\operatorname{Jac} F$ is the Jacobian matrix of F. Then there exist a neighborhood $U \subset \Omega$ of p_0 and a neighborhood $V \subset \mathbb{R}^n$ of $F(p_0)$ such that $F|_U\colon U \to V$ is a diffeomorphism of class C^k.*

LEMMA 2.1. *Let $\varphi\colon U \to \mathbb{R}^3$ be an immersed surface, where $U \subseteq \mathbb{R}^2$ is open. Then for all $x_0 \in U$ there exist an open set $\Omega \subseteq \mathbb{R}^3$ of $(x_0, 0) \in U \times \mathbb{R}$, an open neighborhood $W \subseteq \mathbb{R}^3$ of $\varphi(x_0)$, and a diffeomorphism $G\colon \Omega \to W$ such that $G(x, 0) = \varphi(x)$ for all $(x, 0) \in \Omega \cap (U \times \{0\})$.*

PROOF. By definition of immersed surface, the differential in x_0 of the map $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ has rank 2; so the Jacobian matrix of φ computed in x_0 has a 2×2 minor with nonzero determinant. Up to reordering the coordinates, we may assume that this minor is obtained by discarding the third row, that is, we can assume that

$$\det \left(\frac{\partial \varphi_i}{\partial x_j}(x_0) \right)_{i,j=1,2} \neq 0 \; .$$

Let then $G\colon U \times \mathbb{R} \to \mathbb{R}^3$ be given by

$$G(x_1, x_2, t) = \varphi(x_1, x_2) + (0, 0, t) \; ;$$

note that if to find the minor with nonzero determinant,we had discarded the j-th row, then G would be defined by adding $t\vec{e}_j$ to φ, where $\vec{e}_j$ is the j-th vector of the canonical basis of $\mathbb{R}^3$.

Clearly, $G(x, 0) = \varphi(x)$ for all $x \in U$, and

$$\det \operatorname{Jac} G\,(x_0, O) = \det \left(\frac{\partial \varphi_i}{\partial x_j}(x_0) \right)_{i,j=1,2} \neq 0 \; ;$$

So the inverse function theorem (Theorem 2.1) gives us a neighborhood $\Omega \subseteq U \times \mathbb{R}$ of (x_0, O) and a neighborhood $W \subseteq \mathbb{R}^3$ of $\varphi(x_0)$ such that $G|_\Omega$ is a diffeomorphism between Ω and W, as required. □

In particular, we have

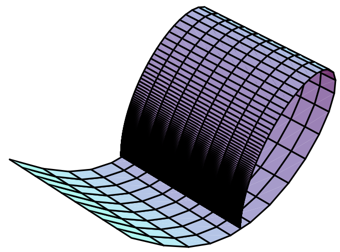


FIGURE 1

COROLLARY 2.1. *Let $\varphi\colon U \to \mathbb{R}^3$ be an immersed surface. Then every $x_0 \in U$ has a neighborhood $U_1 \subseteq U$ such that $\varphi|_{U_1}\colon U_1 \to \mathbb{R}^3$ is a homeomorphism with its image.*

PROOF. Let $G\colon \Omega \to W$ be the diffeomorphism provided by the previous lemma, $\pi\colon \mathbb{R}^3 \to \mathbb{R}^2$ the projection on the first two coordinates, and set

$$U_1 = \pi\bigl(\Omega \cap (U \times \{0\})\bigr) .$$

Then $\varphi(x) = G(x,0)$ for all $x \in U_1$, and so $\varphi|_{U_1}$ is a homeomorphism with its image, as required. □

However, it is important to remember that, in general, immersed surfaces are *not* homeomorphisms with their images:

EXAMPLE 2.1. For $U = (-1,+\infty) \times \mathbb{R}$, let $\varphi\colon U \to \mathbb{R}^3$ be given by

$$\varphi(x,y) = \left(\frac{3x}{1+x^3}, \frac{3x^2}{1+x^3}, y\right) ;$$

see Fig. 1. It is easy to verify that φ is an injective immersed surface, but is not a homeomorphism with its image, as $\varphi\bigl((-1,1)\times(-1,1)\bigr)$ is not open in $\varphi(U)$.

A careful consideration of the material seen in the previous chapters will show that the approach based on defining a curve as an equivalence class of maps was effective because of the existence of a canonical representative element defined from such a fundamental geometric concept as length. The drawback (or the advantage, depending on which half of the glass you prefer) of the theory of surfaces with respect to the theory of curves is that for surfaces this cannot be done, and cannot be done because of intrinsic, unavoidable reasons.

Of course, we may define two immersed surfaces $\varphi\colon U \to \mathbb{R}^3$ and $\psi\colon V \to \mathbb{R}^3$ to be *equivalent* if there exists a diffeomorphism $h\colon U \to V$ such that $\varphi = \psi \circ h$. However, the problem we face with such a definition is that the procedure we followed in the case of curves to choose in each equivalence class an (essentially) unique representative element does not work anymore.

In the case of curves, we have chosen a canonical representative element, the parametrization by arc length, by using the geometric notion of length. Two equivalent parametrizations by arc length have to differ by a diffeomorphism h that preserves lengths; and this implies (see the proof of Proposition 1.4) that $|h'| \equiv 1$, so h is an affine isometry and the parametrization by arc length is unique up to parameter translations (and orientation changes).

In the case of surfaces, it is natural to try using area instead of length. Two equivalent "parametrizations by area" should differ by a diffeomorphism h of open sets in the plane that preserves areas. But Calculus experts teach us that a diffeomorphism h preserves areas if and only if $|\det \mathrm{Jac}(h)| \equiv 1$, which is a far weaker condition than $|h'| \equiv 1$. For instance, *all* diffeomorphisms of the form $h(x, y) = \big(x + f(y), y\big)$, where f is *any* smooth function of one variable, preserve areas; so using this method there is no hope of identifying an essentially unique representative element.

But the obstruction is even more fundamental than this. Arc length parametrization works because it is a (local) isometry between an interval and the curve; on the other hand, we shall see towards the end of Chapter 3 (with Gauss' theorema egregium Theorem 3.5) that, except for very particular cases, *isometries between an open set in the plane and a surface do not exist.* A notion equivalent to parametrization by arc length to study the metric structure of surface cannot possibly exist. Moreover, even the topological structure of surface is far more complex than that of open subsets of the plane (see Remark 2.6); to try and study it by using a single map would be hopeless.

Historically, the most successful — for its effectiveness both in dealing with local questions and in studying global problems — definition of a surface tries, in a sense, to take the best from both worlds. It emphasizes the support, that is, the subset of $\mathbb{R}^3$ considered as such; but the idea that a surface has to be a set locally built like an open subset of the plane is made concrete and formal by using immersed surfaces (which work well locally, as we have seen).

Enough chatting: it is now time to give the official definition of surface in space.

DEFINITION 2.2. A connected subset $S \subset \mathbb{R}^3$ is a *(regular* or *embedded) surface* in space if for all $p \in S$ there exists a map $\varphi\colon U \to \mathbb{R}^3$ of class C^∞, where $U \subseteq \mathbb{R}^2$ is an open subset, such that:

(a) $\varphi(U) \subseteq S$ is an open neighborhood of p in S (or, equivalently, there exists an open neighborhood $W \subseteq \mathbb{R}^3$ of p in $\mathbb{R}^3$ such that $\varphi(U) = W \cap S$);
(b) φ is a homeomorphism with its image;
(c) the differential $d\varphi_x\colon \mathbb{R}^2 \to \mathbb{R}^3$ is injective (that is, it has maximum rank, i.e., 2) for all $x \in U$.

Any map φ satisfying (a)–(c) is a *local* (or *regular*) *parametrization* in p; if $O \in U$ and $\varphi(O) = p$ we say that the local parametrization is *centered* in p. The inverse map $\varphi^{-1}\colon \varphi(U) \to U$ is called *local chart* in p; the neighborhood $\varphi(U)$ of p in S is called a *coordinate neighborhood*, the coordinates $\big(x_1(p), x_2(p)\big) = \varphi^{-1}(p)$ are called *local coordinates* of p; and, for $j = 1, 2$, the curve $t \mapsto \varphi(x_o + t\vec{e}_j)$ is the j-th *coordinate curve* (or *line*) through $\varphi(x_o)$.

DEFINITION 2.3. An *atlas* for a regular surface $S \subset \mathbb{R}^3$ is a family $\mathcal{A} = \{\varphi_\alpha\}$ of local parametrizations $\varphi_\alpha\colon U_\alpha \to S$ such that $S = \bigcup_\alpha \varphi_\alpha(U_\alpha)$.

REMARK 2.3. Clearly, a local parametrization $\varphi\colon U \to \mathbb{R}^3$ of a surface S carries the topology of the open subset U of the plane to the topology of the open set $\varphi(U)$ of S, since φ is a homeomorphism between U and $\varphi(U)$. But to work with surfaces it is important to keep in mind that φ carries another fundamental thing from U to S: a coordinate system. As shown in Fig. 2, the local parametrization φ assigns to

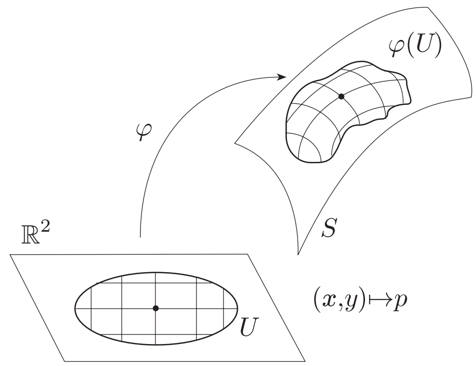


FIGURE 2. A local parametrization

each point $p \in \varphi(U)$ a pair of real numbers $(x, y) = \varphi^{-1}(p) \in U$, which will play the role of coordinates of p in S, in analogy to the role played by the usual Cartesian coordinates for points in the plane. In a sense, choosing a local parametrization of a surface amounts to constructing a geographical map of a part of the surface; and this is the reason (historically too) of the use of geographical terminology in this context. *Warning:* Different local parametrizations provide different local coordinates (charts)! In the next section we shall describe the connection between coordinates induced by different parametrizations (Theorem 2.2).

REMARK 2.4. If $\varphi\colon U \to S$ is a local parametrization of a surface $S \subset \mathbb{R}^3$, and $\chi\colon U_1 \to U$ is a diffeomorphism, where U_1 is another open subset of $\mathbb{R}^2$, then $\tilde{\varphi} = \varphi \circ \chi$ is another local parametrization of S (why?). In particular, if $p = \varphi(x_0) \in S$ and χ is the translation $\chi(x) = x + x_0$ then $\tilde{\varphi} = \varphi \circ \chi$ is a local parametrization of S centered at p.

REMARK 2.5. If $\varphi\colon U \to S$ is a local parametrization of a surface $S \subset \mathbb{R}^3$, and $V \subset U$ is an open subset of $\mathbb{R}^2$ then $\varphi|_V$ also is a local parametrization of S (why?). In particular, we may find local parametrizations with arbitrarily small domain.

As we shall see, the philosophy beneath the theory of surfaces is to use local parametrizations to transfer notions, properties, and proofs from open subsets of the plane to open sets of the surfaces, and vice versa. But let us see for now some examples of surface.

EXAMPLE 2.2. The plane $S \subset \mathbb{R}^3$ through $p_0 \in \mathbb{R}^3$ and parallel to the linearly independent vectors $\vec{v}_1$, $\vec{v}_2 \in \mathbb{R}^3$ is a regular surface, with an atlas consisting of a single local parametrization $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^3$ given by $\varphi(x) = p_0 + x_1\vec{v}_1 + x_2\vec{v}_2$.

EXAMPLE 2.3. Let $U \subseteq \mathbb{R}^2$ be an open set, and $f \in C^\infty(U)$ an arbitrary function. Then the *graph* $\Gamma_f = \big\{\big(x, f(x)\big) \in \mathbb{R}^3 \bigm| x \in U\big\}$ of f is a regular surface, with an atlas consisting of a single local parametrization $\varphi\colon U \to \mathbb{R}^3$ given by $\varphi(x) = \big(x, f(x)\big)$. Indeed, condition (a) of the definition of a surface is clearly satisfied. The restriction to Γ_f of the projection on the first two coordinates is the

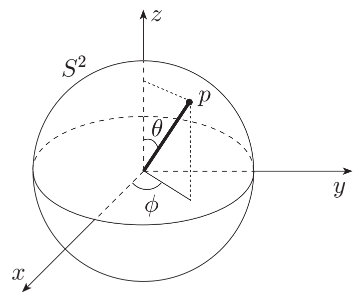


FIGURE 3. Spherical coordinates

(continuous) inverse of φ, so condition (b) is satisfied as well. Finally,

$$\operatorname{Jac}\varphi(x) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial f}{\partial x_1}(x) & \frac{\partial f}{\partial x_2}(x) \end{vmatrix}$$

has rank 2 everywhere, and we are done.

EXAMPLE 2.4. The support S of an immersed surface φ that is a homeomorphism with its image is a regular surface with atlas $\mathcal{A} = \{\varphi\}$. In this case we shall say that φ is a *global parametrization* of S.

EXAMPLE 2.5. We want to show that the *sphere*

$$S^2 = \{p \in \mathbb{R}^3 \mid \|p\| = 1\}$$

with center in the origin and radius 1 is a regular surface by finding an atlas. Let $U = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ be the open unit disc in the plane, and define $\varphi_1, \ldots, \varphi_6 : U \to \mathbb{R}^3$ by setting

$$\varphi_1(x,y) = \left(x, y, \sqrt{1-x^2-y^2}\right), \qquad \varphi_2(x,y) = \left(x, y, -\sqrt{1-x^2-y^2}\right),$$
$$\varphi_3(x,y) = \left(x, \sqrt{1-x^2-y^2}, y\right), \qquad \varphi_4(x,y) = \left(x, -\sqrt{1-x^2-y^2}, y\right),$$
$$\varphi_5(x,y) = \left(\sqrt{1-x^2-y^2}, x, y\right), \qquad \varphi_6(x,y) = \left(-\sqrt{1-x^2-y^2}, x, y\right).$$

Arguing as in Example 2.3, it is easy to see that all the maps φ_j are local parametrizations of S^2; moreover, $S^2 = \varphi_1(U) \cup \cdots \cup \varphi_6(U)$, and so $\{\varphi_1, \ldots, \varphi_6\}$ is an atlas for S^2. Note that if we omit even one of these local parametrizations we do not cover the whole sphere.

EXAMPLE 2.6. We now describe another atlas for the sphere. Set

$$U = \{(\theta, \phi) \in \mathbb{R}^2 \mid 0 < \theta < \pi, 0 < \phi < 2\pi\},$$

and let $\varphi_1 : U \to \mathbb{R}^3$ be given by

$$\varphi_1(\theta, \phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta);$$

we want to prove that φ_1 is a local parametrization of the sphere. The parameter θ is usually called *colatitude* (the *latitude* is $\pi/2 - \theta$), while ϕ is the *longitude*. The local coordinates (θ, ϕ) are called *spherical coordinates*; see Fig. 3.

First of all,

$$\varphi_1(U) = S^2 \setminus \{(x,y,z) \in \mathbb{R}^3 \mid y = 0, x \geq 0\}$$

is an open subset of S^2, so condition (a) is satisfied. Next,

$$\operatorname{Jac}\varphi_1(\theta,\phi) = \begin{vmatrix} \cos\theta\cos\phi & -\sin\theta\sin\phi \\ \cos\theta\sin\phi & \sin\theta\cos\phi \\ -\sin\theta & 0 \end{vmatrix} ,$$

and it is straightforward to verify that this matrix has rank 2 everywhere (since $\sin\theta \neq 0$ when $(\theta,\phi) \in U$), so condition (c) is satisfied. Moreover, if we take an arbitrary $(x,y,z) = \varphi(\theta,\phi) \in \varphi_1(U)$, we find $\theta = \arccos z \in (0,\pi)$; being $\sin\theta \neq 0$, we recover $(\cos\phi, \sin\phi) \in S^1$ and consequently $\phi \in (0, 2\pi)$ in terms of x, y and z, so φ_1 is globally injective. To conclude, we should prove that φ_1 is a homeomorphism with its image (i.e., that φ_1^{-1} is continuous); but we shall see shortly (Proposition 2.3) that this is a consequence of the fact that we already know that S^2 is a surface, so we leave this as an exercise (but see also Example 2.8). Finally, let $\varphi_2 \colon U \to \mathbb{R}^3$ be given by

$$\varphi_2(\theta,\phi) = (-\sin\theta\cos\phi, \cos\theta, -\sin\theta\sin\phi) .$$

Arguing as above, we see that φ_2 is also a local parametrization, with

$$\varphi_2(U) = S^2 \setminus \{(x,y,z) \in \mathbb{R}^3 \mid z = 0, x \leq 0\} ,$$

so $\{\varphi_1, \varphi_2\}$ is an atlas for S^2.

Exercise 2.4 describes a third possible atlas for the sphere.

Example 2.7. Let $S \subset \mathbb{R}^3$ be a surface, and $S_1 \subseteq S$ an open subset of S. Then S_1 is a surface as well. Indeed, choose $p \in S_1$ and let $\varphi \colon U \to \mathbb{R}^3$ be a local parametrization of S at p. Then $U_1 = \varphi^{-1}(S_1)$ is open in $\mathbb{R}^2$ and $\varphi_1 = \varphi|_{U_1} \colon U_1 \to \mathbb{R}^3$ is a local parametrization of S_1 at p.

If $\chi \colon \Omega \to \mathbb{R}^3$ is a diffeomorphism with its image defined on an open neighborhood Ω of S, then $\chi(S)$ is a surface. Indeed, if φ is a local parametrization of S at $p \in S$, the map $\chi \circ \varphi$ is a local parametrization of $\chi(S)$ at $\chi(p)$.

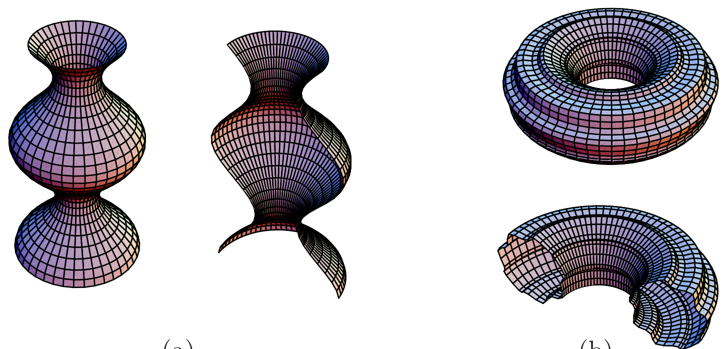
Example 2.8 (Surfaces of revolution). Let $H \subset \mathbb{R}^3$ be a plane, $C \subset H$ the support of an open Jordan arc or of a Jordan curve of class C^∞, and $\ell \subset H$ a straight line disjoint from C. We want to prove that the set $S \subset \mathbb{R}^3$ obtained by rotating C around ℓ is a regular surface, called *surface of revolution* having C as its *generatrix* and ℓ as its *axis.*

Without loss of generality, we may assume that H is the plane xz, that ℓ is the z-axis, and that C lies in the half-plane $\{x > 0\}$. If C is the support of an open Jordan arc, we have by definition a global parametrization $\sigma \colon I \to \mathbb{R}^3$ that is a homeomorphism with its image, where $I \subseteq \mathbb{R}$ is an open interval. Since all open intervals are diffeomorphic to $\mathbb{R}$ (Exercise 1.5), we may assume without loss of generality that $I = \mathbb{R}$. If, on the other hand, C is the support of a Jordan curve, take a periodic parametrization $\sigma \colon \mathbb{R} \to \mathbb{R}^3$ of C. In both cases, we may write $\sigma(t) = \big(\alpha(t), 0, \beta(t)\big)$ with $\alpha(t) > 0$ for all $t \in \mathbb{R}$, so

$$S = \Big\{ \big(\alpha(t)\cos\theta, \alpha(t)\sin\theta, \beta(t)\big) \;\Big|\; t, \theta \in \mathbb{R} \Big\} .$$

Define now $\varphi \colon \mathbb{R}^2 \to \mathbb{R}^3$ by setting

$$\varphi(t,\theta) = \big(\alpha(t)\cos\theta, \alpha(t)\sin\theta, \beta(t)\big) ,$$



(a) (b)

Figure 4. Surfaces of revolution (whole and sections)

so that $S = \varphi(\mathbb{R}^2)$. If we fix $t_0 \in \mathbb{R}$, the curve $\theta \mapsto \varphi(t_0, \theta)$ is a *parallel* of S; it is the circle with radius $\alpha(t_0)$ obtained by rotating the point $\sigma(t_0)$ around ℓ. If we fix $\theta_0 \in \mathbb{R}$, the curve $t \mapsto \varphi(t, \theta_0)$ is a *meridian* of S; it is obtained rotating C by an angle θ_0 around ℓ.

Now we have

$$\operatorname{Jac} \varphi(t,\theta) = \begin{vmatrix} \alpha'(t)\cos\theta & -\alpha(t)\sin\theta \\ \alpha'(t)\sin\theta & \alpha(t)\cos\theta \\ \beta'(t) & 0 \end{vmatrix} .$$

So $\operatorname{Jac} \varphi(t,\theta)$ has rank less than 2 if and only if

$$\begin{cases} \alpha'(t)\alpha(t) = 0 \ , \\ \alpha(t)\beta'(t)\sin\theta = 0 \ , \\ \alpha(t)\beta'(t)\cos\theta = 0 \ , \end{cases}$$

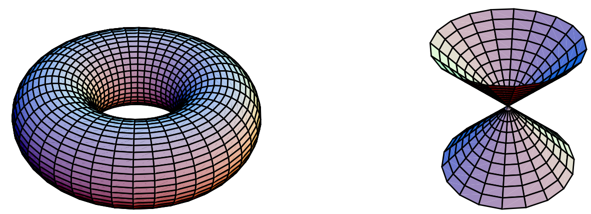
and this never happens since α is always positive and σ is regular. In particular, φ is an immersed surface having S as its support.

This, however, is not enough to prove that S is a regular surface. To conclude, we have to consider two cases.

(a) *C is not compact, and σ is a global parametrization*: see Fig. 4.(a). In this case, we set $\varphi_1 = \varphi|_{\mathbb{R}\times(0,2\pi)}$ and $\varphi_2 = \varphi|_{\mathbb{R}\times(-\pi,\pi)}$; since the union of the supports of φ_1 and φ_2 is S, if we prove that φ_1 and φ_2 are local parametrizations we are done. Since

$$\varphi_1\big(\mathbb{R}\times(0,2\pi)\big) = S \setminus \{(x,y,z) \in \mathbb{R}^3 \mid y = 0, x \geq 0\}$$

is open in S and φ_1 is the restriction of an immersed surface, to show that φ_1 is a local parametrization it suffices to prove that it is a homeomorphism with its image. From $\varphi_1(t,\theta) = (x,y,z)$ we find $\beta(t) = z$ and $\alpha(t) = \sqrt{x^2+y^2}$. As σ is injective, from this we can find a unique $t \in I$, and hence a unique $\theta \in (0,2\pi)$, such that $x = \alpha(t)\cos\theta$ and $y = \alpha(t)\sin\theta$; thus, φ_1 is invertible. Furthermore, since σ is a homeomorphism with its image, the coordinate t depends continuously on z and $\sqrt{x^2+y^2}$; if we prove that θ also depends continuously on (x,y,z) we have proved that



(a) (b)

FIGURE 5. (**a**) a torus; (**b**) a two-sheeted cone

φ_1^{-1} is continuous. Now, if $(x, y, z) \in S$ is such that $y > 0$ we have

$$0 < \frac{y}{x + \sqrt{x^2 + y^2}} = \frac{y/\alpha(t)}{1 + x/\alpha(t)} = \frac{\sin\theta}{1 + \cos\theta} = \frac{\sin(\theta/2)}{\cos(\theta/2)} = \tan\frac{\theta}{2} ,$$

so

$$\theta = 2\arctan\left(\frac{y}{x + \sqrt{x^2 + y^2}}\right) \in (0, \pi)$$

depends continuously on (x, y, z). Analogously, is $(x, y, z) \in S$ is such that $y < 0$ we find

$$\theta = 2\pi + 2\arctan\left(\frac{y}{x + \sqrt{x^2 + y^2}}\right) \in (\pi, 2\pi) ,$$

and in this case too we are done. Finally, in order to verify that φ_1^{-1} is continuous in a neighborhood of a point $(x_0, 0, z_0) \in \varphi_1\big(\mathbb{R} \times (0, 2\pi)\big)$ note that $x_0 < 0$ necessarily, and that if $(x, y, z) \in S$ with $x < 0$ then

$$\frac{y}{\sqrt{x^2 + y^2} - x} = \frac{y/\alpha(t)}{1 - x/\alpha(t)} = \frac{\sin\theta}{1 - \cos\theta} = \frac{\cos(\theta/2)}{\sin(\theta/2)} = \operatorname{cotan}\frac{\theta}{2} ,$$

so

$$\theta = 2\operatorname{arccotan}\left(\frac{y}{-x + \sqrt{x^2 + y^2}}\right) \in (\pi/2, 3\pi/2) ,$$

and in this case we are done as well. The proof that φ_2 is a local parametrization is completely analogous, so S is a regular surface.

(b) *C is compact, and σ is a periodic parametrization with period $2r > 0$*; see Fig. 4.(b). In this case set $\varphi_1 = \varphi|_{(0,2r)\times(0,2\pi)}$, $\varphi_2 = \varphi|_{(0,2r)\times(-\pi,\pi)}$, $\varphi_3 = \varphi|_{(-r,r)\times(0,2\pi)}$, and $\varphi_4 = \varphi|_{(-r,r)\times(-\pi,\pi)}$; then, arguing as in the previous case, we immediately see that $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$ is an atlas for S.

Another way to prove that surfaces of revolution are regular surfaces is outlined in Exercise 2.23.

EXAMPLE 2.9. A *torus* is a surface obtained by rotating a circle around an axis (contained in the plane of the circle) not intersecting it. For instance, if C is the circle with center $(x_0, 0, z_0)$ and radius $0 < r_0 < |x_0|$ in the xz-plane, then the torus obtained by rotating C around the z-axis is the support of the immersed surface $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$\varphi(t, \theta) = \big((r\cos t + x_0)\cos\theta, (r\cos t + x_0)\sin\theta, r\sin t + z_0\big) ;$$

see Fig. 5.(a).

EXAMPLE 2.10. Let us see now an example of a subset of $\mathbb{R}^3$ that is not a regular surface. The *two-sheeted (infinite) cone* is the set

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2\} ;$$

see Fig. 5.(b). The set S cannot be a regular surface: indeed, if the origin $O \in S$ had in S a neighborhood homeomorphic to an open subset of the plane, then $S \setminus \{O\}$ should be connected (why?), but this is not the case. We shall see shortly (Example 2.13) that the one-sheeted infinite cone $S \cap \{z \geq 0\}$ is not a regular surface too, whereas either connected component of $S \setminus \{O\}$ is (Exercise 2.9).

REMARK 2.6. We can now show that there are non-compact surfaces that cannot be the support of a single immersed surface which is also a homeomorphism with its image. In other words, there exist non-compact regular surfaces not homeomorphic to an open subset of the plane. Let $S \subset \mathbb{R}^3$ be the non-compact surface obtained by removing a point from a torus (Examples 2.7 and 2.9). Then S contains Jordan curves (the meridians of the torus) that do not disconnect it, so it cannot be homeomorphic to an open subset of the plane without contradicting the *Jordan curve theorem*: the complement of a closed simple curve in the plane is not connected.

We give now a general procedure for building regular surfaces. Let us begin with a definition:

DEFINITION 2.4. Let $V \subseteq \mathbb{R}^n$ be an open set, and $F: V \to \mathbb{R}^m$ a C^∞ map. We shall say that $p \in V$ is a *critical point* of F if $\mathrm{d}F_p: \mathbb{R}^n \to \mathbb{R}^m$ is not surjective. We shall denote the set of critical points of F by $\mathrm{Crit}(F)$. If $p \in V$ is a critical point $F(p) \in \mathbb{R}^m$ will be called a *critical value*. A point $y \in F(V) \subseteq \mathbb{R}^m$ that is not a critical value is a *regular value*.

REMARK 2.7. If $f: V \to \mathbb{R}$ is a C^∞ function defined on an open subset $V \subset \mathbb{R}^n$ and $p \in V$ then $\mathrm{d}f_p: \mathbb{R}^n \to \mathbb{R}$ is not surjective if and only if it is everywhere zero. In other words, $p \in V$ is a critical point of f if and only if the gradient of f is zero in p.

REMARK 2.8. In a very precise sense, almost every point of the image of a C^∞ map is a regular value. Indeed, it can be shown that if $F: V \to \mathbb{R}^m$ is a function of class C^∞, where V is an open subset of $\mathbb{R}^n$, then the measure of the set of critical values of F in $\mathbb{R}^m$ is zero (*Sard's theorem*).

The previous remark explains the vast applicability of the following result (see also Exercise 2.9):

PROPOSITION 2.1. *Let $V \subseteq \mathbb{R}^3$ be an open set, and $f \in C^\infty(V)$. If $a \in \mathbb{R}$ is a regular value of f then every connected component of the level set*

$$f^{-1}(a) = \{p \in V \mid f(p) = a\}$$

is a regular surface.

PROOF. Let $p_0 = (x_0, y_0, z_0) \in f^{-1}(a)$. Since a is a regular value for f, the gradient of f is not zero in p_0; so, up to permuting the coordinates, we may assume

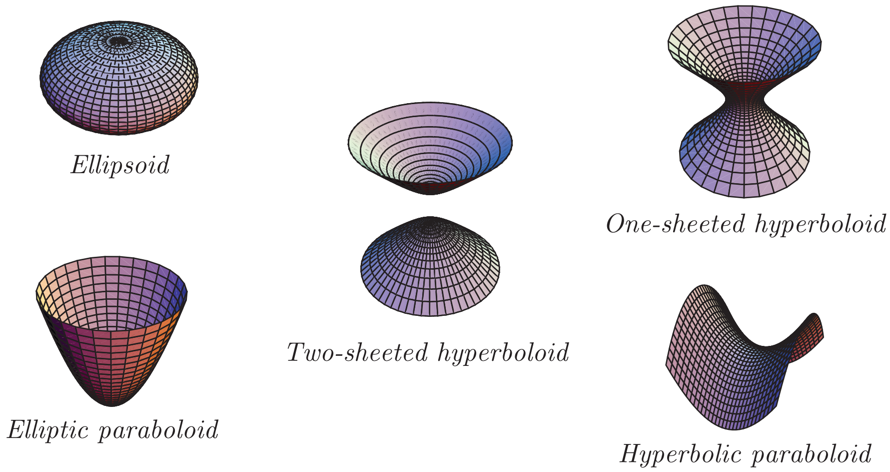


FIGURE 6. Quadrics

that $\partial f/\partial z(p_0) \neq 0$. Let now $F\colon V \to \mathbb{R}^3$ be given by $F(x,y,z) = \big(x, y, f(x,y,z)\big)$. Clearly,

$$\det \operatorname{Jac} F(p_0) = \frac{\partial f}{\partial z}(p_0) \neq 0 \; .$$

Thus we may apply the inverse function theorem (Theorem 2.1) to get neighborhoods $\tilde{V} \subseteq V$ of p_0 and $W \subseteq \mathbb{R}^3$ of $F(p_0)$ such that $F|_{\tilde{V}}\colon \tilde{V} \to W$ is a diffeomorphism. Setting $G = (g_1, g_2, g_3) = F^{-1}$ we have

$$(u,v,w) = F \circ G(u,v,w) = \Big(g_1(u,v,w), g_2(u,v,w), f\big(G(u,v,w)\big)\Big) \; .$$

So $g_1(u,v,w) \equiv u$, $g_2(u,v,w) \equiv v$, and

$$\forall (u,v,w) \in W \qquad f\big(G(u,v,w)\big) \equiv w \; . \tag{20}$$

Clearly, the set $U = \big\{(u,v) \in \mathbb{R}^2 \mid (u,v,a) \in W\big\}$ is an open subset of $\mathbb{R}^2$, and we may define $\varphi\colon U \to \mathbb{R}^3$ with

$$\varphi(u,v) = G(u,v,a) = \big(u, v, g_3(u,v,a)\big) \; .$$

By (20), we know (why?) that $\varphi(U) = f^{-1}(a) \cap \tilde{V}$, and it is straightforward to verify that φ is a local parametrization of $f^{-1}(a)$ at p_0. □

DEFINITION 2.5. Let $V \subseteq \mathbb{R}^3$ be an open set and $f \in C^\infty(V)$. Every component of $f^{-1}(a)$, where $a \in \mathbb{R}$ is a regular value for f, is a *level surface* of f.

EXAMPLE 2.11. The *ellipsoid* having equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

with a, b, $c > 0$ is a regular surface. Indeed, it is of the form $f^{-1}(1)$, where $f\colon \mathbb{R}^3 \to \mathbb{R}$ is given by

$$f(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \; .$$

Since $\nabla f = \big(2x/a^2, 2y/b^2, 2z/c^2\big)$, the only critical point of f is the origin, the only critical value of f is 0, and so $f^{-1}(1)$ is a level surface.

Example 2.12. More in general, a *quadric* is the subset of $\mathbb{R}^3$ of the points that are solutions of an equation of the form $p(x,y,z)=0$, where p is a polynomial of degree 2. Not all quadrics are regular surfaces (see Example 2.10 and Problem 2.4), but the components of those that are provide a good repertory of examples of surfaces. Besides the ellipsoid, we have the *two-sheeted* (or *elliptic*) *hyperboloid* of equation $(x/a)^2+(y/b)^2-(z/c)^2+1=0$, the *one-sheeted* (or *hyperbolic*) *hyperboloid* of equation $(x/a)^2+(y/b)^2-(z/c)^2-1=0$, the *elliptic paraboloid* having equation $(x/a)^2+(y/b)^2-z=0$, the *hyperbolic paraboloid* having equation $(x/a)^2-(y/b)^2-z=0$, and cylinders having a conic section as generatrix (see Problem 2.3). Fig. 6 shows some quadrics.

We end this section with two general results.

Proposition 2.2. *Every regular surface is locally a graph. In other words, if $S\subset\mathbb{R}^3$ is a regular surface and $p\in S$ then there exists a local parametrization $\varphi\colon U\to S$ in p of one of the following forms:*

$$\varphi(x,y)=\begin{cases}\big(x,y,f(x,y)\big)\,, & \textit{or}\\ \big(x,f(x,y),y\big)\,, & \textit{or}\\ \big(f(x,y),x,y\big)\,, & \end{cases}$$

for a suitable $f\in C^\infty(U)$. In particular, there is always an open neighborhood $\Omega\subseteq\mathbb{R}^3$ of S such that S is closed in Ω.

Proof. Let $\psi=(\psi_1,\psi_2,\psi_3)\colon U_1\to\mathbb{R}^3$ be a local parametrization centered at p. Up to permuting coordinates, we may assume that

$$\det\left(\frac{\partial\psi_h}{\partial x_k}(O)\right)_{h,k=1,2}\neq 0\,;$$

so, setting $F=(\psi_1,\psi_2)$ we may find a neighborhood $V\subseteq U_1$ of O and a neighborhood $U\subseteq\mathbb{R}^2$ of $F(O)$ such that $F|_V\colon V\to U$ is a diffeomorphism. Let $F^{-1}\colon U\to V$ be the inverse map, and set $f=\psi_3\circ F^{-1}\colon U\to\mathbb{R}$. Since we have $F\circ F^{-1}=\mathrm{id}_U$, we get

$$\psi\circ F^{-1}(u,v)=\big(u,v,f(u,v)\big)\,,$$

so $\varphi=\psi\circ F^{-1}\colon U\to\mathbb{R}^3$ is a local parametrization of S at p of the required form. Finally, for all $p\in S$ let $W_p\subset\mathbb{R}^3$ be an open neighborhood of p such that $W_p\cap S$ is a graph. Then $W_p\cap S$ is closed in W_p, and so S is closed (why?) in $\Omega=\bigcup_{p\in S}W_p$. □

The converse of this result holds too: every set that is locally a graph is a regular surface (Exercise 2.11).

Example 2.13. The one-sheeted infinite cone

$$S=\{(x,y,z)\in\mathbb{R}^3\mid z=\sqrt{x^2+y^2}\}$$

is not a regular surface. If it were, it should be the graph of a C^∞ function in a neighborhood of $(0,0,0)$. As the projections on the xz-plane and yz-plane are not injective, it should be a graph over the xy-plane; but in this case it should be the graph of the function $\sqrt{x^2+y^2}$, which is not of class C^∞.

And at last, here is the result promised in Example 2.6:

Proposition 2.3. *Let $S\subset\mathbb{R}^3$ be a regular surface, $U\subseteq\mathbb{R}^2$ an open subset, and $\varphi\colon U\to\mathbb{R}^3$ an immersed surface with support contained in S. Then:*

(i) $\varphi(U)$ *is open in* S;
(ii) *if* φ *is injective then for all* $p \in \varphi(U)$ *there exist a neighborhood* $W \subset \mathbb{R}^3$ *of* p *in* $\mathbb{R}^3$ *with* $W \cap S \subseteq \varphi(U)$, *and a map* $\Phi\colon W \to \mathbb{R}^2$ *of class* C^∞ *such that* $\Phi(W) \subseteq U$ *and* $\Phi|_{W\cap S} \equiv \varphi^{-1}|_{W\cap S}$. *In particular,* $\varphi^{-1}\colon \varphi(U) \to U$ *is continuous, so* φ *is a local parametrization of* S.

PROOF. Let $p = \varphi(x_0, y_0) \in \varphi(U)$. As S is a surface, we can find a neighborhood W_0 of p in $\mathbb{R}^3$ such that $W_0 \cap S$ is a graph; to fix ideas, say that $W_0 \cap S$ is the graph over the xy-plane of a function f. If $\pi\colon \mathbb{R}^3 \to \mathbb{R}^2$ is the projection on the xy-plane, set $U_0 = \varphi^{-1}(W_0) \subseteq U$ and $h = \pi \circ \varphi\colon U_0 \to \mathbb{R}^2$. For $(x, y) \in U_0$ we have $\varphi_3(x, y) = f\bigl(\varphi_1(x, y), \varphi_2(x, y)\bigr)$, and so the third row of the Jacobian matrix of φ in (x, y) is a linear combination of the first two. Since the differential of φ is supposed to have rank 2 everywhere, it follows that the first two rows of the Jacobian matrix of φ have to be linearly independent, and so $\operatorname{Jac} h(x, y)$ is invertible. The inverse function theorem (Theorem 2.1) then yields a neighborhood $U_1 \subseteq U_0$ of (x_0, y_0) and a neighborhood $V_1 \subseteq \mathbb{R}^2$ of $h(x_0, y_0) = \pi(p)$ such that $h|_{U_1}\colon U_1 \to V_1$ is a diffeomorphism. In particular, $\varphi(U_1) = \varphi \circ h|_{U_1}^{-1}(V_1) = (\pi|_{S\cap W_0})^{-1}(V_1)$ is open in S, so $\varphi(U)$ is a neighborhood of p in S. Since p is arbitrary, it follows that $\varphi(U)$ is open in S, and (i) is proved.

Suppose now that φ is injective, so $\varphi^{-1}\colon \varphi(U) \to U$ is defined. As $\varphi(U)$ is open in S, up to restricting W_0 we may assume that $W_0 \cap S \subseteq \varphi(U)$. Set $W = W_0 \cap \pi^{-1}(V_1)$ and $\Phi = h|_{U_1}^{-1} \circ \pi$; to complete the proof of (ii) it remains to show that $\Phi|_{W\cap S} \equiv \varphi^{-1}|_{W\cap S}$.

Let $q \in W \cap S$. As $q \in W_0 \cap \pi^{-1}(V_1)$, we can find a point $(u, v) \in V_1$ such that $q = \bigl(u, v, f(u, v)\bigr)$. On the other hand, being $q \in \varphi(U)$ there is a unique point $(x, y) \in U$ such that $q = \varphi(x, y)$. But then $(u, v) = \pi(q) = h(x, y)$; so $(x, y) = h|_{U_1}^{-1}(u, v) \in U_1$ and $\varphi^{-1}(q) = (x, y) = h|_{U_1}^{-1} \circ \pi(q) = \Phi(q)$, as required. □

In other words, if we already know that S is a surface, to verify whether a map $\varphi\colon U \to \mathbb{R}^3$ from an open subset U of $\mathbb{R}^2$ to S is a local parametrization it suffices to check that φ is injective and that $d\varphi_x$ has rank 2 for all $x \in U$.

REMARK 2.9. The previous proposition and Lemma 2.1 might suggest that a claim along the following lines might be true: "Let $\varphi\colon U \to \mathbb{R}^3$ be an injective immersed surface with support $S = \varphi(U)$. Then for all $p \in \varphi(U)$ we can find a neighborhood $W \subset \mathbb{R}^3$ of p in $\mathbb{R}^3$ and a map $\Phi\colon W \to \mathbb{R}^2$ of class C^∞ such that $\Phi(W) \subseteq U$ and $\Phi|_{W\cap S} \equiv \varphi^{-1}|_{W\cap S}$. In particular, $\varphi^{-1}\colon \varphi(U) \to U$ is continuous, and S is a regular surface." We have even a "proof" of this claim: "Since, by assumption, φ is an immersed surface, we may apply Lemma 2.1. Let $p = \varphi(x_0) \in \varphi(U)$, and $G\colon \Omega \to W$ the diffeomorphism provided by Lemma 2.1; up to restricting Ω, we may also assume that $\Omega = U_1 \times (-\delta, \delta)$, where $\delta > 0$ and $U_1 \subseteq U$ is a suitable neighborhood of x_0. Then $\Phi = \pi \circ G^{-1}$, where $\pi\colon \mathbb{R}^3 \to \mathbb{R}^2$ is the projection on the first two coordinates, is as required. Indeed, for all $q \in W \cap \varphi(U)$ the point $G^{-1}(q) = (y, t) \in \Omega$ is the only one satisfying $G(y, t) = q$. But $G\bigl(\varphi^{-1}(q), 0\bigr) = \varphi\bigl(\varphi^{-1}(q)\bigr) = q$, so $G^{-1}(q) = \bigl(\varphi^{-1}(q), 0\bigr)$, and we are done." However, *the claim is false and this proof is wrong.*

The (subtle) error in the proof is that if $q \in W \cap \varphi(U)$ then $\varphi^{-1}(q)$ *does not necessarily belong to* U_1, and so $\bigl(\varphi^{-1}(q), 0\bigr)$ does not belong to the domain of G; hence we cannot say that $G\bigl(\varphi^{-1}(q), 0\bigr) = \varphi\bigl(\varphi^{-1}(q)\bigr) = q$ or deduce that $G^{-1}(q) = \bigl(\varphi^{-1}(q), 0\bigr)$. Of course, the claim might be true even if this particular proof is wrong. But the claim is false indeed, and in Example 2.14 you'll find a counterexample.

Summing up, we may deduce the continuity of the inverse of a globally injective immersed surface φ *only if we already know that the image of* φ *lies within a regular surface*; otherwise, it might be false.

EXAMPLE 2.14. Let $\varphi\colon(-1,+\infty)\times\mathbb{R}\to\mathbb{R}^3$ be the immersed surface of Example 2.1. We have already remarked that φ is an injective immersed surface that is not a homeomorphism with its image, and it is immediate to notice that its support S is not a regular surface, since in a neighborhood of the point $(0,0,0)\in S$ none of the three projections on the coordinate planes is injective, and so S cannot be locally a graph.

2.2. Smooth functions

Local parametrizations are the tool that allows us to give concrete form to the idea that a surface locally resembles an open subset of the plane; let us see how to use them to determine when a function defined on a surface is smooth.

DEFINITION 2.6. Let $S\subset\mathbb{R}^3$ be a surface, and $p\in S$. A function $f\colon S\to\mathbb{R}$ is of *class* C^∞ (or *smooth*) at p if there exists a local parametrization $\varphi\colon U\to S$ at p such that $f\circ\varphi\colon U\to\mathbb{R}$ is of class C^∞ in a neighborhood of $\varphi^{-1}(p)$. We shall say that f is of *class* C^∞ (or *smooth*) if it is so at every point. The space of C^∞ functions on S will be denoted by $C^\infty(S)$.

REMARK 2.10. A smooth function $f\colon S\to\mathbb{R}$ is automatically continuous. Indeed, let $I\subseteq\mathbb{R}$ be an open interval, and $p\in f^{-1}(I)$. By assumption, there is a local parametrization $\varphi\colon U\to S$ at p such that $f\circ\varphi$ is of class C^∞ (and thus continuous) in a neighborhood of $\varphi^{-1}(p)$. Then $(f\circ\varphi)^{-1}(I)=\varphi^{-1}\bigl(f^{-1}(I)\bigr)$ is a neighborhood of $\varphi^{-1}(p)$. But φ is a homeomorphism with its image; so $f^{-1}(I)$ has to be a neighborhood of $\varphi\bigl(\varphi^{-1}(p)\bigr)=p$. Since p was arbitrary, it follows that $f^{-1}(I)$ is open in S, and so f is continuous.

A possible problem with this definition is that it might depend on the particular local parametrization we have chosen: a priori, there might be another local parametrization ψ at p such that $f\circ\psi$ is *not* smooth in $\psi^{-1}(p)$. Luckily, the following theorem implies that this cannot happen.

THEOREM 2.2. *Let S be a surface, and let $\varphi\colon U\to S$, $\psi\colon V\to S$ be two local parametrizations with $\Omega=\varphi(U)\cap\psi(V)\neq\varnothing$. Then the map*

$$h=\varphi^{-1}\circ\psi|_{\psi^{-1}(\Omega)}\colon\psi^{-1}(\Omega)\to\varphi^{-1}(\Omega)$$

is a diffeomorphism.

PROOF. The map h is a homeomorphism, as it is a composition of homeomorphisms; we have to show that it and its inverse are of class C^∞.

Let $x_0\in\psi^{-1}(\Omega)$, $y_0=h(x_0)\in\varphi^{-1}(\Omega)$, and $p=\psi(x_0)=\varphi(y_0)\in\Omega$. Proposition 2.3 provides us with a neighborhood W of $p\in\mathbb{R}^3$ and a map $\Phi\colon W\to\mathbb{R}^2$ of class C^∞ such that $\Phi|_{W\cap S}\equiv\varphi^{-1}$. Now, by the continuity of ψ, there is a neighborhood $V_1\subset\psi^{-1}(\Omega)$ of x_0 such that $\psi(V_1)\subset W$. Then $h|_{V_1}=\Phi\circ\psi|_{V_1}$, and so h is of class C^∞ in x_0. Since x_0 is an arbitrary element, h is of class C^∞ everywhere. In an analogous way it can be proved that h^{-1} is of class C^∞, and so h is a diffeomorphism. $\square$

COROLLARY 2.2. *Let $S \subset \mathbb{R}^3$ be a surface, $f\colon S \to \mathbb{R}$ a function, and $p \in S$. If there is a local parametrization $\varphi\colon U \to S$ at p such that $f \circ \varphi$ is of class C^∞ in a neighborhood of $\varphi^{-1}(p)$, then $f \circ \psi$ is of class C^∞ in a neighborhood of $\psi^{-1}(p)$ for all local parametrization $\psi\colon V \to S$ of S at p.*

PROOF. We may write

$$f \circ \psi = (f \circ \varphi) \circ (\varphi^{-1} \circ \psi) ,$$

and thus the previous theorem implies that $f \circ \psi$ is of class C^∞ in a neighborhood of $\psi^{-1}(p)$ if and only if $f \circ \varphi$ is of class C^∞ in a neighborhood of $\varphi^{-1}(p)$. □

So the being smooth on a surface is a property of the function, and does not depend on local parametrizations; to test whether a function is smooth we may use an arbitrary local parametrization.

Using the same approach, we may define the notion of smooth map between two surfaces:

DEFINITION 2.7. If S_1, $S_2 \subset \mathbb{R}^3$ are two surfaces, we shall say that a map $F\colon S_1 \to S_2$ is of *class* C^∞ (or *smooth*) at $p \in S_1$ if there exist a local parametrization $\varphi_1\colon U_1 \to S_1$ in p and a local parametrization $\varphi_2\colon U_2 \to S_2$ in $F(p)$ such that $\varphi_2^{-1} \circ F \circ \varphi_1$ is of class C^∞ (where defined). We shall say that F is *of class* C^∞ (or *smooth*) if it so at every point. If F is of class C^∞ and invertible with inverse of class C^∞ we shall say that F is a *diffeomorphism*, and that S_1 and S_2 are *diffeomorphic*.

REMARK 2.11. The notion of smooth map defined on an open subset of $\mathbb{R}^n$ with values in a surface, or from a surface with values in $\mathbb{R}^n$, can be introduced in an analogous way (see Exercise 2.20).

It is easy to prove that the definition of smooth map does not depend on the local parametrizations used (Exercise 2.39), that smooth maps are continuous (Exercise 2.38), and that a composition of smooth maps is smooth:

PROPOSITION 2.4. *If $F\colon S_1 \to S_2$ and $G\colon S_2 \to S_3$ are smooth maps between surfaces, then the composition $G \circ F\colon S_1 \to S_3$ is smooth as well.*

PROOF. Fix $p \in S_1$ and choose an arbitrary local parametrization $\varphi_1\colon U_1 \to S_1$ of S_1 at p, a local parametrization $\varphi_2\colon U_2 \to S_2$ of S_2 at $F(p)$, and a local parametrization $\varphi_3\colon U_3 \to S_3$ of S_3 at $G\big(F(p)\big)$. Then

$$\varphi_3^{-1} \circ (G \circ F) \circ \varphi_1 = (\varphi_3^{-1} \circ G \circ \varphi_2) \circ (\varphi_2^{-1} \circ F \circ \varphi_1)$$

is of class C^∞ where defined, and we are done. □

EXAMPLE 2.15. A local parametrization $\varphi\colon U \to \varphi(U) \subset S$ is a diffeomorphism between U and $\varphi(U)$. Indeed, first of all, it is invertible by definition. Next, to test the differentiability of φ and φ^{-1} we can use the identity map id as local parametrization of U, and φ itself as local parametrization of S. So it suffices to verify that $\varphi^{-1} \circ \varphi \circ \mathrm{id}$ and $\mathrm{id} \circ \varphi^{-1} \circ \varphi$ are of class C^∞, which is straightforward.

EXAMPLE 2.16. If $U \subset \mathbb{R}^n$ is open and $F\colon U \to \mathbb{R}^3$ is a C^∞ map whose image is contained in a surface S then F is of class C^∞ as an S-valued map as well. Indeed, let ψ be a local parametrization at a point $p \in F(U)$; Proposition 2.3 tells us that there exists a function Ψ of class C^∞ defined in a neighborhood of p such that $\psi^{-1} \circ F = \Psi \circ F$, and the latter composition is of class C^∞.

EXAMPLE 2.17. If $S \subset \mathbb{R}^3$ is a surface, then the inclusion $\iota\colon S \hookrightarrow \mathbb{R}^3$ is of class C^∞. Indeed, saying that ι is of class C^∞ is exactly equivalent (why?) to saying that local parametrizations are of class C^∞ when considered as maps with values in $\mathbb{R}^3$.

EXAMPLE 2.18. If $\Omega \subseteq \mathbb{R}^3$ is an open subset of $\mathbb{R}^3$ that contains the surface S, and $\tilde{f} \in C^\infty(\Omega)$, then the restriction $f = \tilde{f}|_S$ is of class C^∞ on S. Indeed, $f \circ \varphi = \tilde{f} \circ \varphi$ is of class C^∞ for every local parametrization φ.

Actually, it is possible to prove that the previous example provides all C^∞ functions on a surface S. However, for our purposes, it is sufficient a local version of this result:

PROPOSITION 2.5. *Let $S \subset \mathbb{R}^3$ be a surface, and take $p \in S$. Then a function $f\colon S \to \mathbb{R}$ is of class C^∞ at p if and only if there exist an open neighborhood $W \subseteq \mathbb{R}^3$ of p in $\mathbb{R}^3$ and a function $\tilde{f} \in C^\infty(W)$ such that $\tilde{f}|_{W\cap S} \equiv f|_{W\cap S}$.*

PROOF. One implication is given by Example 2.18. For the converse, suppose that f is of class C^∞ at p, and let $\varphi\colon U \to S$ be a local parametrization centered at p. Proposition 2.3.(ii) provides us with a neighborhood W of p in $\mathbb{R}^3$ and a map $\Phi\colon W \to \mathbb{R}^2$ of class C^∞ such that $\Phi(W) \subseteq U$ and $\Phi_{W\cap S} \equiv \varphi^{-1}|_{W\cap S}$. Then the function $\tilde{f} = (f \circ \varphi) \circ \Phi \in C^\infty(W)$ is as required. □

2.3. Tangent plane

We have seen that tangent vectors play a major role in the study of curves. In this section we intend to define the notion of a tangent vector to a surface at a point. The geometrically simplest way is as follows:

DEFINITION 2.8. Let $S \subseteq \mathbb{R}^3$ be a set, and $p \in S$. A *tangent vector* to S at p is a vector of the form $\sigma'(0)$, where $\sigma\colon (-\varepsilon, \varepsilon) \to \mathbb{R}^3$ is a curve of class C^∞ whose support lies in S and such that $\sigma(0) = p$. The set of all possible tangent vectors to S at p is the *tangent cone* T_pS to S at p.

REMARK 2.12. A *cone* (with the origin as vertex) in a vector space V is a subset $C \subseteq V$ such that $av \in C$ for all $a \in \mathbb{R}$ and $v \in C$. It is not difficult to verify that the tangent cone to a set is in fact a cone in this sense. Indeed, first of all, the zero vector is the tangent vector to a constant curve, so $O \in T_pS$ for all $p \in S$. Next, if $a \in \mathbb{R}^*$ and $O \neq v \in T_pS$, if we choose a curve $\sigma\colon (-\varepsilon, \varepsilon) \to S$ with $\sigma(0) = p$ and $\sigma'(0) = v$, then the curve $\sigma_a\colon (-\varepsilon/|a|, \varepsilon/|a|) \to S$ given by $\sigma_a(t) = \sigma(at)$ is such that $\sigma_a(0) = p$ and $\sigma_a'(0) = av$; so $av \in T_pS$ as required by the definition of cone.

EXAMPLE 2.19. If $S \subset \mathbb{R}^3$ is the union of two straight lines through the origin, it is straightforward to verify (check it) that $T_OS = S$.

The advantage of this definition of tangent vector is the clear geometric meaning. If S is a surface, however, our geometric intuition tells us that T_pS should be a plane, not just a cone. Unfortunately, this is not so evident from the definition: the sum of two curves in S is not necessarily a curve in S, and so the "obvious" way of proving that the sum of two tangent vectors is a tangent vector does not work. On the other hand, the previous examples shows that if S is not a surface the tangent cone has no reason to be a plane; so, in order to get such a result, we have to fully exploit the definition of a surface, that is, we must involve local parametrizations.

Let us begin by seeing what happens in the simplest case, that of open sets in the plane:

EXAMPLE 2.20. Let $U \subseteq \mathbb{R}^2$ be an open set, and $p \in U$. Every curve contained in U is plane, and so the tangent vectors to U at p lie necessarily in $\mathbb{R}^2$. Conversely, if $v \in \mathbb{R}^2$ then the curve $\sigma\colon (-\varepsilon, \varepsilon) \to V$ given by $\sigma(t) = p + tv$ has its support within U for ε small enough, and has v as its tangent vector. So we have proved that $T_pU = \mathbb{R}^2$.

Applying the usual strategy of using local parametrizations to carry notions from open subsets of the plane to surfaces, we get the following:

PROPOSITION 2.6. *Let $S \subset \mathbb{R}^3$ be a surface, $p \in S$, and $\varphi\colon U \to S$ a local parametrization at p with $\varphi(x_o) = p$. Then $\mathrm{d}\varphi_{x_o}$ is an isomorphism between $\mathbb{R}^2$ and T_pS. In particular, $T_pS = \mathrm{d}\varphi_{x_0}(\mathbb{R}^2)$ is always a vector space of dimension 2, and $\mathrm{d}\varphi_{x_o}(\mathbb{R}^2)$ does not depend on φ but only on S and p.*

PROOF. Given $v \in \mathbb{R}^2$, we may find $\varepsilon > 0$ such that $x_o + tv \in U$ for all $t \in (-\varepsilon, \varepsilon)$; so the curve $\sigma_v\colon (-\varepsilon, \varepsilon) \to S$ given by $\sigma_v(t) = \varphi(x_o + tv)$ is well defined. Since $\sigma_v(0) = p$ and $\sigma_v'(0) = \mathrm{d}\varphi_{x_o}(v)$, it follows that $\mathrm{d}\varphi_{x_o}(\mathbb{R}^2) \subseteq T_pS$.

Vice versa, let $\sigma\colon (-\varepsilon, \varepsilon) \to S$ be a curve such that $\sigma(0) = p$; up to taking a smaller ε, we may assume that the support of σ is contained in $\varphi(U)$. Proposition 2.3.(ii) ensures that the composition $\sigma_o = \varphi^{-1} \circ \sigma$ is a C^∞ curve in U such that $\sigma_o(0) = x_o$; set $v = \sigma_o'(0) \in \mathbb{R}^2$. Then

$$\mathrm{d}\varphi_{x_o}(v) = \frac{\mathrm{d}(\varphi \circ \sigma_o)}{\mathrm{d}t}(0) = \sigma'(0) \;,$$

and so $T_pS \subseteq \mathrm{d}\varphi_{x_o}(\mathbb{R}^2)$. Hence, $\mathrm{d}\varphi_{x_o}\colon \mathbb{R}^2 \to T_pS$ is surjective; since it is injective too, it is an isomorphism between $\mathbb{R}^2$ and T_pS. □

DEFINITION 2.9. Let $S \subset \mathbb{R}^3$ be a surface, and take $p \in S$. The vector space $T_pS \subset \mathbb{R}^3$ is the *tangent plane* to S at p.

REMARK 2.13. Warning: according to our definition, the tangent plane is a vector subspace of $\mathbb{R}^3$, and so it passes through the origin, no matter where the point $p \in S$ is. When we draw the tangent plane as a plane resting on the surface, we are not actually drawing T_pS, but rather the plane $p + T_pS$ parallel to it, which is the *affine tangent plane* through p.

REMARK 2.14. It is apparent from the definition that if $S \subset \mathbb{R}^3$ is a surface, $p \in S$ and $U \subseteq S$ is an open subset of S containing p, then $T_pU = T_pS$. In particular, if $S = \mathbb{R}^2$ then $T_pU = T_p\mathbb{R}^2 = \mathbb{R}^2$ for every open set U of the plane and every $p \in U$.

The isomorphism between $\mathbb{R}^2$ and T_pS provided by the local parametrizations allows us to consider special bases of the tangent plane:

DEFINITION 2.10. Let $S \subset \mathbb{R}^3$ be a surface, and $p \in S$. If $\varphi\colon U \to S$ is a local parametrization centered at p, and $\{\vec{e}_1, \vec{e}_2\}$ is the canonical basis of $\mathbb{R}^2$, then the tangent vectors $\partial/\partial x_1|_p$, $\partial/\partial x_2|_p \in T_pS$ (the reasons behind this notation will be made clear in Remark 2.19) are defined by setting

$$\left.\frac{\partial}{\partial x_j}\right|_p = \mathrm{d}\varphi_O(\vec{e}_j) = \frac{\partial \varphi}{\partial x_j}(O) = \begin{vmatrix} \frac{\partial \varphi_1}{\partial x_j}(O) \\ \frac{\partial \varphi_2}{\partial x_j}(O) \\ \frac{\partial \varphi_3}{\partial x_j}(O) \end{vmatrix} .$$

We shall often write $\partial_j|_p$ (or even, when no confusion may arise, simply ∂_j) rather than $\partial/\partial x_j|_p$. Clearly, $\{\partial_1|_p, \partial_2|_p\}$ is a basis of T_pS, the basis *induced* by the local parametrization φ. Note that $\partial_1|_p$ and $\partial_2|_p$ are just the two columns of the Jacobian matrix of φ computed in $O = \varphi^{-1}(p)$. Finally, a curve in S tangent to $\partial_j|_p$ is the j-th coordinate curve $\sigma\colon (-\varepsilon, \varepsilon) \to S$ given by $\sigma(t) = \varphi(t\vec{e}_j)$ for ε small enough.

We have seen that a possible way to define surfaces is as level surfaces of a smooth function. The following proposition tells us how to find the tangent plane in this case:

Proposition 2.7. *Let $U \subseteq \mathbb{R}^3$ an open set, and $a \in \mathbb{R}$ a regular value of a function $f \in C^\infty(U)$. If S is a connected component of $f^{-1}(a)$ and $p \in S$, the tangent plane T_pS is the subspace of $\mathbb{R}^3$ orthogonal to $\nabla f(p)$.*

Proof. Take $v = (v_1, v_2, v_3) \in T_pS$ and let $\sigma\colon (-\varepsilon, \varepsilon) \to S$ be a curve with $\sigma(0) = p$ and $\sigma'(0) = v$. Differentiating $f \circ \sigma \equiv a$ and evaluating in 0 we find

$$\frac{\partial f}{\partial x_1}(p)v_1 + \frac{\partial f}{\partial x_2}(p)v_2 + \frac{\partial f}{\partial x_3}(p)v_3 = 0 ,$$

and so v is orthogonal to $\nabla f(p)$. Hence T_pS is contained in the subspace orthogonal to $\nabla f(p)$; but both spaces have dimension 2, and so they coincide. $\square$

Let us now see some examples of tangent planes.

Example 2.21. Let $H \subset \mathbb{R}^3$ be a plane through a point $p_0 \in \mathbb{R}^3$, and denote by $H_0 = H - p_0 \subset \mathbb{R}^3$ the plane through the origin and parallel to H. Since the tangent vectors to curves with support in H must belong to H_0 (see the proof of Proposition 1.4), we obtain $T_{p_0}H = H_0$.

Example 2.22. Let $p_0 = (x_0, y_0, z_0) \in S^2$ be a point of the unit sphere S^2 (see Example 2.5). If we set $f(x, y, z) = x^2 + y^2 + z^2$, by Proposition 2.7 $T_{p_0}S^2$ is the subspace orthogonal to $\nabla f(p_0) = (2x_0, 2y_0, 2z_0) = 2p_0$. So, *the tangent plane to a sphere at a point is always orthogonal to the radius in that point.* If $z_0 > 0$, by using the local parametrization φ_1 from Example 2.5, we find that a basis of $T_{p_0}S^2$ consists of the vectors

$$\left.\frac{\partial}{\partial x}\right|_{p_0} = \frac{\partial \varphi_1}{\partial x}(x_0, y_0) = \begin{vmatrix} 1 \\ 0 \\ \frac{-x_0}{\sqrt{1-x_0^2-y_0^2}} \end{vmatrix} , \quad \left.\frac{\partial}{\partial y}\right|_{p_0} = \frac{\partial \varphi_1}{\partial y}(x_0, y_0) = \begin{vmatrix} 0 \\ 1 \\ \frac{-y_0}{\sqrt{1-x_0^2-y_0^2}} \end{vmatrix} .$$

The basis induced by the local parametrization given by the spherical coordinates (Example 2.6), on the other hand, consists of the vectors

$$\left.\frac{\partial}{\partial \theta}\right|_{p_0} = \begin{vmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{vmatrix} \quad \text{and} \quad \left.\frac{\partial}{\partial \phi}\right|_{p_0} = \begin{vmatrix} -\sin\theta\sin\phi \\ \sin\theta\cos\phi \\ 0 \end{vmatrix} .$$

EXAMPLE 2.23. Let $\Gamma_f \subset \mathbb{R}^3$ be the graph of a function $f \in C^\infty(U)$. Using Proposition 2.6 and the local parametrization from Example 2.3 we see that a basis of the tangent plane to Γ_f at the point $p = \big(x_1, x_2, f(x_1, x_2)\big) \in \Gamma_f$ consist of the vectors

$$\left.\frac{\partial}{\partial x_1}\right|_p = \begin{vmatrix} 1 \\ 0 \\ \frac{\partial f}{\partial x_1}(x_1, x_2) \end{vmatrix}, \qquad \left.\frac{\partial}{\partial x_2}\right|_p = \begin{vmatrix} 0 \\ 1 \\ \frac{\partial f}{\partial x_2}(x_1, x_2) \end{vmatrix}.$$

EXAMPLE 2.24. Let $S \subset \mathbb{R}^3$ be the surface of revolution obtained by rotating around the z-axis a Jordan curve (or open arc) C contained in the right half-plane of the xz-plane. Let $\sigma\colon \mathbb{R} \to \mathbb{R}^3$ be a global or periodic parametrization of C of the form $\sigma(t) = \big(\alpha(t), 0, \beta(t)\big)$, and $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^3$ the immersed surface with support S introduced in Example 2.8. By Proposition 2.3.(ii), we know that every restriction of φ to an open set on which it is injective is a local parametrization of S; so Proposition 2.6 implies that $T_pS = d\varphi_{(t,\theta)}(\mathbb{R}^2)$ for all $p = \varphi(t, \theta) \in S$. In particular, a basis of the tangent plane at p consists of the vectors

$$\left.\frac{\partial}{\partial t}\right|_p = \frac{\partial \varphi}{\partial t}(t, \theta) = \begin{vmatrix} \alpha'(t)\cos\theta \\ \alpha'(t)\sin\theta \\ \beta'(t) \end{vmatrix}, \qquad \left.\frac{\partial}{\partial \theta}\right|_p = \frac{\partial \varphi}{\partial \theta}(t, \theta) = \begin{vmatrix} -\alpha(t)\sin\theta \\ \alpha(t)\cos\theta \\ 0 \end{vmatrix}.$$

EXAMPLE 2.25. A second degree polynomial p in three variables can always be written in the form $p(x) = x^T\mathbf{A}x + 2b^Tx + c$, where $\mathbf{A} = (a_{ij}) \in M_{3,3}(\mathbb{R})$ is a symmetric matrix, $b \in \mathbb{R}^3$ (we are writing vectors in $\mathbb{R}^3$ as column vectors), and $c \in \mathbb{R}$. In particular, $\nabla p(x) = 2(\mathbf{A}x + b)$. So, if $S \subset \mathbb{R}^3$ is the component of the quadric having equation $p(x) = 0$ that contains the point $x_0 \notin \mathrm{Crit}(p)$, the tangent plane $T_{x_0}S$ to the surface S (see Exercise 2.9) at x_0 is given by

$$T_{x_0}S = \{v \in \mathbb{R}^3 \mid \langle \mathbf{A}x_0 + b, v\rangle = 0\}.$$

For instance, the tangent plane at the point $x_0 = (1, 0, 1)$ to the one-sheeted hyperboloid with equation $x^2+y^2-z^2-1 = 0$ is the plane $\{v = (v_1, v_2, v_3) \in \mathbb{R}^3 \mid v_1 = v_3\}$.

2.4. Tangent vectors and derivations

Definition 2.9 of tangent plane is not completely satisfactory: it strongly depends on the fact that the surface S is contained in $\mathbb{R}^3$, while it would be nice to have a notion of tangent vector intrinsic to S, independent of its embedding in the Euclidean space. In other words, we would like to have a definition of T_pS not as a subspace of $\mathbb{R}^3$, but as an abstract vector space, depending only on S and p. Moreover, since we are dealing with "differential geometry", sooner or later we shall have to find a way to differentiate on a surface.

Surprisingly enough, we may solve both these problems at the same time. The main idea is contained in the following example.

EXAMPLE 2.26. Let $U \subseteq \mathbb{R}^2$ be an open set, and $p \in U$. Then we can associate with each tangent vector $v \in T_pU = \mathbb{R}^2$ a partial derivative:

$$v = (v_1, v_2) \mapsto \left.\frac{\partial}{\partial v}\right|_p = v_1 \left.\frac{\partial}{\partial x_1}\right|_p + v_2 \left.\frac{\partial}{\partial x_2}\right|_p,$$

and all partial derivatives are of this kind. So, in a sense, we may identify T_pU with the set of partial derivatives.

Our aim will then be to find a way for identifying, for general surfaces, tangent vectors with the right kind of partial derivative. To do so, we must first of all understand better which objects we want to differentiate. The key observation is that to differentiate a function in a point it suffices to know its behaviour in a neighborhood of the point; if our goal is just to differentiate at p, two functions that coincide in some neighborhood of p are completely equivalent. This remark suggests the following

DEFINITION 2.11. Let $S \subset \mathbb{R}^3$ be a surface, and $p \in S$. Denote by $\mathcal{F}$ the set of pairs (U, f), where $U \subseteq S$ is an open neighborhood of p in S and $f \in C^\infty(U)$. We define an equivalence relation $\sim$ on $\mathcal{F}$ as follows: $(U, f) \sim (V, g)$ if there exists an open neighborhood $W \subseteq U \cap V$ of p such that $f|_W \equiv g|_W$. The quotient space $C^\infty(p) = \mathcal{F}/\sim$ is the *space (or stalk) of germs of C^∞ functions* at p, and an element $\mathbf{f} \in C^\infty(p)$ is a *germ* at p. An element (U, f) of the equivalence class $\mathbf{f}$ is a *representative* of $\mathbf{f}$. If it is necessary to remind the surface on which we are working, we shall write $C^\infty_S(p)$ rather than $C^\infty(p)$.

REMARK 2.15. If $U \subseteq S$ is an open subset of a surface S and $p \in U$ then we clearly have $C^\infty_U(p) = C^\infty_S(p)$.

So, what we really want to differentiate are germs of C^∞ functions. Before seeing how to do this, note that $C^\infty(p)$ has a natural algebraic structure.

DEFINITION 2.12. An *algebra* over a field $\mathbb{K}$ is a set A equipped with an addition $+$, a multiplication $\cdot$ and a multiplication by scalars $\lambda\cdot$, such that $(A, +, \cdot)$ is a ring, $(A, +, \lambda\cdot)$ is a vector space, and the associative property $(\lambda f)g = \lambda(fg) = f(\lambda g)$ holds, for all $\lambda \in \mathbb{K}$ and f, $g \in A$.

LEMMA 2.2. *Let $S \subset \mathbb{R}^3$ be a surface, $p \in S$, and $\mathbf{f}$, $\mathbf{g} \in C^\infty(p)$ two germs at p. Let also (U_1, f_1), (U_2, f_2) be two representatives of $\mathbf{f}$, and (V_1, g_1), (V_2, g_2) two representatives of $\mathbf{g}$. Then:*

(i) *$(U_1 \cap V_1, f_1 + g_1)$ is equivalent to $(U_2 \cap V_2, f_2 + g_2)$;*
(ii) *$(U_1 \cap V_1, f_1 g_1)$ is equivalent to $(U_2 \cap V_2, f_2 g_2)$;*
(iii) *$(U_1, \lambda f_1)$ is equivalent to $(U_2, \lambda f_2)$ for all $\lambda \in \mathbb{R}$;*
(iv) *$f_1(p) = f_2(p)$.*

PROOF. Let us begin with (i). Since $(U_1, f_1) \sim (U_2, f_2)$, there exists an open neighborhood $W_f \subseteq U_1 \cap U_2$ of p such that $f_1|_{W_f} \equiv f_2|_{W_f}$. Analogously, since $(V_1, g_1) \sim (V_2, g_2)$, there exists an open neighborhood $W_g \subseteq V_1 \cap V_2$ di p such that $g_1|_{W_g} \equiv g_2|_{W_g}$. But then $(f_1 + f_2)|_{W_f \cap W_g} \equiv (g_1 + g_2)|_{W_f \cap W_g}$, and so $(U_1 \cap V_1, f_1 + g_1) \sim (U_2 \cap V_2, f_2 + g_2)$ as $W_f \cap W_g \subseteq U_1 \cap V_1 \cap U_2 \cap V_2$.

The proof of (ii) is analogous, and (iii) and (iv) are straightforward. □

DEFINITION 2.13. Let $\mathbf{f}$, $\mathbf{g} \in C^\infty(p)$ be two germs at a point $p \in S$. We shall denote by $\mathbf{f} + \mathbf{g} \in C^\infty(p)$ the germ represented by $(U \cap V, f + g)$, where (U, f) is an arbitrary representative of $\mathbf{f}$ and (V, g) is an arbitrary representative of $\mathbf{g}$. Analogously, we denote by $\mathbf{fg} \in C^\infty(p)$ the germ represented by $(U \cap V, fg)$, and, given $\lambda \in \mathbb{R}$, by $\lambda\mathbf{f} \in C^\infty(p)$ the germ represented by $(U, \lambda f)$. Lemma 2.2 tells us that these objects are well defined, and it is straightforward (why?) to verify that $C^\infty(p)$ with these operations is an algebra. Finally, for all $\mathbf{f} \in C^\infty(p)$ we define its value $\mathbf{f}(p) \in \mathbb{R}$ in p by setting $\mathbf{f}(p) = f(p)$ for an arbitrary representative (U, f) of $\mathbf{f}$; Lemma 2.2 again implies that $\mathbf{f}(p)$ is well defined.

The fact that the composition of smooth maps is itself a smooth map allows us to compare stalks in different points of different surfaces. Indeed, let $F\colon S_1 \to S_2$ be a C^∞ map between surfaces, and let (V_1, g_1) and (V_2, g_2) be two representatives of a germ $\mathbf{g} \in C^\infty\big(F(p)\big)$. Then it is clear (exercise) that $\big(F^{-1}(V_1), g_1 \circ F\big)$ and $\big(F^{-1}(V_2), g_2 \circ F\big)$ represent the same germ at p, which, then, depends on $\mathbf{g}$ (and on F) only.

Definition 2.14. Let $F\colon S_1 \to S_2$ be a smooth map between surfaces, and take $p \in S_1$. We shall denote by $F_p^*\colon C^\infty_{S_2}\big(F(p)\big) \to C^\infty_{S_1}(p)$ the map associating with a germ $\mathbf{g} \in C^\infty_{S_2}\big(F(p)\big)$ having (V, g) as a representative the germ $F_p^*(\mathbf{g}) \in C^\infty_{S_1}(p)$ having $\big(F^{-1}(V), g \circ F\big)$ as a representative. We shall sometimes write $\mathbf{g} \circ F$ rather than $F_p^*(\mathbf{g})$. It is immediate to see (exercise) that F_p^* is an algebra homomorphism.

Remark 2.16. A very common (and very useful) convention in contemporary mathematics consists in denoting by a star written as a superscript (as in F_p^*) a map associated in a canonical way with a given map, but going in the opposite direction: F is a function from S_1 to S_2, whereas F^* is a function from the germs in S_2 to the germs in S_1. The same convention uses a star as a subscript (as in F_*) to denote an associated map going in the same direction as the given one (see for instance Definitions 2.16 and 2.17 later on).

Lemma 2.3.

(i) *We have* $(\mathrm{id}_S)_p^* = \mathrm{id}$ *for all points p of a surface S.*

(ii) *Let $F\colon S_1 \to S_2$ and $G\colon S_2 \to S_3$ be two C^∞ maps between surfaces. Then $(G \circ F)_p^* = F_p^* \circ G_{F(p)}^*$ for all $p \in S_1$.*

(iii) *If $F\colon S_1 \to S_2$ is a diffeomorphism, then $F_p^*\colon C^\infty\big(F(p)\big) \to C^\infty(p)$ is an algebra isomorphism for all $p \in S_1$. In particular, if $\varphi\colon U \to S$ is a local parametrization with $\varphi(x_o) = p \in S$, then $\varphi_{x_o}^*\colon C^\infty_S(p) \to C^\infty_U(x_o)$ is an algebra isomorphism.*

Proof. (i) Obvious.

(ii) Follows immediately (exercise) from the equality $g \circ (G \circ F) = (g \circ G) \circ F$.

(iii) Indeed $(F^{-1})_{F(p)}^*$ is the inverse of F_p^*, by (i) and (ii). □

Now we can define what we mean by a partial derivative on a surface.

Definition 2.15. Let $S \subset \mathbb{R}^3$ be a surface, and $p \in S$. A *derivation* at p is a $\mathbb{R}$-linear function $D\colon C^\infty(p) \to \mathbb{R}$ satisfying a Leibniz (or product) rule:

$$D(\mathbf{fg}) = \mathbf{f}(p)D(\mathbf{g}) + \mathbf{g}(p)D(\mathbf{f}) \ .$$

It is immediate to verify (exercise) that the set $\mathcal{D}\big(C^\infty(p)\big)$ of derivations of $C^\infty(p)$ is a vector subspace of the dual space (as a vector space) of $C^\infty(p)$.

Example 2.27. Let $U \subset \mathbb{R}^2$ be an open subset of the plane, and $p \in U$. We have already remarked that $T_pU = \mathbb{R}^2$. On the other hand, the partial derivatives at p are clearly derivations of $C^\infty(p)$; so we may introduce a natural linear map $\alpha\colon T_pU \to \mathcal{D}\big(C^\infty(p)\big)$ by setting

$$\alpha(v) = \left.\frac{\partial}{\partial v}\right|_p = v_1 \left.\frac{\partial}{\partial x_1}\right|_p + v_2 \left.\frac{\partial}{\partial x_2}\right|_p \ .$$

The key point here is that the map α is actually an isomorphism between T_pU and $\mathcal{D}\big(C^\infty(p)\big)$. Moreover, we shall show that T_pS and $\mathcal{D}\big(C^\infty_S(p)\big)$ are canonically isomorphic for every surface S and for every $p \in S$, and this fact will provide us with the desired intrinsic characterization of the tangent plane. To prove all this we need one more definition and a lemma.

DEFINITION 2.16. Let $S \subset \mathbb{R}^3$ be a surface, and $p \in S$. Given a local parametrization $\varphi\colon U \to S$ in p with $\varphi(x_o) = p \in S$, define a map

$$\varphi_*\colon \mathcal{D}\big(C^\infty(x_o)\big) \to \mathcal{D}\big(C^\infty(p)\big)$$

by setting $\varphi_*(D) = D \circ \varphi^*_{x_o}$, that is,

$$\varphi_*(D)(\mathbf{f}) = D(\mathbf{f} \circ \varphi)$$

for all $\mathbf{f} \in C^\infty(p)$ and $D \in \mathcal{D}\big(C^\infty(x_o)\big)$. It is immediate to verify (check it!) that $\varphi_*(D)$ is a derivation, since $\varphi^*_{x_o}$ is an algebra isomorphism, and so the image of φ_* is actually contained in $\mathcal{D}\big(C^\infty(p)\big)$. Moreover, it is easy to see (exercise) that φ_* is a vector space isomorphism, with $(\varphi_*)^{-1}(D) = D \circ (\varphi^{-1})^*_p$ as its inverse.

REMARK 2.17. We shall see later on (Remark 2.18) that φ_* can be canonically identified with the differential of the local parametrization.

LEMMA 2.4. *Let $U \subseteq \mathbb{R}^n$ be an open domain star-shaped with respect to $x^o \in \mathbb{R}^n$. Then for all $f \in C^\infty(U)$ there exist $g_1, \dots, g_n \in C^\infty(U)$ such that $g_j(x^o) = \frac{\partial f}{\partial x_j}(x^o)$ and*

$$f(x) = f(x^o) + \sum_{j=1}^n (x_j - x_j^o) g_j(x)$$

for all $x \in U$.

PROOF. We have

$$\begin{aligned} f(x) - f(x^o) &= \int_0^1 \frac{\partial}{\partial t} f\big(x^o + t(x - x^o)\big)\,\mathrm{d}t \\ &= \sum_{j=1}^n (x_j - x_j^o) \int_0^1 \frac{\partial f}{\partial x_j}\big(x^o + t(x - x^o)\big)\,\mathrm{d}t\ , \end{aligned}$$

so it suffices to define

$$g_j(x) = \int_0^1 \frac{\partial f}{\partial x_j}\big(x^o + t(x - x^o)\big)\,\mathrm{d}t\ .$$

and we are done. □

We may now prove the characterization of the tangent plane we promised:

THEOREM 2.3. *Let $S \subset \mathbb{R}^3$ be a surface, and $p \in S$. Then the tangent plane T_pS is canonically isomorphic to the space $\mathcal{D}\big(C^\infty(p)\big)$ of derivations of $C^\infty(p)$.*

PROOF. Let $\varphi\colon U \to S$ be a local parametrization centered at p. Let us begin by writing the following commutative diagram:

$$\begin{tikzcd} T_OU = \mathbb{R}^2 \arrow[r, "\alpha"] \arrow[d, "\mathrm{d}\varphi_O"'] & \mathcal{D}\big(C^\infty(O)\big) \arrow[d, "\varphi_*"] \\ T_pS \arrow[r, "\beta"] & \mathcal{D}\big(C^\infty(p)\big) \end{tikzcd}\ , \tag{21}$$

where α is the map defined in Example 2.27, and $\beta = \varphi_* \circ \alpha \circ (\mathrm{d}\varphi_O)^{-1}$.

We shall proceed in two steps: first of all, we shall show that α is an isomorphism. Since $\mathrm{d}\varphi_O$ and φ_* are isomorphisms, this will imply that β is an isomorphism too. We shall prove next that it is possible to express β in a way independent of φ; so β will be a canonical isomorphism, independent of arbitrary choices, and we shall be done.

Let us prove that α is an isomorphism. As it is obviously linear, it suffices to show that it is injective and surjective. If $v = (v_1, v_2) \in \mathbb{R}^2 = T_O U$, we have

$$v_j = v_j \frac{\partial x_j}{\partial x_j}(O) = \alpha(v)(\mathbf{x}_j)$$

for $j = 1$, 2, where $\mathbf{x}_j$ is the germ at the origin of the coordinate function x_j. Innparticular, if $v_j \neq 0$ we have $\alpha(v)(\mathbf{x}_j) \neq 0$; so $v \neq O$ implies $\alpha(v) \neq O$ and α is injective. To show that it is surjective, take $D \in \mathcal{D}\big(C^\infty(O)\big)$; we claim that $D = \alpha(v)$, where $v = (D\mathbf{x}_1, D\mathbf{x}_2)$. First of all, note that

$$D\mathbf{1} = D(\mathbf{1} \cdot \mathbf{1}) = 2D\mathbf{1} \; ,$$

so $D\mathbf{c} = 0$ for any constant $c \in \mathbb{R}$, where $\mathbf{c}$ is the germ represented by $(\mathbb{R}^2, c)$. Take now an arbitrary $\mathbf{f} \in C^\infty(O)$. By applying Lemma 2.4, we find

$$\begin{aligned} D\mathbf{f} &= D\big(\mathbf{f}(O)\big) + D\big(\mathbf{x}_1\mathbf{g}_1 + \mathbf{x}_2\mathbf{g}_2\big) \\ &= \sum_{j=1}^{2} \big[\mathbf{x}_j(O) D\mathbf{g}_j + \mathbf{g}_j(O) D\mathbf{x}_j\big] = \sum_{j=1}^{2} D\mathbf{x}_j \, \frac{\partial \mathbf{f}}{\partial x_j}(O) = \alpha(v)(\mathbf{f}) \; , \end{aligned} \tag{22}$$

where $v = (D\mathbf{x}_1, D\mathbf{x}_2)$ as claimed, and we are done.

So, α and β are isomorphisms; to complete the proof, we only have to show that β does not depend on φ but only on S and p. Let $v \in T_p S$, and choose a curve $\sigma\colon (-\varepsilon, \varepsilon) \to S$ such that $\sigma(0) = p$ and $\sigma'(0) = v$. We want to show that

$$\beta(v)(\mathbf{f}) = (f \circ \sigma)'(0) \tag{23}$$

for all $\mathbf{f} \in C^\infty(p)$ and any representative $(U, f) \in \mathbf{f}$. If we prove this, we are done: indeed, the left-hand side of (23) does not depend on σ nor on the chosen representative of $\mathbf{f}$, while the right-hand side does not depend on any local parametrization. So β does not depend on φ or on σ, and thus it is the canonical isomorphism we were looking for.

Let us then prove (23). Write $\sigma = \varphi \circ \sigma_o$ as in the proof of Proposition 2.6, so that $v = \mathrm{d}\varphi_O(v^o) = v_1^o \partial_1|_p + v_2^o \partial_2|_p$ and $v^o = (v_1^o, v_2^o) = \sigma_o'(0) \in \mathbb{R}^2$. Then

$$\begin{aligned} \beta(v)(\mathbf{f}) &= \big(\varphi_* \circ \alpha \circ (\mathrm{d}\varphi_O)^{-1}\big)(v)(\mathbf{f}) = (\varphi_* \circ \alpha)(v^o)(\mathbf{f}) \\ &= \alpha(v^o)\big(\varphi_O^*(\mathbf{f})\big) = \alpha(v^o)(\mathbf{f} \circ \varphi) \\ &= v_1^o \frac{\partial (f \circ \varphi)}{\partial x_1}(O) + v_2^o \frac{\partial (f \circ \varphi)}{\partial x_2}(O) \\ &= (\sigma_o')_1(0) \frac{\partial (f \circ \varphi)}{\partial x_1}(O) + (\sigma_o')_2(0) \frac{\partial (f \circ \varphi)}{\partial x_2}(O) \\ &= \big((f \circ \varphi) \circ \sigma_o\big)'(0) = (f \circ \sigma)'(0) \; , \end{aligned} \tag{24}$$

and we are done. □

Remark 2.18. A consequence of diagram (21) is that, as anticipated in Remark 2.17, the map φ_* is the exact analogue of the differential of φ when we interpret tangent planes as spaces of derivations.

From now on, we shall always identify T_pS and $\mathcal{D}\big(C^\infty(p)\big)$ without (almost ever) explicitly mentioning the isomorphism β; a tangent vector will be considered both as a vector of $\mathbb{R}^3$ and as a derivation of the space of germs at p without further remarks.

Remark 2.19. Let $\varphi\colon U \to S$ be a local parametrization centered at $p \in S$, and take a tangent vector $v = v_1\partial_1|_p + v_2\partial_2|_p \in T_pS$. Then (24) tells us that the action of v as a derivation is given by

$$v(\mathbf{f}) = v_1\frac{\partial(f\circ\varphi)}{\partial x_1}(O) + v_2\frac{\partial(f\circ\varphi)}{\partial x_2}(O)\ ,$$

for all germs $\mathbf{f} \in C^\infty(p)$ and all representatives (V,f) of $\mathbf{f}$. In particular,

$$\left.\frac{\partial}{\partial x_j}\right|_p(\mathbf{f}) = \frac{\partial(f\circ\varphi)}{\partial x_j}(O)\ ,$$

a formula which explains the notation introduced in Definition 2.10. As a consequence, for any $p \in \mathbb{R}^2$ we shall always identify the vectors $\vec{e}_1, \vec{e}_2$ of the canonical basis of $\mathbb{R}^2$ with the partial derivatives $\partial/\partial x_1|_p$, $\partial/\partial x_2|_p \in T_p\mathbb{R}^2$.

Remark 2.20. In the previous remark we have described the action of a tangent vector on a germ by expressing the tangent vector in terms of the basis induced by a local parametrization. If, on the other hand, we consider $v = (v_1, v_2, v_3) \in T_pS$ as a vector of $\mathbb{R}^3$, we may describe its action as follows: given $\mathbf{f} \in C^\infty(p)$, choose a representative (V,f) of $\mathbf{f}$ and extend it using Proposition 2.5 to a smooth function $\tilde{f}$ defined in a neighborhood W of p in $\mathbb{R}^3$. Finally, let $\sigma\colon(-\varepsilon,\varepsilon) \to S$ be a curve with $\sigma(0) = p$ and $\sigma'(0) = v$. Then:

$$v(\mathbf{f}) = (f\circ\sigma)'(0) = (\tilde{f}\circ\sigma)'(0) = \sum_{j=1}^{3} v_j\frac{\partial\tilde{f}}{\partial x_j}(p)\ .$$

Warning: while the linear combination in the right-hand side of this formula is well defined and only depends on the tangent vector v and on the germ $\mathbf{f}$, the partial derivatives $\partial\tilde{f}/\partial x_j(p)$ taken on their own depend on the particular extension $\tilde{f}$ and not only on $\mathbf{f}$, and thus they have nothing to do with the surface S.

Remark 2.21. If we have two local parametrizations $\varphi\colon U \to S$ and $\hat{\varphi}\colon \hat{U} \to S$ centered at $p \in S$, we obtain two bases $\{\partial_1, \partial_2\}$ and $\{\hat{\partial}_1, \hat{\partial}_2\}$ of T_pS, where we set $\hat{\partial}_j = \partial\hat{\varphi}/\partial\hat{x}_j(O)$, and $(\hat{x}_1, \hat{x}_2)$ are the coordinates in $\hat{U}$; we want to compute the change of basis matrix. If $h = \hat{\varphi}^{-1}\circ\varphi$ is the change of coordinates, we have $\varphi = \hat{\varphi}\circ h$, and so

$$\begin{aligned}\partial_j &= \frac{\partial\varphi}{\partial x_j}(O) = \frac{\partial\hat{\varphi}}{\partial x_1}\big(h(O)\big)\frac{\partial h_1}{\partial x_j}(O) + \frac{\partial\hat{\varphi}}{\partial x_2}\big(h(O)\big)\frac{\partial h_2}{\partial x_j}(O)\\ &= \frac{\partial\hat{x}_1}{\partial x_j}(O)\hat{\partial}_1 + \frac{\partial\hat{x}_2}{\partial x_j}(O)\hat{\partial}_2\ ,\end{aligned}$$

where, to make the formula easier to remember, we have written $\partial\hat{x}_i/\partial x_j$ rather than $\partial h_i/\partial x_j$. So *the change of basis matrix is the Jacobian matrix of the change of coordinates.*

Remark 2.22. The identification of tangent vectors and derivations only holds when working with functions and local parametrizations of class C^∞. The reason is Lemma 2.4. Indeed, if $f \in C^k(U)$ with $k < \infty$, the same proof provides functions $g_1, \ldots, g_n$ which are in $C^{k-1}(U)$ but a priori might not be in $C^k(U)$, and the computation made in (22) might become meaningless. This is an insurmountable obstacle: in fact, the space of derivations of C^k germs with $1 \le k < \infty$ is of infinite dimension (Exercise 2.34), and thus it cannot be isomorphic to a plane.

The way we have introduced the map φ_*, together with its relation with the usual differential, suggests the following definition of a differential for an arbitrary C^∞ map between surfaces:

Definition 2.17. Let $F\colon S_1 \to S_2$ be a C^∞ map between two surfaces. The *differential* of F at $p \in S_1$ is the linear map $\mathrm{d}F_p\colon T_pS_1 \to T_{F(p)}S_2$ defined by

$$\mathrm{d}F_p(D) = D \circ F_p^*$$

for any derivation $D \in T_pS$ of $C^\infty(p)$. We may also write $(F_*)_p$ instead of $\mathrm{d}F_p$.

It is not difficult to see how the differential looks like when applied to vectors seen as tangent vectors to a curve:

Lemma 2.5. *Let $F\colon S_1 \to S_2$ be a C^∞ map between surfaces, and $p \in S_1$. If $\sigma\colon (-\varepsilon, \varepsilon) \to S_1$ is a curve with $\sigma(0) = p$ and $\sigma'(0) = v$, then*

$$\mathrm{d}F_p(v) = (F \circ \sigma)'(0)\ . \tag{25}$$

Proof. Set $w = (F \circ \sigma)'(0) \in T_{F(p)}S_2$. Using the notation introduced in the proof of Theorem 2.3, we have to show that $\mathrm{d}F_p\bigl(\beta(v)\bigr) = \beta(w)$. But for each $\mathbf{f} \in C^\infty\bigl(F(p)\bigr)$ we have

$$\begin{aligned} \mathrm{d}F_p\bigl(\beta(v)\bigr)(\mathbf{f}) &= \beta(v)\bigl(F_p^*(\mathbf{f})\bigr) = \beta(v)(\mathbf{f} \circ F) \\ &= \bigl((f \circ F) \circ \sigma\bigr)'(0) = \bigl(f \circ (F \circ \sigma)\bigr)'(0) = \beta(w)(\mathbf{f})\ , \end{aligned}$$

where (U, f) is a representative of $\vec{f}$, and we have used (23). □

As for the tangent plane, we then have two different ways to define the differential, each one with its own strengths and weaknesses. Formula (25) underlines the geometric meaning of differential, showing how it acts on tangent vectors to curves; Definition 2.17 highlights instead its algebraic properties, such as the fact that the differential is a linear map between tangent planes, and makes it (far) easier to prove its properties. For instance, we obtain a one-line proof of the following proposition:

Proposition 2.8.

(i) *We have $\mathrm{d}(\mathrm{id}_S)_p = \mathrm{id}$ for every surface S and every $p \in S$.*
(ii) *Let $F\colon S_1 \to S_2$ and $G\colon S_2 \to S_3$ be C^∞ maps between surfaces, and take $p \in S_1$. Then $\mathrm{d}(G \circ F)_p = \mathrm{d}G_{F(p)} \circ \mathrm{d}F_p$.*
(iii) *If $F\colon S_1 \to S_2$ is a diffeomorphism then $\mathrm{d}F_p\colon T_pS_1 \to T_{F(p)}S_2$ is invertible and $(\mathrm{d}F_p)^{-1} = \mathrm{d}(F^{-1})_{F(p)}$ for all $p \in S_1$.*

Proof. It is an immediate consequence of Lemma 2.3 and of the definition of differential. □

Formula (25) also suggests how to define the differential of a C^∞ map defined on a surface but with values in $\mathbb{R}^n$:

Definition 2.18. If $F\colon S\to\mathbb{R}^n$ is a C^∞ map, and $p\in S$, the *differential* $\mathrm{d}F_p\colon T_pS\to\mathbb{R}^n$ of F at p is defined by setting $\mathrm{d}F_p(v)=(F\circ\sigma)'(0)$ for all $v\in T_pS$, where $\sigma\colon(-\varepsilon,\varepsilon)\to S$ is an arbitrary curve in S with $\sigma(0)=p$ and $\sigma'(0)=v$. It is not hard (exercise) to verify that $\mathrm{d}F_p(v)$ only depends on v and not on the curve σ, and that $\mathrm{d}F_p$ is a linear map.

Remark 2.23. In particular, if $f\in C^\infty(S)$ and $v\in T_pS$ then we have

$$\mathrm{d}f_p(v)=(f\circ\sigma)'(0)=v(\mathbf{f})\ ,$$

where $\mathbf{f}$ is the germ represented by (S,f) at p. This formula shows that the action of the differential of functions on tangent vectors is dual to the action of tangent vectors on functions.

Remark 2.24. If $F\colon S\to\mathbb{R}^n$ is of class C^∞ and $\varphi\colon U\to S$ is a local parametrization centered at $p\in S$, it is immediate (why?) to see that

$$\mathrm{d}F_p(\partial_j)=\frac{\partial(F\circ\varphi)}{\partial x_j}(O)$$

for $j=1$, 2, where $\{\partial_1,\partial_2\}$ is the basis of T_pS induced by φ. In particular, if $\tilde\varphi$ is another local parametrization of S centered at p and $F=\tilde\varphi\circ\varphi^{-1}$, then

$$dF_p(\partial_j)=\tilde\partial_j \tag{26}$$

for $j=1$, 2, where $\{\tilde\partial_1,\tilde\partial_2\}$ is the basis of T_pS induced by $\tilde\varphi$.

Let us see now how to express the differential in local coordinates. Given a smooth map $F\colon S_1\to S_2$ between surfaces, choose a local parametrization $\varphi\colon U\to S_1$ centered at $p\in S_1$, and a local parametrization $\hat\varphi\colon\hat U\to S_2$ centered at $F(p)\in S_2$ with $F\bigl(\varphi(U)\bigr)\subseteq\hat\varphi(\hat U)$. By definition, *the expression of F in local coordinates* is the map $\hat F=(\hat F_1,\hat F_2)\colon U\to\hat U$ given by

$$\hat F=\hat\varphi^{-1}\circ F\circ\varphi\ .$$

We want to find the matrix that represents dF_p with respect to the bases $\{\partial_1,\partial_2\}$ of T_pS_1 (induced by φ) and $\{\hat\partial_1,\hat\partial_2\}$ of $T_{F(p)}S_2$ (induced by $\hat\varphi$); recall that the columns of this matrix contain the coordinates with respect to the new basis of the images under $\mathrm{d}F_p$ of the vectors of the old basis. We may proceed in any of two ways: either by using curves, or by using derivations.

A curve in S_1, tangent to ∂_j at p, is $\sigma_j(t)=\varphi(t\vec e_j)$; so

$$\mathrm{d}F_p(\partial_j)=(F\circ\sigma_j)'(0)=\frac{\mathrm{d}}{\mathrm{d}t}\bigl(\hat\varphi\circ\hat F(t\vec e_j)\bigr)\Big|_{t=0}=\frac{\partial\hat F_1}{\partial x_j}(O)\hat\partial_1+\frac{\partial\hat F_2}{\partial x_j}(O)\hat\partial_2\ .$$

Hence, *the matrix that represents* $\mathrm{d}F_p$ *with respect to the bases induced by two local parametrizations is exactly the Jacobian matrix of the expression $\hat F$ of F in local coordinates.* In particular, the differential as we have defined it really is a generalization to surfaces of the usual differential of C^∞ maps between open subsets of the plane.

Let us now get again the same result by using derivations. We want to find $a_{ij}\in\mathbb{R}$ such that $\mathrm{d}F_p(\partial_j)=a_{1j}\hat\partial_1+a_{2j}\hat\partial_2$ for $j=1$, 2. If we set $\hat\varphi^{-1}=(\hat x_1,\hat x_2)$, it is immediate to verify that

$$\hat\partial_h(\hat{\mathbf{x}}_k)=\delta_{hk}=\begin{cases}1 & \text{if } h=k\ ,\\ 0 & \text{if } h\neq k\ ,\end{cases}$$

where $\hat{\mathbf{x}}_k$ is the germ at p of the function $\hat{x}_k$. Hence,

$$a_{ij} = \mathrm{d}F_p(\partial_j)(\hat{\mathbf{x}}_i) = \partial_j\bigl(F_p^*(\hat{\mathbf{x}}_i)\bigr) = \frac{\partial(\hat{x}_i \circ F \circ \varphi)}{\partial x_j}(O) = \frac{\partial \hat{F}_i}{\partial x_j}(O)\ ,$$

in accord with what we have already obtained.

Remark 2.25. Warning: the matrix representing the differential of a map between surfaces is a 2×2 matrix (and not a 3×3, or 3×2 or 2×3 matrix), because tangent planes have dimension 2.

We conclude this chapter remarking that the fact that the differential of a map between surfaces is represented by the Jacobian matrix of the map expressed in local coordinates allows us to easily transfer to surfaces classical calculus results. For instance, here is the inverse function theorem (for other results of this kind, see Exercises 2.19, 2.30, and 2.22):

Corollary 2.3. *Let $F\colon S_1 \to S_2$ be a smooth map between surfaces, and $p \in S_1$ a point such that $\mathrm{d}F_p\colon T_pS_1 \to T_{F(p)}S_2$ is an isomorphism. Then there exist a neighborhood $V \subseteq S_1$ of p and a neighborhood $\hat{V} \subseteq S_2$ of $F(p)$ such that the restriction $F|_V\colon V \to \hat{V}$ is a diffeomorphism.*

Proof. Let $\varphi\colon U \to S_1$ be a local parametrization at p, and $\hat{\varphi}\colon \hat{U} \to S_2$ a local parametrization at $F(p)$ with $F\bigl(\varphi(U)\bigr) \subseteq \hat{\varphi}(\hat{U})$. Then the assertion immediately follows (why?) from the classical inverse function theorem (Theorem 2.1) applied to $\hat{\varphi}^{-1} \circ F \circ \varphi$. $\square$

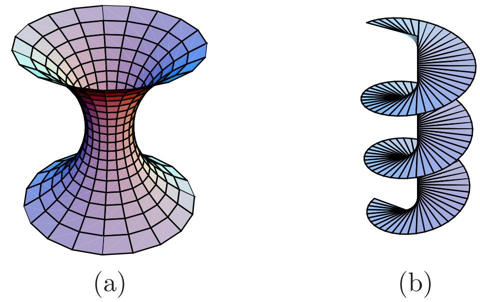


FIGURE 7. (**a**) a catenoid; (**b**) a helicoid

Guided problems

DEFINITION 2.P.1. The *catenoid* is a surface of revolution having a catenary (see Example 1.23) as its generatrix, and axis disjoint from the support of the catenary; see Fig. 7.(a).

PROBLEM 2.1. Let $\sigma\colon \mathbb{R} \to \mathbb{R}^3$ be the parametrization $\sigma(v) = (a \cosh\, v, 0, av)$ of a catenary, and let S be the catenoid obtained by rotating this catenary around the z-axis.

(i) Determine an immersed surface whose support is the catenoid S.
(ii) Determine for each point p of S a basis of the tangent plane T_pS.

SOLUTION. Proceeding as in Example 2.8 we find that an immersed surface $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^3$ having the catenoid as support is

$$\varphi(u, v) = (a \cosh\, v \cos\, u, a \cosh\, v \sin\, u, av)\ .$$

Since σ is a global parametrization of a regular curve whose support does not meet the z-axis, the catenoid is a regular surface. In particular, for all $(u_0, v_0) \in \mathbb{R}^2$ the restriction of φ to a neighborhood of (u_0, v_0) is a local parametrization of S, and so in the point $p = \varphi(u_0, v_0)$ of the catenoid a basis of the tangent plane is given by

$$\begin{aligned} \partial_1|_p &= \frac{\partial\varphi}{\partial u}(u_0, v_0) = \begin{vmatrix} -a \cosh v_0 \sin u_0 \\ a \cosh v_0 \cos u_0 \\ 0 \end{vmatrix}\ , \\ \partial_2|_p &= \frac{\partial\varphi}{\partial v}(u_0, v_0) = \begin{vmatrix} a \sinh\, v_0 \cos\, u_0 \\ a \sinh\, v_0 \sin\, u_0 \\ a \end{vmatrix}\ . \end{aligned}$$

□

DEFINITION 2.P.2. Given a circular helix in $\mathbb{R}^3$, the union of the straight lines issuing from a point of the helix and intersecting orthogonally the axis of the helix is the *helicoid* associated with the given helix; see Fig. 7.(b).

PROBLEM 2.2. Given $a \neq 0$, let $\sigma\colon \mathbb{R} \to \mathbb{R}^3$ be the circular helix parametrized by $\sigma(u) = (\cos\, u, \sin\, u, au)$.

(i) Prove that the helicoid associated with σ is the support of the map $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$\varphi(u,v) = (v\cos u, v\sin u, au) .$$

(ii) Show that φ is a global parametrization and that the helicoid is a regular surface.

(iii) Determine, for every point of the helicoid, a basis of the tangent plane.

Solution. (i) Indeed the straight line issuing from a point $(x_0, y_0, z_0) \in \mathbb{R}^3$ and intersecting orthogonally the z-axis is parametrized by $v \mapsto (vx_0, vy_0, z_0)$.

(ii) The map φ is clearly of class C^∞. Moreover, its differential is injective in every point; indeed, $\frac{\partial\varphi}{\partial u} = (-v\,\sin\,u, v\,\cos\,u, a)$ and $\frac{\partial\varphi}{\partial v} = (\cos\,u, \sin\,u, 0)$, and so

$$\frac{\partial\varphi}{\partial u} \wedge \frac{\partial\varphi}{\partial v} = (-a\,\sin u, a\,\cos\,u, -v)$$

has absolute value $\sqrt{a^2+v^2}$ nowhere zero. Finally, φ is injective and is a homeomorphism with its image. Indeed, the continuous inverse can be constructed as follows: if $(x,y,z) = \varphi(u,v)$, then $u = z/a$ and $v = x/\cos(z/a)$, or $v = y/\sin(z/a)$ if $\cos(z/a) = 0$.

(iii) In the point $p = \varphi(u_0, v_0)$ of the helicoid a basis of the tangent plane is given by

$$\partial_1|_p = \frac{\partial\varphi}{\partial u}(u_0,v_0) = \begin{vmatrix} -v_0\,\sin u_0 \\ v_0\,\cos u_0 \\ a \end{vmatrix} , \quad \partial_2|_p = \frac{\partial\varphi}{\partial v}(u_0,v_0) = \begin{vmatrix} \cos u_0 \\ \sin u_0 \\ 0 \end{vmatrix} .$$

□

Definition 2.P.3. Let $H \subset \mathbb{R}^3$ be a plane, $\ell \subset \mathbb{R}^3$ a straight line not contained in H, and $C \subseteq H$ a subset of H. The *cylinder* with *generatrix* C and *directrix* ℓ is the subset of $\mathbb{R}^3$ consisting of the lines parallel to ℓ issuing from the points of C. If ℓ is orthogonal to H, the cylinder is said to be *right*.

Problem 2.3. Let $C \subset \mathbb{R}^2$ be the support of a Jordan curve (or open arc) of class C^∞ contained in the xy-plane, and $\ell \subset \mathbb{R}^3$ a straight line transversal to the xy-plane. Denote by $S \subset \mathbb{R}^3$ the cylinder having C as generatrix and ℓ as directrix.

(i) Show that S is a regular surface.

(ii) Determine an atlas for S when ℓ is the z-axis and C is the circle of equation $x^2 + y^2 = 1$ in the plane $H = \{z = 0\}$.

(iii) If S is as in (ii), show that the map $G\colon \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$G(x,y,z) = (\mathrm{e}^z x, \mathrm{e}^z y)$$

induces a diffeomorphism $G|_S\colon S \to \mathbb{R}^2 \setminus \{(0,0)\}$.

Solution. (i) Let $\vec{v}$ be a versor parallel to ℓ, denote by H the xy-plane, and let $\sigma\colon \mathbb{R} \to C$ be a global or periodic parametrization of C (see Example 2.8). A point $p = (x,y,z)$ belongs to S if and only if there exist a point $p_0 \in C$ and a real number v such that $p = p_0 + v\vec{v}$. Define then $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^3$ by setting

$$\varphi(t,v) = \sigma(t) + v\vec{v} .$$

Since $\partial\varphi/\partial t(t,v) = \sigma'(t) \in H$ and $\partial\varphi/\partial v(t,v) = \vec{v}$, the differential of φ has rank 2 everywhere, and so φ is an immersed surface with support S.

If C is an open Jordan arc, then σ is a homeomorphism with its image, so we obtain a continuous inverse of φ as follows: if $p = \varphi(t, v)$ then

$$v = \frac{\langle p, \vec{w} \rangle}{\langle \vec{v}, \vec{w} \rangle} \quad \text{and} \quad t = \sigma^{-1}\bigl(p - v\vec{v}\bigr) ,$$

where $\vec{w}$ is a versor orthogonal to H, and $\langle \vec{v}, \vec{w} \rangle \neq 0$ because $\vec{v}$ is transversal to H. So in this case φ is a global parametrization of the regular surface S.

If C is a Jordan curve, the same argument shows that if (a, b) is an interval where σ is a homeomorphism with its image then φ restricted to $(a, b) \times \mathbb{R}$ is a homeomorphism with its image; so, as seen for surfaces of revolution, it turns out that S is a regular surface with an atlas consisting of two charts, obtained by restricting φ to suitable open subsets of the plane.

(ii) In (i) we already constructed an atlas; let us find another one. This particular cylinder is the level surface $f^{-1}(0)$ of the function $f\colon \mathbb{R}^3 \to \mathbb{R}$ given by $f(x, y, z) = x^2 + y^2 - 1$; note that 0 is a regular value of f because the gradient $\nabla f = (2x, 2y, 0)$ of f is nowhere zero on S. We shall find an atlas for S by following the proof of Proposition 2.1, where it is shown that maps of the form $\varphi(u, v) = \bigl(u, g(u, v), v\bigr)$ with g solving the equation $f\bigl(u, g(u, v), v\bigr) = 0$ are local parametrizations at points $p_0 \in S$ where $\partial f/\partial y(p_0) \neq 0$, and that maps of the form $\varphi(u, v) = \bigl(g(u, v), u, v\bigr)$ with g solving the equation $f\bigl(g(u, v), u, v\bigr) = 0$ are local parametrizations at points $p_0 \in S$ where $\partial f/\partial x(p_0) \neq 0$.

In our case, $\nabla f = (2x, 2y, 0)$. So if $y_0 \neq 0$ we must solve the equation $u^2 + g(u, v)^2 - 1 = 0$; therefore $g(u, v) = \pm\sqrt{1 - u^2}$, and setting

$$U = \{(u, v) \in \mathbb{R}^2 \mid -1 < u < 1\}$$

we get the parametrizations φ_+, $\varphi_-\colon U \to \mathbb{R}^3$ at points in $S \cap \{y \neq 0\}$ by setting

$$\varphi_+(u, v) = (u, \sqrt{1 - u^2}, v) , \quad \varphi_-(u, v) = (u, -\sqrt{1 - u^2}, v) .$$

Analogously, we construct the local parametrizations ψ_+, $\psi_-\colon U \to \mathbb{R}^3$ at points in $S \cap \{x \neq 0\}$ by setting

$$\psi_+(u, v) = (\sqrt{1 - u^2}, u, v) , \quad \psi_-(u, v) = (-\sqrt{1 - u^2}, u, v) .$$

It is then easy to see that $\{\varphi_+, \varphi_-, \psi_+, \psi_-\}$ is an atlas of S, because every point of S is contained in the image of at least one of them.

(iii) The map $G|_S$ is the restriction to S of the map G which is of class C^∞ on the whole $\mathbb{R}^3$, so it is of class C^∞ on S. So, to prove that $G|_S$ is a diffeomorphism it suffices to find a map $H\colon \mathbb{R}^2 \setminus \{(0, 0)\} \to S$ of class C^∞ that is the inverse of $G|_S$. First of all, note that the image of $G|_S$ lies in $\mathbb{R}^2 \setminus \{(0, 0)\}$. Moreover, for all a, $b \in \mathbb{R}$ with $a^2 + b^2 = 1$ the restriction of $G|_S$ to the straight line $\{(a, b, v) \in S \mid v \in \mathbb{R}\} \subset S$ is a bijection with the half-line $\{\mathrm{e}^v(a, b) \mid v \in \mathbb{R}\} \subset \mathbb{R}^2 \setminus \{(0, 0)\}$. So $G|_S$ is a bijection between S and $\mathbb{R}^2 \setminus \{(0, 0)\}$, and the inverse H we are looking for is given by

$$H(a, b) = \left(\frac{a}{\sqrt{a^2 + b^2}}, \frac{b}{\sqrt{a^2 + b^2}}, \log\sqrt{a^2 + b^2} \right) .$$

Note that H is of class C^∞, since it is a C^∞ map from $\mathbb{R}^2 \setminus \{(0, 0)\}$ to $\mathbb{R}^3$ having S as its image. □

Problem 2.4. As in Example 2.25, write a quadratic polynomial p in three variables in the form $p(x) = x^T \mathbf{A} x + 2b^T x + c$, where $\mathbf{A} = (a_{ij}) \in M_{3,3}(\mathbb{R})$ is a symmetric matrix, $b \in \mathbb{R}^3$ (we are writing the elements of $\mathbb{R}^3$ as column vectors), and $c \in \mathbb{R}$. Let now S be the quadric in $\mathbb{R}^3$ defined by the equation $p(x) = 0$. Remember that the quadric S is said to be *central* if the linear system $\mathbf{A}x + b = O$ has a solution (called *center* of the quadric), and is a *paraboloid* otherwise (see [**2**, p. 149]).

(i) Prove that (the connected components of) the paraboloids and the central quadrics not containing any of their centers are regular surfaces.

(ii) Given the symmetric matrix

$$\mathbf{B} = \begin{pmatrix} \mathbf{A} & b \\ b^T & c \end{pmatrix} \in M_{4,4}(\mathbb{R}) ,$$

show that a point $x \in \mathbb{R}^3$ belongs to the quadric if and only if

$$\begin{pmatrix} x^T & 1 \end{pmatrix} \mathbf{B} \begin{pmatrix} x \\ 1 \end{pmatrix} = 0 . \tag{27}$$

(iii) Prove that if $\det \mathbf{B} \neq 0$ then the connected components of the quadric S are (either empty or) regular surfaces.

(iv) Show that if S is a central quadric containing one of its centers, then its components are regular surfaces if and only if S is a plane if and only if $\operatorname{rg} \mathbf{A} = 1$.

Solution. (i) In Example 2.25 we saw that $\nabla p(x) = 2(\mathbf{A}x + b)$, so the critical points of f are exactly the centers of S. So, if S is a paraboloid or it does not contain its centers then 0 is a regular value of p, and the components of $S = p^{-1}(0)$ (if non-empty) are regular surfaces by Proposition 2.1.

(ii) The product in the left-hand side of (27) is exactly equal to $p(x)$.

(iii) Assume by contradiction that S is a central quadric containing a center x_0. From $p(x_0) = 0$ and $\mathbf{A}x_0 + b = O$ we immediately deduce (because $x_0^T b = b^T x_0$) that $\begin{pmatrix} x_0 \\ 1 \end{pmatrix}$ is a non-zero element of the kernel of $\mathbf{B}$, and hence $\det \mathbf{B} = 0$. The assertion then follows from (i).

(iv) Suppose that $x_0 \in S$ is a center of S. Since the centers of S are exactly the critical points of p, the property of containing one of its own centers is preserved under translations or linear transformations on $\mathbb{R}^3$; hence, up to a translation, we may assume without loss of generality that $x_0 = O$. Now, the origin is a center if and only if $b = O$, and it belongs to S if and only if $c = O$. This means that $O \in S$ is a center of S if and only if $p(x) = x^T \mathbf{A} x$. By Sylvester's law of inertia (see [**2**, Vol. II, Theorem 13.4.7, p. 98]), we only have the following cases:

(a) if $\det \mathbf{A} \neq 0$, then up to a linear transformation we may assume that $p(x) = x_1^2 + x_2^2 \pm x_3^2$, so either S is a single point or it is a two-sheeted cone, and in both cases it is not a regular surface;

(b) if $\operatorname{rg} \mathbf{A} = 2$, then up to a linear transformation we may assume that $p(x) = x_1^2 \pm x_2^2$, so either S is a straight line or it is the union of two incident planes, and in both cases it is not a regular surface;

(c) if $\operatorname{rg} \mathbf{A} = 1$, then up to a linear transformation we have $p(x) = x_1^2$, and so S is a plane. $\square$

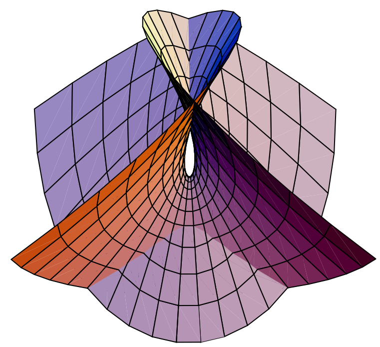


FIGURE 8. Enneper's surface

Exercises

IMMERSED SURFACES AND REGULAR SURFACES

2.1. Show that the map $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$\varphi(u,v) = \left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2 \right).$$

is an injective immersed surface (*Enneper's surface*; see Fig. 8). Is it a homeomorphism with its image as well?

2.2. Prove that the map $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$\varphi(u,v) = \left(\frac{u+v}{2}, \frac{u-v}{2}, uv \right)$$

is a global parametrization of the one-sheeted hyperboloid, and describe its coordinate curves, $v \mapsto \varphi(u_0, v)$ with u_0 fixed and $u \mapsto \varphi(u, v_0)$, with v_0 fixed.

2.3. Let $U = \{(u,v) \in \mathbb{R}^2 \mid u > 0\}$. Show that the map $\varphi\colon U \to \mathbb{R}^3$ given by $\varphi(u,v) = (u + v\cos u, u^2 + v \sin u, u^3)$ is an immersed surface.

2.4. Let $S^2 \subset \mathbb{R}^3$ be the sphere of equation $x^2 + y^2 + z^2 = 1$, denote by N the point of coordinates $(0,0,1)$, and let H be the plane of equation $z = 0$, which we shall identify with $\mathbb{R}^2$ by the projection $(u,v,0) \mapsto (u,v)$. The *stereographic projection* $\pi_N\colon S^2 \setminus \{N\} \to \mathbb{R}^2$ from the point N onto the plane H maps $p = (x,y,z) \in S^2 \setminus \{N\}$ to the intersection point $\pi_N(p)$ between H and the line joining N and p.

(i) Show that the map π_N is bijective and continuous, with continuous inverse $\pi_N^{-1}\colon \mathbb{R}^2 \to S \setminus \{N\}$ given by

$$\pi_N^{-1}(u,v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1} \right).$$

(ii) Show that π_N^{-1} is a local parametrization of S^2.

(iii) Determine, in an analogous way, the stereographic projection π_S of S^2 from the point $S = (0, 0, -1)$ onto the plane H.
(iv) Show that $\{\pi_N^{-1}, \pi_S^{-1}\}$ is an atlas for S^2 consisting of two charts.

2.5. Let $S \subset \mathbb{R}^3$ be a connected subset of $\mathbb{R}^3$ such that there exists a family $\{S_\alpha\}$ of surfaces with $S = \bigcup_\alpha S_\alpha$ and such that every S_α is open in S. Prove that S is a surface.

2.6. Find an atlas for the ellipsoid of equation $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$ (see also Example 2.11).

2.7. Consider a Jordan curve C of class C^∞ contained in a plane $H \subset \mathbb{R}^3$, take a straight line $\ell \subset H$ not containing C, and suppose that C is symmetric with respect to ℓ (that is, $\rho(C) = C$, where $\rho\colon H \to H$ is the reflection with respect to ℓ). Prove that the set obtained by rotating C around ℓ is a regular surface. In particular, this shows again that the sphere is a regular surface.

2.8. Prove that the set of critical points of a map $F\colon U \to \mathbb{R}^m$ of class C^∞, where $U \subset \mathbb{R}^n$ is open, is a closed subset of U.

2.9. Let $V \subseteq \mathbb{R}^3$ be an open subset and $f \in C^\infty(V)$. Prove that for all $a \in \mathbb{R}$ the connected components of the set $f^{-1}(a) \setminus \mathrm{Crit}(f)$ are regular surfaces. Deduce that each component of complement of the vertex in a double-sheeted cone is a regular surface.

2.10. Prove using Proposition 2.1 that the torus of equation

$$z^2 = r^2 - (\sqrt{x^2 + y^2} - a)^2,$$

obtained by rotating the circle with radius $r < a$ and center $(a, 0, 0)$ around the z-axis is a regular surface.

2.11. Let $S \subset \mathbb{R}^3$ be a subset such that for all $p \in S$ there exists an open neighborhood W of p in $\mathbb{R}^3$ such that $W \cap S$ is a graph with respect to one of the three coordinate planes. Prove that S is a regular surface.

2.12. Show that if $\sigma\colon I \to U \subset \mathbb{R}^2$ is the parametrization of a regular C^∞ curve whose support is contained in an open set $U \subset \mathbb{R}^2$, and if $\varphi\colon U \to S$ is a local parametrization of a surface S then the composition $\varphi \circ \sigma$ parametrizes a C^∞ curve in S.

2.13. Prove that the set $S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^3 = 1\}$ is a regular surface, and find an atlas for it.

2.14. Let $\varphi\colon \mathbb{R} \times (0, \pi) \to S^2$ be the immersed surface given by

$$\varphi(u, v) = (\cos u \, \sin v, \sin u \, \sin v, \cos v) \; ,$$

and let $\sigma\colon (0, 1) \to S^2$ be the curve defined by $\sigma(t) = \varphi(\log t, 2 \arctan t)$. Show that the tangent vector to σ at $\sigma(t)$ forms a constant angle of $\pi/4$ with the tangent vector to the meridian passing through $\sigma(t)$, where the meridians are characterized by the condition $u = \text{const}$.

2.15. Show that the surface $S_1 \subset \mathbb{R}^3$ of equation $x^2 + y^2 z^2 = 1$ is not compact, while the surface $S_2 \subset \mathbb{R}^3$ of equation $x^2 + y^4 + z^6 = 1$ is compact.

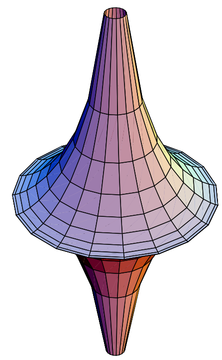


FIGURE 9. The pseudosphere

2.16. Consider a map $f: U \to \mathbb{R}^3$, defined on an open subset $U \subset \mathbb{R}^2$ and of class C^∞, and let $\varphi: \mathbb{R}^2 \to R^3$ be given by $\varphi(u,v) = (u, v, f(u,v))$. Prove that φ is a diffeomorphism between U and $S = \varphi(U)$.

2.17. Show that, for any real numbers a, b, $c > 0$, the following maps are local parametrizations of quadrics in $\mathbb{R}^3$, with equations analogous to those given in Example 2.12:

$\varphi_1(u,v) = (a \sin u \cos v, b \sin u \sin v, c \cos u)$	ellipsoid,
$\varphi_2(u,v) = (a \sinh u \cos v, b \sinh u \sin v, c \cosh u)$	two-sheeted hyperboloid,
$\varphi_3(u,v) = (a \sinh u \sinh v, b \sinh u \cosh v, c \sinh u)$	one-sheeted hyperboloid,
$\varphi_4(u,v) = (au \cos v, bu \sin v, u^2)$	elliptic paraboloid,
$\varphi_5(u,v) = (au \cosh v, bu \sinh v, u^2)$	hyperbolic paraboloid.

Is it possible to choose a, b, c in such a way that the surface is a surface of revolution with respect to one of the coordinate axes? Consider each case separately.

2.18. Let $S \subset \mathbb{R}^3$ be the set (called *pseudosphere*) obtained by rotating around the z-axis the support of the tractrix $\sigma: (0, \pi) \to \mathbb{R}^3$ given by

$$\sigma(t) = \big(\sin t, 0, \cos t + \log \tan(t/2)\big) ;$$

see Fig. 9. Denote by $H \subset \mathbb{R}^3$ the plane $\{z = 0\}$. Prove that S is not a regular surface, whereas each connected component of $S \setminus H$ is; see Example 3.37 and Problem 3.8.

SMOOTH FUNCTIONS

2.19. Let $S \subset \mathbb{R}^3$ be a surface. Prove that if $p \in S$ is a local minimum or a local maximum of a function $f \in C^\infty(S)$ then $\mathrm{d}f_p \equiv 0$.

2.20. Define the notions of a C^∞ map from an open subset of $\mathbb{R}^n$ to a surface, and of a C^∞ map from a surface to an Euclidean space $\mathbb{R}^m$.

2.21. Let $S \subset \mathbb{R}^3$ be a surface, and $p \in S$. Prove that there exists an open set $W \subseteq \mathbb{R}^3$ of p in $\mathbb{R}^3$, a function $f \in C^\infty(W)$ and a regular value $a \in \mathbb{R}$ of f such that $S \cap W = f^{-1}(a)$.

2.22. Given a surface $S \subset \mathbb{R}^3$, take a function $f \in C^\infty(S)$ and a regular value $a \in \mathbb{R}$ of f, in the sense that $\mathrm{d}f_p \not\equiv O$ for all $p \in f^{-1}(a)$. Prove that $f^{-1}(a)$ is locally the support of a simple curve of class C^∞.

2.23. Let $C \subset \mathbb{R}^2$ be the support of a Jordan curve (or open arc) of class C^∞ contained in the half-plane $\{x > 0\}$. Identify $\mathbb{R}^2$ with the xz-plane in $\mathbb{R}^3$, and let S be the set obtained by rotating C around the z-axis, which we shall denote by ℓ.

(i) Let $\Phi\colon \mathbb{R}^+ \times \mathbb{R} \times S^1 \to \mathbb{R}^3$ be given by $\Phi\bigl(x, z, (s,t)\bigr) = (xs, xt, z)$ for all $x > 0$, $z \in \mathbb{R}$ and $(s,t) \in S^1$. Prove that Φ is a homeomorphism between $\mathbb{R}^+ \times \mathbb{R} \times S^1$ and $\mathbb{R}^3 \setminus \ell$, and deduce that S is homeomorphic to $C \times S^1$.

(ii) Let $\Psi\colon \mathbb{R}^2 \to C \times S^1$ be given by $\Psi(t,\theta) = \bigl(\sigma(t), (\cos\theta, \sin\theta)\bigr)$, where $\sigma\colon \mathbb{R} \to \mathbb{R}^2$ is a global or periodic parametrization of C, and let $I \subseteq \mathbb{R}$ be an open interval where σ is injective. Prove that $\Psi|_{I \times (\theta_0, \theta_0 + 2\pi)}$ is a homeomorphism with its image for all $\theta_0 \in \mathbb{R}$.

(iii) Use (i) and (ii) to prove that S is a regular surface.

TANGENT PLANE

2.24. Let $S \subset \mathbb{R}^3$ be a surface, $p \in S$ and $\{v_1, v_2\}$ a basis of T_pS. Prove that there is a local parametrization $\varphi\colon U \to S$ centered at p such that $\partial_1|_p = v_1$ and $\partial_2|_p = v_2$.

2.25. Given an open set $W \subseteq \mathbb{R}^3$ and a function $f \in C^\infty(W)$, take $a \in \mathbb{R}$ and let S be a connected component of $f^{-1}(a) \setminus \mathrm{Crit}(f)$. Prove that for all $p \in S$ the tangent plane T_pS coincides with the subspace of $\mathbb{R}^3$ orthogonal to $\nabla f(p)$.

2.26. Show that the tangent plane at a point $p = (x_0, y_0, z_0)$ of a level surface $f(x,y,z) = 0$ corresponding to the regular value 0 of a C^∞ function $f\colon \mathbb{R}^3 \to \mathbb{R}$ is given by the equation

$$\frac{\partial f}{\partial x}(p)\, x + \frac{\partial f}{\partial y}(p)\, y + \frac{\partial f}{\partial z}(p)\, z = 0\ ,$$

while the equation of the affine tangent plane, parallel to the tangent plane and passing through p, is given by

$$\frac{\partial f}{\partial x}(p)\,(x - x_0) + \frac{\partial f}{\partial y}(p)\,(y - y_0) + \frac{\partial f}{\partial z}(p)\,(z - z_0) = 0\ .$$

2.27. Determine the tangent plane at every point of the hyperbolic paraboloid with global parametrization $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^3$ given by $\varphi(u,v) = (u, v, u^2 - v^2)$.

2.28. Let $S \subset \mathbb{R}^3$ be a surface, and $p \in S$. Prove that

$$\mathfrak{m} = \{\mathbf{f} \in C^\infty(p) \mid \mathbf{f}(p) = 0\}$$

is the unique maximal ideal of $C^\infty(p)$, and that T_pS is canonically isomorphic to the dual (as vector space) of $\mathfrak{m}/\mathfrak{m}^2$.

2.29. Let $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^3$ be given by $\varphi(u,v) = (u, v^3, u - v)$, and let $\sigma\colon \mathbb{R} \to \mathbb{R}^3$ be the curve parametrized by $\sigma(t) = (3t, t^6, 3t - t^2)$.

(i) Prove that $S = \varphi(\mathbb{R}^2)$ is a regular surface.

(ii) Show that σ is regular and has support contained in S.

(iii) Determine the curve $\sigma_o\colon \mathbb{R} \to \mathbb{R}^2$ such that $\sigma = \varphi \circ \sigma_o$.

(iv) Write the tangent versor to σ at $O=\sigma(0)$ as a combination of the basis ∂_1 and ∂_2 of the tangent plane T_OS to S at O induced by φ.

2.30. Let $S\subset\mathbb{R}^3$ be a surface. Prove that a C^∞ function $F\colon S\to\mathbb{R}^m$ satisfies $\mathrm{d}F_p\equiv O$ for all $p\in S$ if and only if F is constant.

2.31. Two surfaces S_1, $S_2\subset\mathbb{R}^3$ are *transversal* if $S_1\cap S_2\neq\varnothing$ and $T_pS_1\neq T_pS_2$ for all $p\in S_1\cap S_2$. Prove that if S_1 and S_2 are transversal, then each component of $S_1\cap S_2$ is locally the support of a simple regular C^∞ curve.

2.32. Let $H\subset\mathbb{R}^3$ be a plane, $\ell\subset\mathbb{R}^3$ a straight line not contained in H, and $C\subseteq H$ a subset of H. Consider the cylinder S with generatrix C and directrix ℓ. Show that the tangent plane to S is constant at the points of S belonging to a line parallel to the directrix ℓ.

2.33. Let $\varphi\colon\mathbb{R}^3\to\mathbb{R}^3$ be the global parametrization of the regular surface $S=\varphi(\mathbb{R}^2)$ given by $\varphi(u,v)=(u-v,u^2+v,u-v^3)$. Determine the Cartesian equation of the tangent plane to S at $p=(0,2,0)=\varphi(1,1)$.

2.34. Prove that the space of derivations of germs of C^k functions has infinite dimension if $1\leq k<\infty$.

2.35. Prove that the space of derivations of germs of continuous functions consists of just the zero derivation.

Definition 2.E.1. Let S_1 and S_2 be regular surfaces in $\mathbb{R}^3$ having in common a point p. We say that S_1 and S_2 have *contact of order at least* 1 at p if there exist parametrizations φ_1 of S_1 and φ_2 of S_2, centered at p, such that $\partial\varphi_1/\partial u(O)=\partial\varphi_2/\partial u(O)$ and $\partial\varphi_1/\partial v(O)=\partial\varphi_2/\partial v(O)$. Moreover, the surfaces are said to have *contact of order at least* 2 at p if there is a pair of parametrizations centered at p for which all the second order partial derivatives coincide too.

2.36. Show that two surfaces have contact of order at least 1 at p if and only if they have the same tangent plane at p. In particular, the tangent plane at p is the only plane having contact of order at least 1 with a regular surface.

2.37. Show that if the intersection between a regular surface S and a plane H consists of a single point p_0, then H is the tangent plane to S at p_0.

SMOOTH MAPS BETWEEN SURFACES

2.38. Prove that a smooth map between surfaces is necessarily continuous.

2.39. Let $F\colon S_1\to S_2$ be a map between surfaces, and $p\in S_1$. Prove that if there exist a local parametrization $\varphi_1\colon U_1\to S_1$ at p and a local parametrization $\varphi_2\colon U_2\to S_2$ at $F(p)$ such that $\varphi_2^{-1}\circ F\circ\varphi_1$ is of class C^∞ in a neighborhood of $\varphi_1^{-1}(p)$, then $\psi_2^{-1}\circ F\circ\psi_1$ is of class C^∞ in a neighborhood of $\psi_1^{-1}(p)$ for any local parametrization $\psi_1\colon V_1\to S_1$ of S at p and any local parametrization $\psi_2\colon V_2\to S_2$ of S at $F(p)$.

2.40. Show that the relation "S_1 is diffeomorphic to S_2" is an equivalence relation on the set of regular surfaces in $\mathbb{R}^3$.

2.41. Let $F\colon S^2\to\mathbb{R}^3$ be defined by

$$F(p)=(x^2-y^2,xy,yz)$$

for all $p=(x,y,z)\in S^2$. Set $N=(0,0,1)$ and $E=(1,0,0)$.

(i) Prove that $\mathrm{d}F_p$ is injective on T_pS^2 for all $p \in S^2 \setminus \{\pm N, \pm E\}$.

(ii) Prove that $S_1 = F(S^2 \setminus \{y = 0\})$ is a regular surface, and find a basis of T_qS_1 for all $q \in S_1$.

(iii) Given $p = (0, 1, 0)$ and $q = F(p)$, choose a local parametrization of S^2 at p, a local parametrization of S_1 at q, and write the matrix representing the linear map $\mathrm{d}F_p \colon T_pS^2 \to T_qS_1$ with respect to the bases of T_pS^2 and T_qS_1 determined by the local coordinates you have chosen.

2.42. Show that the antipodal map $F \colon S^2 \to S^2$ defined by $F(p) = -p$ is a diffeomorphism.

2.43. Determine an explicit diffeomorphism between the portion of a cylinder defined by $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1,\ -1 < z < 1\}$ and $S^2 \setminus \{N, S\}$, where $N = (0, 0, 1)$ and $S = (0, 0, -1)$.

2.44. Determine a diffeomorphism between the unit sphere $S^2 \subset \mathbb{R}^3$ and the ellipsoid of equation $4x^2 + 9y^2 + 25z^2 = 1$.

2.45. Let C_1 and C_2 be supports of two regular curves contained in a surface S that are tangent at a point p_0, that is, having the same tangent line at a common point p_0. Show that if $F \colon S \to S$ is a diffeomorphism then $F(C_1)$ and $F(C_2)$ are the supports of regular curves tangent at $F(p_0)$.

2.46. Let $f \colon S_1 \to S_2$ be a smooth map between connected regular surfaces. Show that f is constant if and only if $\mathrm{d}f \equiv 0$.

2.47. Prove that every surface of revolution having as its generatrix an open Jordan arc is diffeomorphic to a circular cylinder.

2.48. Show that a rotation of an angle θ of $\mathbb{R}^3$ around the z-axis induces a diffeomorphism on a regular surface of revolution obtained by rotating a curve around the z-axis.

2.49. Let $S \subset \mathbb{R}^n$ be a regular surface and $p_0 \notin S$. Prove that the function $d \colon S \to \mathbb{R}$ defined by $d(p) = \|p - p_0\|$, i.e., the distance from p_0, is of class C^∞.

2.50. Construct an explicit diffeomorphism F between the one-sheeted hyperboloid of equation $(x/a)^2 + (y/b)^2 - (z/c)^2 = 1$ and the right circular cylinder of equation $x^2 + y^2 = 1$, determine its differential $\mathrm{d}F_p$ at every point, and describe the inverse of F in local coordinates.

2.51. Construct a diffeomorphism between the right circular cylinder of equation $x^2 + y^2 = 4$ and the plane $\mathbb{R}^2$ with the origin removed.

CHAPTER 3

Curvatures

One of the main goals of differential geometry consists in finding an effective and meaningful way of measuring the curvature of non-flat objects (curves and surfaces). For curves we have seen that it is sufficient to measure the changes in tangent versors: in the case of surface things are, understandably, more complicated. The first obvious problem is that a surface can curve differently in different directions; so we need a measure of curvature related to tangent directions, that is, a way of measuring the variation of tangent planes.

To solve this problem we have to introduce several new tools. First of all, we need to know the length of vectors tangent to the surface. As explained in Section 3.1, for this it suffices to restrict to each tangent plane the canonical scalar product in $\mathbb{R}^3$. In this way, we get a positive definite quadratic form on each tangent plane (the *first fundamental form*), which allows us to measure the length of tangent vectors to the surface (and, as we shall see in Section 3.2, the area of regions of the surfaces as well). It is worthwhile to notice right now that the first fundamental form is an *intrinsic* object associated with the surface: we may compute it while remaining within the surface itself, without having to go out to $\mathbb{R}^3$.

A tangent plane, being a plane in $\mathbb{R}^3$, is completely determined as soon as we know an orthogonal versor. So a family of tangent planes can be described by the *Gauss map,* associating each point of the surface with a versor normal to the tangent plane at that point. In Section 3.3 we shall see that the Gauss map always exists locally, and exists globally only on orientable surfaces (that is, surfaces where we can distinguish an interior and an exterior).

In Section 3.4 we shall at last define the curvature of a surface along a tangent direction. We shall do so in two ways: geometrically (as the curvature of the curve obtained by intersecting the surface with an orthogonal plane) and analytically, by using the differential of the Gauss map and an associated quadratic form (the *second fundamental form*). In particular, in Section 3.5 we shall introduce the *Gaussian curvature* of a surface as the determinant of the differential of the Gauss map, and we shall see that the Gaussian curvature summarize the main curvature properties of a surface. Furthermore, in Section 3.6, we shall prove Gauss' *Theorema Egregium*, showing that, although the definition involves explicitly the ambient space $\mathbb{R}^3$, the Gaussian curvature actually is an *intrinsic* quantity, that is, it can be measured while remaining inside the surface. This, for instance, allows us to determine that the Earth is not flat without resorting to satellite photos, since it is possible to ascertain that the Earth has non zero Gaussian curvature with measurements made at sea level.

3.1. The first fundamental form

As mentioned in the introduction to this chapter, we begin our journey among surfaces' curvatures by measuring the length of tangent vectors.

The Euclidean space $\mathbb{R}^3$ is intrinsically provided with the canonical scalar product. If $S \subset \mathbb{R}^3$ is a surface, and $p \in S$, the tangent plane T_pS may be thought of as a vector subspace of $\mathbb{R}^3$, and so we may compute the canonical scalar product of two tangent vectors to S at p.

Definition 3.1. Let $S \subset \mathbb{R}^3$ be a surface. For all $p \in S$ we shall denote by $\langle \cdot\,, \cdot \rangle_p$ the positive definite scalar product on T_pS induced by the canonical scalar product of $\mathbb{R}^3$. The *first fundamental form* $I_p \colon T_pS \to \mathbb{R}$ is the (positive definite) quadratic form associated with this scalar product:

$$\forall v \in T_pS \qquad\qquad I_p(v) = \langle v, v \rangle_p \geq 0 \; .$$

Remark 3.1. The knowledge of the first fundamental form I_p is equivalent to the knowledge of the scalar product $\langle \cdot\,, \cdot \rangle_p$: indeed,

$$\langle v, w \rangle_p = \frac{1}{2}\big[I_p(v+w) - I_p(v) - I_p(w)\big] = \frac{1}{4}\big[I_p(v+w) - I_p(v-w)\big] \; .$$

If we forget that the surface lives in the ambient space $\mathbb{R}^3$, and that the first fundamental form is induced by the constant canonical scalar product of $\mathbb{R}^3$, limiting ourselves to try and understand what can be seen from within the surface, we immediately notice that it is natural to consider $\langle \cdot\,, \cdot \rangle_p$ as a scalar product defined on the tangent plane T_pS which varies with p (and with the tangent plane).

A way to quantify this variability consists in using local parametrizations and the bases they induce on the tangent planes to deduce the (variable!) matrix representing this scalar product. Let then $\varphi \colon U \to S$ be a local parametrization at $p \in S$, and $\{\partial_1, \partial_2\}$ the basis of T_pS induced by φ. If we take two tangent vectors v, $w \in T_pS$ and we write them as linear combination of basis vectors, that is $v = v_1\partial_1 + v_2\partial_2$ and $w = w_1\partial_1 + w_2\partial_2 \in T_pS$, we may express $\langle v, w \rangle_p$ in coordinates:

$$\langle v, w \rangle_p = v_1 w_1 \langle \partial_1, \partial_1 \rangle_p + [v_1 w_2 + v_2 w_1] \langle \partial_1, \partial_2 \rangle_p + v_2 w_2 \langle \partial_2, \partial_2 \rangle_p \; .$$

Definition 3.2. Let $\varphi \colon U \to S$ be a local parametrization of a surface S. Then the *metric coefficients* of S with respect to φ are the functions E, F, $G \colon U \to \mathbb{R}$ given by

$$E(x) = \langle \partial_1, \partial_1 \rangle_{\varphi(x)} \; , \quad F(x) = \langle \partial_1, \partial_2 \rangle_{\varphi(x)} \; , \quad G(x) = \langle \partial_2, \partial_2 \rangle_{\varphi(x)} \; ,$$

for all $x \in U$.

Clearly, the metric coefficients are (why?) C^∞ functions on U, and they completely determine the first fundamental form:

$$I_p(v) = E(x)v_1^2 + 2F(x)v_1v_2 + G(x)v_2^2 = \begin{vmatrix} v_1 & v_2 \end{vmatrix} \begin{vmatrix} E(x) & F(x) \\ F(x) & G(x) \end{vmatrix} \begin{vmatrix} v_1 \\ v_2 \end{vmatrix}$$

for all $p = \varphi(x) \in \varphi(U)$ and $v = v_1\partial_1 + v_2\partial_2 \in T_pS$.

Remark 3.2. The notation E, F and G, which we shall systematically use, was introduced by Gauss in the early 19th century. In a more modern notation we may write $E = g_{11}$, $F = g_{12} = g_{21}$ and $G = g_{22}$, so as to get

$$\langle v, w \rangle_p = \sum_{h,k=1}^{2} g_{hk}(p) v_h w_k \ .$$

REMARK 3.3. We have introduced E, F and G as functions defined on U. However, it will sometimes be more convenient to consider them as functions defined on $\varphi(U)$, that is, to replace them with $E \circ \varphi^{-1}$, $F \circ \varphi^{-1}$ and $G \circ \varphi^{-1}$, respectively. You might have noticed that we have performed just this substitution in the last formula.

REMARK 3.4. Warning: the metric coefficients depend strongly on the local parametrization! Example 3.4 will show how much they can change, even in a very simple case, when choosing a different local parametrization.

EXAMPLE 3.1. Let $S \subset \mathbb{R}^3$ be the plane passing through $p_0 \in \mathbb{R}^3$ and parallel to the linearly independent vectors $\vec{v}_1$, $\vec{v}_2 \in \mathbb{R}^3$. In Example 2.2 we have seen that a local parametrization of S is the map $\varphi \colon \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$\varphi(x_1, x_2) = p_0 + x_1 \vec{v}_1 + x_2 \vec{v}_2 \ .$$

For all $p \in S$ the basis of T_pS induced by φ is $\partial_1 = \vec{v}_1$ and $\partial_2 = \vec{v}_2$, so the metric coefficients of the plane with respect to φ are given by $E \equiv \|\vec{v}_1\|^2$, $F \equiv \langle \vec{v}_1, \vec{v}_2 \rangle$ and $G \equiv \|\vec{v}_2\|^2$. In particular, if $\vec{v}_1$ and $\vec{v}_2$ are orthonormal versors, we find

$$E \equiv 1 \ , \qquad F \equiv 0 \ , \qquad G \equiv 1 \ .$$

EXAMPLE 3.2. Let $U \subseteq \mathbb{R}^2$ be an open set, $h \in C^\infty(U)$, and $\varphi \colon U \to \mathbb{R}^3$ the local parametrization of the graph Γ_h given by $\varphi(x) = \big(x, h(x)\big)$. Recalling Example 2.23 we see that the metric coefficients of Γ_h with respect to φ are given by

$$E = 1 + \left| \frac{\partial h}{\partial x_1} \right|^2 \ , \quad F = \frac{\partial h}{\partial x_1} \frac{\partial h}{\partial x_2} \ , \quad G = 1 + \left| \frac{\partial h}{\partial x_2} \right|^2 \ .$$

EXAMPLE 3.3. Let $S \subset \mathbb{R}^3$ be the right circular cylinder with radius 1 centered on the z-axis. A local parametrization $\varphi \colon (0, 2\pi) \times \mathbb{R} \to \mathbb{R}^3$ is given by $\varphi(x_1, x_2) = (\cos x_1, \sin x_1, x_2)$. The basis induced by this parametrization is given by $\partial_1 = (-\sin x_1, \cos x_1, 0)$ e $\partial_2 = (0, 0, 1)$, and so

$$E \equiv 1 \ , \qquad F \equiv 0 \ , \qquad G \equiv 1 \ .$$

EXAMPLE 3.4. Using the local parametrization $\varphi \colon U \to \mathbb{R}^3$ of the unit sphere S^2 given by $\varphi(x, y) = \big(x, y, \sqrt{1 - x^2 - y^2}\big)$ and recalling the local basis computed in Example 2.22, we get

$$E = \frac{1 - y^2}{1 - x^2 - y^2} \ , \quad F = \frac{xy}{1 - x^2 - y^2} \ , \quad G = \frac{1 - x^2}{1 - x^2 - y^2} \ .$$

On the other hand, the second local basis in Example 2.22 computed using the parametrization $\psi(\theta, \phi) = (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta)$ gives us

$$E \equiv 1 \ , \quad F \equiv 0 \ , \quad G = \sin^2 \theta \ .$$

EXAMPLE 3.5. Let $S \subset \mathbb{R}^3$ be the helicoid with the local parametrization $\varphi \colon \mathbb{R}^2 \to \mathbb{R}^3$ given by $\varphi(x, y) = (y \cos x, y \sin x, ax)$ for some $a \in \mathbb{R}^*$. Then, recalling the local basis computed in Problem 2.2, we find

$$E = y^2 + a^2, \quad F \equiv 0, \quad G \equiv 1 \ .$$

EXAMPLE 3.6. Let $S \subset \mathbb{R}^3$ be the catenoid with the local parametrization $\psi\colon \mathbb{R} \times (0, 2\pi) \to \mathbb{R}^3$ given by $\psi(x, y) = (a \cosh x \cos y, a \cosh x \sin y, ax)$ for some $a \in \mathbb{R}^*$. Then, recalling the local basis computed in Problem 2.1, we find

$$E = a^2 \cosh^2 x \ , \quad F \equiv 0 \ , \quad G = a^2 \cosh^2 x \ .$$

EXAMPLE 3.7. More in general, let $\varphi\colon I \times J \to \mathbb{R}^3$, given by

$$\varphi(t, \theta) = \bigl(\alpha(t) \cos\theta, \alpha(t) \sin\theta, \beta(t)\bigr) \ ,$$

be a local parametrization of a surface of revolution S obtained as described in Example 2.8 (where I and J are suitable open intervals). Then, using the local basis computed in Example 2.24, we get

$$E = (\alpha')^2 + (\beta')^2 \ , \quad F \equiv 0 \ , \quad G = \alpha^2 \ .$$

For instance, if S is the torus studied in Example 2.9 then

$$E \equiv r^2 \ , \quad F \equiv 0 \ , \quad G = (r \cos t + x_0)^2 \ .$$

The first fundamental form allows us to compute the length of curves on the surface. Indeed, if $\sigma\colon [a, b] \to S$ is a curve whose image is contained in the surface S, we have

$$L(\sigma) = \int_a^b \sqrt{I_{\sigma(t)}\bigl(\sigma'(t)\bigr)}\, \mathrm{d}t \ .$$

Conversely, if we can compute the length of curves with support on the surface S, we may retrieve the first fundamental form as follows: given $p \in S$ and $v \in T_pS$ let $\sigma\colon (-\varepsilon, \varepsilon) \to S$ be a curve with $\sigma(0) = p$ and $\sigma'(0) = v$, and set $\ell(t) = L(\sigma|_{[0,t]})$. Then (check it!):

$$I_p(v) = \left|\frac{\mathrm{d}\ell}{\mathrm{d}t}(0)\right|^2 \ .$$

So, in a sense, the first fundamental form is related to the *intrinsic* metric properties of the surface, properties that do not depend on the way the surface is immersed in $\mathbb{R}^3$. Staying within the surface, we may measure the length of a curve, and so we may compute the first fundamental form, without having to pop our head into $\mathbb{R}^3$; moreover, a diffeomorphism that preserves the length of curves also preserves the first fundamental form. For this reason, properties of the surface that only depend on the first fundamental form are called *intrinsic properties*. For instance, we shall see in the next few sections that the value of a particular curvature (the Gaussian curvature) is an intrinsic property which will allow us to determine, without leaving our planet, whether the Earth is flat or not.

The maps between surfaces preserving the first fundamental form deserve a special name:

DEFINITION 3.3. Let $H\colon S_1 \to S_2$ be a C^∞ map between two surfaces. We say that H is an *isometry at* $p \in S_1$ if for all $v \in T_pS_1$ we have

$$I_{H(p)}\bigl(\mathrm{d}H_p(v)\bigr) = I_p(v) \ ;$$

clearly (why?) this implies that

$$\bigl\langle \mathrm{d}H_p(v), \mathrm{d}H_p(w)\bigr\rangle_{H(p)} = \langle v, w\rangle_p$$

for all v, $w \in T_pS_1$. We say that H is a *local isometry* at $p \in S_1$ if p has a neighborhood U such that H is an isometry at each point of U; and that H is a *local isometry* if it is so at each point of S_1. Finally, we say that H is an *isometry* if it is both a global diffeomorphism and a local isometry.

REMARK 3.5. If $H\colon S_1 \to S_2$ is an isometry at $p \in S_1$, the differential of H at p is invertible, and so H is a diffeomorphism of a neighborhood of p in S_1 with a neighborhood of $H(p)$ in S_2.

REMARK 3.6. Isometries preserve the lenght of curves, and consequently all intrinsic properties of surfaces.

EXAMPLE 3.8. Denote by $S_1 \subset \mathbb{R}^3$ the plane $\{z = 0\}$, by $S_2 \subset \mathbb{R}^3$ the cylinder of equation $x^2 + y^2 = 1$, and let $H\colon S_1 \to S_2$ be the map given by $H(x, y, 0) = (\cos x, \sin x, y)$. As seen in Example 2.21, the tangent plane to S_1 at any of its points is S_1 itself. Moreover, we have

$$\mathrm{d}H_p(v) = v_1 \frac{\partial H}{\partial x}(p) + v_2 \frac{\partial H}{\partial y}(p) = (-v_1 \sin x, v_1 \cos x, v_2)$$

for all $p = (x, y, 0) \in S_1$ and $v = (v_1, v_2, 0) \in T_pS_1$. Hence,

$$I_{H(p)}\big(\mathrm{d}H_p(v)\big) = \|\mathrm{d}H_p(v)\|^2 = v_1^2 + v_2^2 = \|v\|^2 = I_p(v)\ ,$$

and so H is a local isometry. On the other hand, H is not an isometry, because it is not injective.

DEFINITION 3.4. We shall say that a surface $S_1$is *locally isometric* to a surface S_2 if for all $p \in S_1$ there exists an isometry of a neighborhood of p in S_1 with an open subset of S_2.

REMARK 3.7. Warning: being locally isometric is *not* an equivalence relation; see Exercise 3.8.

Two surfaces are locally isometric if and only if they have (in suitable local parametrizations) the same metric coefficients:

PROPOSITION 3.1. *Let S, $\tilde{S} \subset \mathbb{R}^3$ be two surfaces. Then S is locally isometric to $\tilde{S}$ if and only if for every point $p \in S$ there exist a point $\tilde{p} \in \tilde{S}$, an open subset $U \subseteq \mathbb{R}^2$, a local parametrization $\varphi\colon U \to S$ of S centered at p, and a local parametrization $\tilde{\varphi}\colon U \to \tilde{S}$ of $\tilde{S}$ centered at $\tilde{p}$ such that $E \equiv \tilde{E}$, $F \equiv \tilde{F}$ and $G \equiv \tilde{G}$, where E, F, G (respectively $\tilde{E}$, $\tilde{F}$, $\tilde{G}$) are the metric coefficients of S with respect to φ (respectively, of $\tilde{S}$ with respect to $\tilde{\varphi}$).*

PROOF. Assume that S is locally isometric to $\tilde{S}$. Then, given $p \in S$, we may find a neighborhood V of p and an isometry $H\colon V \to H(V) \subseteq \tilde{S}$. Let $\varphi\colon U \to S$ a local parametrization centered at p and such that $\varphi(U) \subset V$; then $\tilde{\varphi} = H \circ \varphi$ is a local parametrization of $\tilde{S}$ centered at $\tilde{p} = H(p)$ with the required properties (check, please).

Conversely, assume that there exist two local parametrizations φ and $\tilde{\varphi}$ as stated, and set $H = \tilde{\varphi} \circ \varphi^{-1}\colon \varphi(U) \to \tilde{\varphi}(U)$. Clearly, H is a diffeomorphism with its image; we have to prove that it is an isometry. Take $q \in \varphi(U)$ and $v \in T_qS$, and write $v = v_1\partial_1 + v_2\partial_2$. By construction (see Remark 2.24) we have $\mathrm{d}H_q(\partial_j) = \tilde{\partial}_j$; so $\mathrm{d}H_q(v) = v_1\tilde{\partial}_1 + v_2\tilde{\partial}_2$; hence

$$\begin{aligned} I_{H(q)}\big(\mathrm{d}H_q(v)\big) &= v_1^2\tilde{E}\big(\tilde{\varphi}^{-1} \circ H(q)\big) + 2v_1v_2\tilde{F}\big(\tilde{\varphi}^{-1} \circ H(q)\big) + v_2^2\tilde{G}\big(\tilde{\varphi}^{-1} \circ H(q)\big) \\ &= v_1^2 E\big(\varphi^{-1}(q)\big) + 2v_1v_2 F\big(\varphi^{-1}(q)\big) + v_2^2 G\big(\varphi^{-1}(q)\big) = I_q(v)\ , \end{aligned}$$

and so H is an isometry, as required. $\square$

Example 3.9. As a consequence, a plane and a right circular cylinder are locally isometric, due to the previous proposition and Examples 3.1 and 3.3 (see also Example 3.8). On the other hand, they cannot be globally isometric, since they are not even homeomorphic (a parallel of the cylinder disconnects it into two components neither of which has compact closure, a thing impossible in the plane due to the Jordan curve theorem).

If you are surprised to find out that the plane and the cylinder are locally isometric, wait till you see next example:

Example 3.10. Every helicoid is locally isometric to a catenoid. Indeed, let S be a helicoid parametrized as in Example 3.5, and let $\tilde{S}$ be the catenoid corresponding to the same value of the parameter $a \in \mathbb{R}^*$, parametrized as in Example 3.6. Choose a point $p_0 = \varphi(x_0, y_0) \in S$, and let $\chi\colon \mathbb{R} \times (0, 2\pi) \to \mathbb{R}^2$ be given by $\chi(x, y) = (y - \pi + x_0, a \sinh x)$. Clearly, χ is a diffeomorphism with its image, so $\varphi \circ \chi$ is a local parametrization at p of the helicoid. The metric coefficients with respect to this parametrization are

$$E = a^2 \cosh^2 x, \quad F \equiv 0, \quad G = a^2 \cosh^2 x \ ,$$

so Proposition 3.1 ensures that the helicoid is locally isometric to the catenoid. In an analogous way (exercise) it can be proved that the catenoid is locally isometric to the helicoid.

So surfaces having a completely different appearance from outside may well be isometric, and so intrinsically indistinguishable. But if so, how do we tell that two surfaces are *not* locally isometric? Could even the plane and the sphere turn out to be locally isometric? One of the main goals of this chapter is to give a first answer to such questions: we shall construct a function, the Gaussian curvature, defined independently of any local parametrization, measuring intrinsic properties of the surface, so surfaces with significantly different Gaussian curvatures cannot be even locally isometric.

By the way, we would like to remind you that one of the problems that prompted the development of differential geometry was the creation of geographical maps. In our language, a geographical map is a diffeomorphism between an open subset of a surface and an open subset of the plane (in other words, the inverse of a local parametrization) preserving some metric properties of the surface. For instance, a geographical map with a 1:1 scale (a full-scale map) is an isometry of an open subset of the surface with an open subset of the plane. Of course, full-scale maps are not terribly practical; usually we prefer smaller-scale maps. This suggests the following

Definition 3.5. A *similitude* with scale factor $r > 0$ between two surfaces is a diffeomorphism $H\colon S_1 \to S_2$ such that

$$I_{H(p)}\bigl(\mathrm{d}H_p(v)\bigr) = r^2 I_p(v)$$

for all $p \in S_1$ and $v \in T_pS_1$.

A similitude multiplies the length of curves by a constant factor, the scale factor, and so it is ideal for road maps. Unfortunately, as we shall see (Corollary 3.2), similitudes between open subsets of surfaces and open subsets of the plane are very rare. In particular, we shall prove that there exist no similitudes between open subsets of the sphere and open subsets of the plane, so a perfect road map is impossible (the maps we normally use are just approximations). A possible replacement (which is actually used in map-making) is given by *conformal maps*, that is, diffeomorphisms preserving angles; see Exercises 3.61 and 3.62.

While we are at it, let us conclude this section talking about angles:

DEFINITION 3.6. Let $S \subset \mathbb{R}^3$ be a surface, and $p \in S$. A *determination of the angle* between two tangent vectors v_1, $v_2 \in T_pS$ is a $\theta \in \mathbb{R}$ such that

$$\cos\theta = \frac{\langle v_1, v_2\rangle_p}{\sqrt{I_p(v_1)I_p(v_2)}} .$$

Moreover, if σ_1, $\sigma_2\colon (-\varepsilon,\varepsilon) \to S$ are curves with $\sigma_1(0) = \sigma_2(0) = p$, we shall call (determination of the) *angle* between σ_1 and σ_2 at p the angle between $\sigma_1'(0)$ and $\sigma_2'(0)$.

In the plane the Cartesian axes meet (usually) at a right angle. Local parametrizations with an analogous property are very useful, and deserve a special name:

DEFINITION 3.7. We shall say that a local parametrization φ of a surface S is *orthogonal* if its coordinate curves meet at a right angle, that is, if $\partial_1|_p$ and $\partial_2|_p$ are orthogonal for each p in the image of φ.

REMARK 3.8. The tangent vectors to coordinate curves are ∂_1 and ∂_2; so the cosine of the angle between two coordinate curves is given by $F/\sqrt{EG}$, and a local parametrization is orthogonal if and only if $F \equiv 0$. It is possible to show that orthogonal parametrizations always exist.

EXAMPLE 3.11. Parallels and meridians are the coordinate curves of the local parametrizations of the surfaces of revolution seen in Example 2.8, and so these parametrizations are orthogonal thanks to Example 3.7.

3.2. Area

The first fundamental form also allows us to compute the area of bounded regions of a regular surface. For the sake of simplicity, we shall confine our treatment to the case of regions contained in the image of a local parametrization.

Let us begin by defining the regions whose area we want to measure.

DEFINITION 3.8. Let $\sigma\colon [a,b] \to S$ be a piecewise regular curve parametrized by arc length in a surface $S \subset \mathbb{R}^3$, and let $a = s_0 < s_1 < \cdots < s_k = b$ be a partition of $[a,b]$ such that $\sigma|_{[s_{j-1},s_j]}$ is regular for $j = 1,\dots,k$. As for plane curves, we set

$$\dot\sigma(s_j^-) = \lim_{s\to s_j^-}\dot\sigma(s) \qquad \text{and} \qquad \dot\sigma(s_j^+) = \lim_{s\to s_j^+}\dot\sigma(s) ;$$

$\dot\sigma(s_j^-)$ and $\dot\sigma(s_j^+)$ are (in general) distinct vectors of $T_{\sigma(s_j)}S$. Of course, $\dot\sigma(s_0^-)$ and $\dot\sigma(s_k^+)$ are not defined unless the curve is closed, in which case we set $\dot\sigma(s_0^-) = \dot\sigma(s_k^-)$ and $\dot\sigma(s_k^+) = \dot\sigma(s_0^+)$. We shall say that $\sigma(s_j)$ is a *vertex* of σ if $\dot\sigma(s_j^-) \neq \dot\sigma(s_j^+)$, and that it is a *cusp* of σ if $\dot\sigma(s_j^-) = -\dot\sigma(s_j^+)$. A *curvilinear polygon* in S is a closed simple piecewise regular curve parametrized by arc length without cusps.

DEFINITION 3.9. A *regular region* $R \subseteq S$ of a surface S is a connected compact subset of S obtained as the closure of its interior $\mathring{R}$ and whose boundary is parametrized by finitely many curvilinear polygons with disjoint supports. If S is compact, then $R = S$ is a regular region of S with empty boundary.

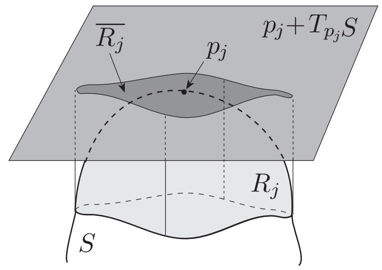


FIGURE 1

To define the length of a curve we approximated it with a polygonal closed curve; to define the area of a region we shall proceed in a similar way.

DEFINITION 3.10. Let $R \subseteq S$ be a regular region of a surface S. A *partition* of R is a finite family $\mathcal{R} = \{R_1, \ldots, R_n\}$ of regular regions contained in R with $R_i \cap R_j \subseteq \partial R_i \cap \partial R_j$ for all $1 \le i \ne j \le n$ and such that $R = R_1 \cup \cdots \cup R_n$. The *diameter* $\|\mathcal{R}\|$ of a partition $\mathcal{R}$ is the maximum of the diameters (in $\mathbb{R}^3$) of the elements of $\mathcal{R}$. Another partition $\tilde{\mathcal{R}} = \{\tilde{R}_1, \ldots, \tilde{R}_m\}$ of R is said to be a *refinement* of $\mathcal{R}$ if for all $i = 1, \ldots, m$ there exists a $1 \le j \le n$ such that $\tilde{R}_i \subseteq R_j$. Finally, a *pointed partition* of R is given by a partition $\mathcal{R} = \{R_1, \ldots, R_n\}$ of R and a n-tuple $\vec{p} = (p_1, \ldots, p_n)$ of points of R such that $p_j \in R_j$ for $j = 1, \ldots, n$.

DEFINITION 3.11. Let $R \subseteq S$ be a regular region of a surface S, and $(\mathcal{R}, \vec{p})$ a pointed partition of R. For all $R_j \in \mathcal{R}$, denote by $\overline{R_j}$ the orthogonal projection of R_j on the affine tangent plane $p_j + T_{p_j}S$ (see Fig. 1), and by $\mathrm{Area}(\overline{R_j})$ its area. The *area* of the pointed partition $(\mathcal{R}, \vec{p})$ is defined as

$$\mathrm{Area}(\mathcal{R}, \vec{p}) = \sum_{R_j \in \mathcal{R}} \mathrm{Area}(\overline{R_j}) \, .$$

We say that the region R is *rectifiable* if the limit

$$\mathrm{Area}(R) = \lim_{\|\mathcal{R}\| \to 0} \mathrm{Area}(\mathcal{R}, \vec{p})$$

exists and is finite. This limit shall be the *area* of R.

To prove that every regular region contained in the image of a local parametrization is rectifiable we shall need the classical Change of Variables Theorem for multiple integrals (see [**3**, Theorem 5.8, p. 211]):

THEOREM 3.1. *Let $h\colon \tilde{\Omega} \to \Omega$ be a diffeomorphism between open sets of $\mathbb{R}^n$. Then, for each regular region $R \subset \Omega$ and each continuous function $f\colon R \to \mathbb{R}$ we have*

$$\int_{h^{-1}(R)} (f \circ h)\, |\det \mathrm{Jac}(h)| \, \mathrm{d}x_1 \cdots \mathrm{d}x_n = \int_R f \, \mathrm{d}x_1 \cdots \mathrm{d}x_n \, .$$

We shall also need an interesting General Topology result:

THEOREM 3.2. *Let $\mathfrak{U} = \{U_\alpha\}_{\alpha \in A}$ be an open cover of a compact metric space (X, d). Then there exists a number $\delta > 0$ such that for all $x \in X$ there is $\alpha \in A$ such that $B_d(x, \delta) \subset U_\alpha$, where $B_d(x, \delta)$ is the open ball with center x and radius δ with respect to distance d.*

PROOF. Let $\{U_1, \ldots, U_n\}$ be a fixed finite subcover of $\mathfrak{U}$. For all $\alpha = 1, \ldots, n$ define the continuous function $f_\alpha \colon X \to \mathbb{R}$ by setting

$$f_\alpha(x) = d(x, X \setminus U_\alpha) ,$$

and set $f = \max\{f_1, \ldots, f_n\}$. The function f is continuous; moreover, for each $x \in X$ we can find $1 \le \alpha \le n$ such that $x \in U_\alpha$, and so $f(x) \ge f_\alpha(x) > 0$. Hence, $f > 0$ everywhere; let $\delta > 0$ be the minimum of f in X. Then for all $x \in X$ we can find $1 \le \alpha \le n$ such that $f_\alpha(x) \ge \delta$, and so the open ball with center x and radius δ is completely contained in U_α, as required. □

Then:

THEOREM 3.3. *Let $R \subseteq S$ be a regular region contained in the image of a local parametrization $\varphi \colon U \to S$ of a surface S. Then R is a rectifiable and*

$$\text{Area}(R) = \int_{\varphi^{-1}(R)} \sqrt{EG - F^2}\, dx_1\, dx_2 . \tag{28}$$

PROOF. Let $R_0 \subseteq R$ be a regular region contained in R, and consider a point $p_0 \in R_0$; our first goal is to describe the orthogonal projection $\overline{R_0}$ of R_0 in $p_0 + T_{p_0}S$. If $p_0 = \varphi(x_0)$, an orthonormal basis of $T_{p_0}S$ is given by the vectors

$$\begin{aligned}
\vec{\epsilon}_1 &= \frac{1}{\sqrt{E(x_0)}} \partial_1(x_0) , \\
\vec{\epsilon}_2 &= \sqrt{\frac{E(x_0)}{E(x_0)G(x_0) - F(x_0)^2}} \left(\partial_2(x_0) - \frac{F(x_0)}{E(x_0)} \partial_1(x_0) \right) .
\end{aligned}$$

It follows (exercise) that the orthogonal projection $\pi_{x_0} \colon \mathbb{R}^3 \to p_0 + T_{p_0}S$ is given by the formula

$$\begin{aligned}
\pi_{x_0}(q) = p_0 \quad &+ \quad \frac{1}{\sqrt{E(x_0)}} \langle q - p_0, \partial_1(x_0) \rangle \vec{\epsilon}_1 \\
&+ \sqrt{\frac{E(x_0)}{E(x_0)G(x_0) - F(x_0)^2}} \left\langle q - p_0, \partial_2(x_0) - \frac{F(x_0)}{E(x_0)} \partial_1(x_0) \right\rangle \vec{\epsilon}_2 .
\end{aligned}$$

Denote now by $\psi_{x_0} \colon p_0 + T_{p_0}S \to \mathbb{R}^2$ the map sending each $p \in p_0 + T_{p_0}S$ to the coordinates of $p - p_0$ with respect to the basis $\{\vec{\epsilon}_1, \vec{\epsilon}_2\}$; since the latter is an orthonormal basis, ψ_{x_0} preserves areas.

Set now $h_x = \psi_x \circ \pi_x \circ \varphi$; let $\Phi \colon U \times U \to \mathbb{R}^2 \times U$ be the map $\Phi(x, y) = \big(h_x(y), x\big)$. It is immediate to verify that

$$\det \text{Jac}(\Phi)(x_0, x_0) = \det \text{Jac}(h_{x_0})(x_0) = \sqrt{E(x_0)G(x_0) - F(x_0)^2} > 0 ; \tag{29}$$

so, for all $x_0 \in U$ there exists a neighborhood $V_{x_0} \subseteq U$ of x_0 such that $\Phi|_{V_{x_0} \times V_{x_0}}$ is a diffeomorphism with its image. Recalling the definition of Φ, this implies that $h_x|_{V_{x_0}}$ is a diffeomorphism with its image for all $x \in V_{x_0}$. In particular, if $R_0 = \varphi(Q_0) \subset \varphi(U)$ is a regular region with $Q_0 \subset V_{x_0}$ and $x \in Q_0$ then the orthogonal projection $\overline{R_0}$ of R_0 on $\varphi(x) + T_{\varphi(x)}S$ is given by $\pi_x \circ \varphi(Q_0)$ and, since ψ_x preserves areas, Theorem 3.1 implies

$$\text{Area}(\overline{R_0}) = \text{Area}\big(h_x(Q_0)\big) = \int_{Q_0} |\det \text{Jac}(h_x)|\, dy_1 dy_2 . \tag{30}$$

Let $R \subset \varphi(U)$ be an arbitrary regular region, and $Q = \varphi^{-1}(R)$. Given $\varepsilon > 0$, we want to find a $\delta > 0$ such that for each pointed partition $(\mathcal{R}, \vec{p})$ of R with diameter less than δ we have

$$\left| \mathrm{Area}(\mathcal{R}, \vec{p}) - \int_Q \sqrt{EG - F^2}\, \mathrm{d}y_1\, \mathrm{d}y_2 \right| < \varepsilon \;.$$

The family $\mathfrak{V} = \{V_x \mid x \in Q\}$ is an open cover of the compact set Q; let $\delta_0 > 0$ be the Lebesgue number (Theorem 3.2) of $\mathfrak{V}$. Let now $\Psi\colon Q \times Q \to \mathbb{R}$ be given by

$$\Psi(x, y) = |\det \mathrm{Jac}(h_x)(y)| - \sqrt{E(y)G(y) - F(y)^2} \;.$$

By (29) we know that $\Psi(x, x) \equiv 0$; so the uniform continuity provides us with a $\delta_1 > 0$ such that

$$|y - x| < \delta_1 \quad \Longrightarrow \quad |\Psi(x, y)| < \varepsilon / \mathrm{Area}(Q) \;.$$

Finally, the uniform continuity of $\varphi^{-1}|_R$ provides us with a $\delta > 0$ such that if $R_0 \subseteq R$ has diameter less than δ then $\varphi^{-1}(R_0)$ has diameter less than $\min\{\delta_0, \delta_1\}$.

Let then $(\mathcal{R}, \vec{p})$, with $\mathcal{R} = \{R_1, \ldots, R_n\}$ and $\vec{p} = (p_1, \ldots, p_n)$, be a pointed partition of R with diameter less than δ, and set $Q_j = \varphi^{-1}(R_j)$ and $x_j = \varphi^{-1}(p_j)$. Since each Q_j has diameter less than δ_0, we may use formula (30) to compute the area of each $\overline{R_j}$. Hence,

$$\begin{aligned} &\left| \mathrm{Area}(\mathcal{R}, \vec{p}) - \int_Q \sqrt{EG - F^2}\, \mathrm{d}y_1\, \mathrm{d}y_2 \right| \\ &\quad = \left| \sum_{j=1}^n \int_{Q_j} |\det \mathrm{Jac}(h_{x_j})|\, \mathrm{d}y_1 \mathrm{d}y_2 - \int_Q \sqrt{EG - F^2}\, \mathrm{d}y_1\, \mathrm{d}y_2 \right| \\ &\quad \le \sum_{j=1}^n \int_{Q_j} |\Psi(x_j, y)|\, \mathrm{d}y_1\, \mathrm{d}y_2 < \sum_{j=1}^n \frac{\varepsilon}{\mathrm{Area}(Q)} \mathrm{Area}(Q_j) = \varepsilon \;, \end{aligned}$$

since each Q_j has diameter less than δ_1, and we are done. □

A consequence of this result is that the value of the integral in the right hand side of (28) is independent of the local parametrization whose image contains R. We are going to conclude this section by generalizing this result in a way that will allow us to integrate functions on a surface. We shall need a lemma containing two formulas that will be useful again later on:

Lemma 3.1. *Let $\varphi\colon U \to S$ be a local parametrization of a surface S. Then*

$$\|\partial_1 \wedge \partial_2\| = \sqrt{EG - F^2} \;, \tag{31}$$

where $\wedge$ is the vector product in $\mathbb{R}^3$. Moreover, if $\hat{\varphi}\colon \hat{U} \to S$ is another local parametrization with $V = \hat{\varphi}(\hat{U}) \cap \varphi(U) \neq \varnothing$, and $h = \hat{\varphi}^{-1} \circ \varphi|_{\varphi^{-1}(V)}$, then

$$\partial_1 \wedge \partial_2|_{\varphi(x)} = \det \mathrm{Jac}(h)(x)\, \hat{\partial}_1 \wedge \hat{\partial}_2|_{\hat{\varphi} \circ h(x)} \tag{32}$$

for all $x \in \varphi^{-1}(V)$, where $\{\hat{\partial}_1, \hat{\partial}_2\}$ is the basis induced by $\hat{\varphi}$.

Proof. Formula (31) follows from equality

$$\|\vec{v} \wedge \vec{w}\|^2 = \|\vec{v}\|^2 \|\vec{w}\|^2 - |\langle \vec{v}, \vec{w} \rangle|^2,$$

which holds for any pair $\vec{v}$, $\vec{w}$ of vectors of $\mathbb{R}^3$.

Furthermore, we have seen (Remark 2.21) that

$$\partial_j|_{\varphi(x)} = \frac{\partial \hat{x}_1}{\partial x_j}\,\hat{\partial}_1|_{\varphi(x)} + \frac{\partial \hat{x}_2}{\partial x_j}\,\hat{\partial}_2|_{\varphi(x)}\ ,$$

and so (32) immediately follows. □

As a consequence we find:

Proposition 3.2. *Let $R \subseteq S$ be a regular region of a surface S, and $f\colon R \to \mathbb{R}$ a continuous function. Assume that there exists a local parametrization $\varphi\colon U \to S$ of S such that $R \subset \varphi(U)$. Then the integral*

$$\int_{\varphi^{-1}(R)} (f \circ \varphi)\sqrt{EG - F^2}\,\mathrm{d}x_1\,\mathrm{d}x_2$$

does not depend on φ.

Proof. Assume that $\hat{\varphi}\colon \tilde{U} \to S$ is another local parametrization such that $R \subset \hat{\varphi}(\tilde{U})$, and set $h = \hat{\varphi}^{-1} \circ \varphi$. Then the previous lemma and Theorem 3.1 yield

$$\begin{aligned}\int_{\varphi^{-1}(R)} (f \circ \varphi) & \sqrt{EG - F^2}\,\mathrm{d}x_1\,\mathrm{d}x_2 = \int_{\varphi^{-1}(R)} (f \circ \varphi)\|\partial_1 \wedge \partial_2\|\,\mathrm{d}x_1\,\mathrm{d}x_2 \\ &= \int_{\varphi^{-1}(R)} \big[(f \circ \hat{\varphi})\|\hat{\partial}_1 \wedge \hat{\partial}_2\|\big] \circ h\,|\det \mathrm{Jac}(h)|\,\mathrm{d}x_1\,\mathrm{d}x_2 \\ &= \int_{\hat{\varphi}^{-1}(R)} (f \circ \hat{\varphi})\sqrt{\hat{E}\hat{G} - \hat{F}^2}\,\mathrm{d}x_1\,\mathrm{d}x_2\ .\end{aligned}$$

□

We may then give the following definition of integral on a surface:

Definition 3.12. Let $R \subseteq S$ be a regular region of a surface S contained in the image of a local parametrization $\varphi\colon U \to S$. Then for all continuous functions $f\colon R \to \mathbb{R}$ the *integral of f on R* is the number

$$\int_R f\,\mathrm{d}\nu = \int_{\varphi^{-1}(R)} (f \circ \varphi)\sqrt{EG - F^2}\,\mathrm{d}x_1\,\mathrm{d}x_2\ .$$

We conclude this section by proving an analogue for surfaces of the Change of Variables Theorem for multiple integrals.

Proposition 3.3. *Let $F\colon \tilde{S} \to S$ be a diffeomorphism between surfaces, and $R \subseteq S$ a regular region contained in the image of a local parametrization $\varphi\colon U \to S$ and such that $F^{-1}(R)$ is also contained in the image of a local parametrization $\tilde{\varphi}\colon \tilde{U} \to \tilde{S}$. Then, for all continuous $f\colon R \to \mathbb{R}$, we have*

$$\int_{F^{-1}(R)} (f \circ F)|\det \mathrm{d}F|\,\mathrm{d}\tilde{\nu} = \int_R f\,\mathrm{d}\nu\ .$$

Proof. Set $\Omega = U$ and $\tilde{\Omega} = \tilde{\varphi}^{-1}\big(F^{-1}\big(\varphi(U)\big)\big)$, so as to get $\varphi^{-1}(R) \subset \Omega$ and $\tilde{\varphi}^{-1}\big(F^{-1}(R)\big) \subset \tilde{\Omega}$; moreover, $h = \varphi^{-1} \circ F \circ \tilde{\varphi}\colon \tilde{\Omega} \to \Omega$ is a diffeomorphism. Set $\hat{\varphi} = F \circ \tilde{\varphi}$. Then $\hat{\varphi}$ is a local parametrization of S, whose local basis $\{\hat{\partial}_1, \hat{\partial}_2\}$ can be obtained from the local basis $\{\tilde{\partial}_1, \tilde{\partial}_2\}$ by the formula $\hat{\partial}_j = \mathrm{d}F(\tilde{\partial}_j)$. In particular,

$$\|\tilde{\partial}_1 \wedge \tilde{\partial}_2\|\,|\det \mathrm{d}F| \circ \tilde{\varphi} = \|\hat{\partial}_1 \wedge \hat{\partial}_2\| \circ \hat{\varphi} = |\det \mathrm{Jac}(h)|\,\|\partial_1 \wedge \partial_2\| \circ h\ ,$$

by (32). Then Theorem 3.1 and (31) imply

$$\begin{aligned}\int_{F^{-1}(R)}(f\circ F)\qquad &|\det \mathrm{d}F|\,\mathrm{d}\tilde\nu\\ &=\int_{\hat\varphi^{-1}(R)}(f\circ\hat\varphi)|\det\mathrm{d}F|\circ\tilde\varphi\sqrt{\tilde E\tilde G-\tilde F^2}\,\mathrm{d}x_1\mathrm{d}x_2\\ &=\int_{h^{-1}(\varphi^{-1}(R))}(f\circ\varphi\circ h)|\,\|\partial_1\wedge\partial_2\|\circ h|\det\mathrm{Jac}(h)|\,\mathrm{d}x_1\,\mathrm{d}x_2\\ &=\int_{\varphi^{-1}(R)}(f\circ\varphi)\sqrt{EG-F^2}\,\mathrm{d}x_1\mathrm{d}x_2=\int_R f\,\mathrm{d}\nu\;.\end{aligned}$$

□

3.3. Orientability

Orientability is an important notion in the theory of surfaces. To put it simply, a surface is orientable if it has two faces, an internal one and an external one, like the sphere, whereas it is non orientable, like the Möbius band (see Example 3.15), if it has only one face, and no well-defined exterior or interior.

There are (at least) two ways to define precisely the notion of orientation: an intrinsic one, and one depending on the embedding of the surface in $\mathbb{R}^3$. To describe the first one, recall that *orienting* a plane means choosing an ordered basis for it (that is, fixing a preferred rotation direction for the angles); two bases determine the same orientation if and only if the change of basis matrix has positive determinant (see [**4**, p. 167] or [**2**, p. 57]). So the idea is that a surface is orientable if we may orient in a consistent way all its tangent planes. Locally this is not a problem: just choose a local parametrization and orient each tangent plane of the support by taking as orientation the one given by the (ordered) basis $\{\partial_1,\partial_2\}$ induced by the parametrization. Since the vectors ∂_1 and ∂_2 vary in a C^∞ way, we can sensibly say that all tangent planes of the support of the parametrization are now oriented consistently. Another parametrization induces the same orientation if and only if the change of basis matrix (that is, the Jacobian matrix of the change of coordinates; see Remark 2.21) has positive determinant. So the following definition becomes natural:

Definition 3.13. Let $S\subset\mathbb{R}^3$ be a surface. We say that two local parametrizations $\varphi_\alpha\colon U_\alpha\to S$ and $\varphi_\beta\colon U_\beta\to S$ *determine the same orientation* (or are *equioriented*) if either $\varphi_\alpha(U_\alpha)\cap\varphi_\beta(U_\beta)=\varnothing$ or $\det\mathrm{Jac}(\varphi_\beta^{-1}\circ\varphi_\alpha)>0$ where it is defined, that is, on $\varphi_\alpha^{-1}\big(\varphi_\alpha(U_\alpha)\cap\varphi_\beta(U_\beta)\big)$. If, on the other hand, $\det\mathrm{Jac}(\varphi_\beta^{-1}\circ\varphi_\alpha)<0$ where it is defined, the two local parametrizations *determine opposite orientations*. The surface S is said to be *orientable* if there exists an atlas $\mathcal{A}=\{\varphi_\alpha\}$ for S consisting of local parametrizations pairwise equioriented (and we shall say that the atlas itself is *oriented*). If we fix such an atlas $\mathcal{A}$, we say that the surface S is *oriented* by the atlas $\mathcal{A}$.

Remark 3.9. Warning: it may happen that a pair of local parametrizations neither determine the same orientation nor opposite orientations. For instance, it may happen that $\varphi_\alpha(U_\alpha)\cap\varphi_\beta(U_\beta)$ has two connected components with $\det\mathrm{Jac}(\varphi_\beta^{-1}\circ\varphi_\alpha)$ positive on one of them and negative on the other one; see Example 3.15.

Recalling what we said, we may conclude that a surface S is orientable if and only if we may simultaneously orient all its tangent planes in a consistent way.

EXAMPLE 3.12. A surface admitting an atlas consisting of a single local parametrization is clearly orientable. For instance, all surfaces given as graphs of functions are orientable.

EXAMPLE 3.13. If a surface has an atlas consisting of two local parametrizations whose images have connected intersection, it is orientable. Indeed, the determinant of the Jacobian matrix of the change of coordinates has (why?) constant sign on the intersection, so up to exchanging the coordinates in the domain of one of the parametrizations (an operation that changes the sign of the determinant of the Jacobian matrix of the change of coordinates), we may always assume that both parametrizations determine the same orientation. For instance, the sphere is orientable (see Example 2.6).

REMARK 3.10. Orientability is a *global* property: we cannot verify if a surface is orientable just by checking what happens on single local parametrizations. The image of a single local parametrization is always orientable; the obstruction to orientability (if any) is related to the way local parametrizations are joined.

This definition of orientation is purely intrinsic: it does not depend on the way the surface is immersed in $\mathbb{R}^3$. In particular, if two surfaces are diffeomorphic, the first one is orientable if and only if the other one is (exercise). As already mentioned, the second definition of orientation will be instead extrinsic: it will strongly depend on the fact that a surface is contained in $\mathbb{R}^3$.

When we studied Jordan curves in the plane, we saw that the normal versor allowed us to distinguish the interior of the curve from its exterior. So, it is natural to try and introduce the notions of interior and exterior of a surface by using normal versors:

DEFINITION 3.14. A *normal vector field* on a surface $S \subset \mathbb{R}^3$ is a C^∞ map $N\colon S \to \mathbb{R}^3$ such that $N(p)$ is orthogonal to T_pS for all $p \in S$; see Fig. 2. If, moreover, $\|N\| \equiv 1$ we shall say that N is *normal versor field* to S.

If N is a normal versor field on a surface S, we may intuitively say that N indicates the external face of the surface, while $-N$ indicates the internal face. But, in contrast to what happens for curves, not every surface as a normal vector field:

PROPOSITION 3.4. *A surface $S \subset \mathbb{R}^3$ is orientable if and only if there exists a normal versor field on S.*

PROOF. We begin with a general remark. Let $\varphi_\alpha\colon U_\alpha \to S$ be a local parametrization of a surface S; for all $p \in \varphi_\alpha(U_\alpha)$ set

$$N_\alpha(p) = \frac{\partial_{1,\alpha} \wedge \partial_{2,\alpha}}{\|\partial_{1,\alpha} \wedge \partial_{2,\alpha}\|}(p) ,$$

where $\partial_{j,\alpha} = \partial\varphi_\alpha/\partial x_j$, as usual. Since $\{\partial_{1,\alpha}, \partial_{2,\alpha}\}$ is a basis of T_pS, the versor $N_\alpha(p)$ is well defined, different from zero, and orthogonal to T_pS; moreover, it clearly is of class C^∞, and so N_α is a normal versor field on $\varphi_\alpha(U_\alpha)$. Notice furthermore that if $\varphi_\beta\colon U_\beta \to S$ is another local parametrization such that $\varphi_\alpha(U_\alpha) \cap \varphi_\beta(U_\beta) \neq \varnothing$ then (32) implies

$$N_\alpha = \operatorname{sgn}\bigl(\det \operatorname{Jac}(\varphi_\beta^{-1} \circ \varphi_\alpha)\bigr)\, N_\beta \tag{33}$$

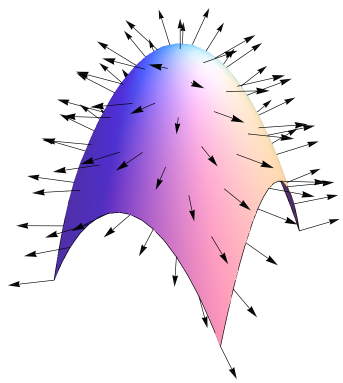


FIGURE 2. A normal vector field

on $\varphi_\alpha(U_\alpha) \cap \varphi_\beta(U_\beta)$.

Assume now S to be orientable, and let $\mathcal{A} = \{\varphi_\alpha\}$ be an oriented atlas. If $p \in \varphi_\alpha(U_\alpha) \cap \varphi_\beta(U_\beta)$, with φ_α, $\varphi_\beta \in \mathcal{A}$, equality (33) tells us that $N_\alpha(p) = N_\beta(p)$; so the map $p \mapsto N_\alpha(p)$ does not depend on the particular local parametrization we have chosen, and defines a normal versor field on S.

Conversely, let $N \colon S \to \mathbb{R}^3$ be a normal versor field on S, and let $\mathcal{A} = \{\varphi_\alpha\}$ be an arbitrary atlas of S such that the domain U_α of each φ_α is connected. By definition of vector product, $N_\alpha(p)$ is orthogonal to T_pS for all $p \in \varphi_\alpha(U_\alpha)$ and $\varphi_\alpha \in \mathcal{A}$; so $\langle N, N_\alpha \rangle \equiv \pm 1$ on each U_α. Since U_α is connected, up to modifying φ_α by exchanging coordinates in U_α, we may assume that all these scalar products are identically equal to 1. Hence,

$$N_\alpha \equiv N$$

on each U_α, and (33) implies that the atlas is oriented. $\square$

DEFINITION 3.15. Let $S \subset \mathbb{R}^3$ be a surface oriented by an atlas $\mathcal{A}$. A normal versor field N will be said to *determine the (assigned) orientation* if

$$N = \frac{\partial_1 \wedge \partial_2}{\|\partial_1 \wedge \partial_2\|}$$

for any local parametrization $\varphi \in \mathcal{A}$.

A consequence of the latter proposition is that if S is an oriented surface then there exists always (why?) a *unique* normal versor field determining the assigned orientation.

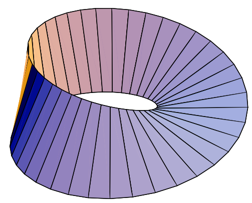


FIGURE 3. The Möbius band

EXAMPLE 3.14. Every surface of revolution S is orientable. Indeed, we may define a normal versor field $N\colon S \to S^2$ by setting

$$\begin{aligned} N(p) &= \left.\frac{\partial}{\partial t}\right|_p \wedge \left.\frac{\partial}{\partial \theta}\right|_p \Bigg/ \left\| \left.\frac{\partial}{\partial t}\right|_p \wedge \left.\frac{\partial}{\partial \theta}\right|_p \right\| \\ &= \frac{1}{\sqrt{\bigl(\alpha'(t)\bigr)^2 + \bigl(\beta'(t)\bigr)^2}} \begin{vmatrix} -\beta'(t)\cos\theta \\ -\beta'(t)\sin\theta \\ \alpha'(t) \end{vmatrix} \end{aligned}$$

for all $p = \varphi(t, \theta) \in S$, where $\varphi\colon \mathbb{R}^2 \to S$ is the immersed surface with support S defined in Example 2.8, and we used Example 2.24.

DEFINITION 3.16. Let $S \subset \mathbb{R}^3$ be an oriented surface, and $N\colon S \to S^2$ a normal versor field that determines the assigned orientation. If $p \in S$, we shall say that a basis $\{v_1, v_2\}$ of T_pS is *positive* (respectively, *negative*) if the basis $\{v_1, v_2, N(p)\}$ of $\mathbb{R}^3$ has the same orientation (respectively, the opposite orientation) as the canonical basis of $\mathbb{R}^3$.

In particular, a local parametrization $\varphi\colon U \to S$ determines the orientation assigned on S if and only if (why?) $\{\partial_1|_p, \partial_2|_p\}$ is a positive basis of T_pS for all $p \in \varphi(U)$.

As mentioned above, not every surface is orientable. The most famous example of non orientable surface is the Möbius band.

EXAMPLE 3.15 (The Möbius band). Let C be the circle in the xy-plane with center in the origin and radius 2, and ℓ_0 the line segment in the yz-plane given by $y = 2$ and $|z| < 1$, with center in the point $c = (0, 2, 0)$. Denote by ℓ_θ the line segment obtained by rotating c clock-wise along C by an angle θ and simultaneously rotating ℓ_0 around c by an angle $\theta/2$. The union $S = \bigcup_{\theta \in [0, 2\pi]} \ell_\theta$ is the *Möbius band* (Fig. 3); we are going to prove that it is a non orientable surface.

Set $U = \{(u, v) \in \mathbb{R}^2 \mid 0 < u < 2\pi, -1 < v < 1\}$, and define φ, $\hat{\varphi}\colon U \to S$ by

$$\begin{aligned} \varphi(u, v) &= \left(\left(2 - v\sin\frac{u}{2}\right)\sin u, \left(2 - v\sin\frac{u}{2}\right)\cos u, v\cos\frac{u}{2}\right), \\ \hat{\varphi}(u, v) &= \left(\left(2 - v\sin\frac{2u+\pi}{4}\right)\cos u, \left(-2 + v\sin\frac{2u+\pi}{4}\right)\sin u, v\cos\frac{2u+\pi}{4}\right). \end{aligned}$$

It is straightforward to verify (exercise) that $\{\varphi, \hat{\varphi}\}$ is an atlas for S, consisting of two local parametrizations whose images have *disconnected* intersection: indeed,

$\varphi(U) \cap \hat{\varphi}(U) = \varphi(W_1) \cup \varphi(W_2)$, with

$$W_1 = \{(u,v) \in U \mid \pi/2 < u < 2\pi\} \quad \text{and} \quad W_2 = \{(u,v) \in U \mid 0 < u < \pi/2\} .$$

Now, if $(u,v) \in W_1$ we have $\varphi(u,v) = \hat{\varphi}(u - \pi/2, v)$, while if $(u,v) \in W_2$ we have $\varphi(u,v) = \hat{\varphi}(u + 3\pi/2, -v)$; so

$$\hat{\varphi}^{-1} \circ \varphi(u,v) = \begin{cases} (u - \pi/2, v) & \text{if } (u,v) \in W_1 , \\ (u + 3\pi/2, -v) & \text{if } (u,v) \in W_2 . \end{cases}$$

In particular,

$$\det \mathrm{Jac}(\hat{\varphi}^{-1} \circ \varphi) \equiv \begin{cases} +1 & \text{on } W_1 , \\ -1 & \text{on } W_2 . \end{cases}$$

Now, assume by contradiction that S is orientable, and let N be a normal versor field on S. Up to inverting the sign of N, we may assume that N is given by $\partial_u \wedge \partial_v / \|\partial_u \wedge \partial_v\|$ on $\varphi(U)$, where $\partial_u = \partial\varphi/\partial u$ and $\partial_v = \partial\varphi/\partial v$. On the other hand, we have $N = \pm\hat{\partial}_u \wedge \hat{\partial}_v / \|\hat{\partial}_u \wedge \hat{\partial}_v\|$ on $\hat{\varphi}(U)$, where $\hat{\partial}_u = \partial\hat{\varphi}/\partial u$ and $\hat{\partial}_v = \partial\hat{\varphi}/\partial v$, and the sign is constant because U is connected. But (33) applied to W_1 tells us that the sign should be $+1$, whereas applied to W_2 yields -1, contradiction.

Let us remark explicitly that the Möbius band is *not* a closed surface in $\mathbb{R}^3$. This is crucial: indeed, it is possible to prove that every closed surface in $\mathbb{R}^3$ is orientable.

Finally, a large family of orientable surfaces is provided by the following

Corollary 3.1. *Let $a \in \mathbb{R}$ be a regular value for a function $f\colon \Omega \to \mathbb{R}$ of class C^∞, where $\Omega \subseteq \mathbb{R}^3$ is an open set. Then every connected component S of $f^{-1}(a)$ is orientable, and a normal versor field is given by $N = \nabla f / \|\nabla f\|$.*

Proof. It immediately follows from Proposition 2.7. □

It is possible to show that a converse of this corollary: if $S \subset \mathbb{R}^3$ is an orientable surface and $\Omega \subseteq \mathbb{R}^3$ an open set containing containing S such that S is closed in Ω with $\Omega \setminus S$ disconnected then there exists a function $f \in C^\infty(\Omega)$ such that S is a level surface for f.

3.4. Normal curvature and second fundamental form

As you have undoubtedly already imagined, one of the main questions differential geometry has to answer is how to measure the curvature of a surface. The situation is quite a bit more complicated than for curves, and as a consequence the answer is not only more complex, but it is not even unique: there are several meaningful ways to measure the curvature of a surface, and we shall explore them in detail in the rest of this chapter.

The first natural remark is that the curvature of a surface, whatever it might be, is not constant in all directions. For instance, a circular cylinder is not curved in the direction of the generatrix, whereas it curves along the directions tangent to the parallels. So it is natural to say that the curvature of the cylinder should be zero in the direction of the generatrix, whereas the curvature in the direction of the parallels should be the same as that of the parallels themselves, that is, the inverse of the radius. And what about other directions? Looking at the cylinder, we would guess that its curvature is maximal in the direction of the parallel, minimal in the direction of the generatrix, and takes intermediate values in the other directions.

To compute it, we might for instance consider a curve contained in the surface, tangent to the direction we have chosen; at the very least, this is an approach that works for generatrices and parallels. The problem is: which curve? A priori (and a posteriori too, as we shall see), if we choose a random curve the curvature might depend on some property of the curve and not only on the surface S and on the tangent direction v we are interested in. So we need a procedure yielding a curve depending only on S and v and representing appropriately the geometry of the surface along that direction. The next lemma describes how to do this:

LEMMA 3.2. *Let S be a surface, $p \in S$ and choose a versor $N(p) \in \mathbb{R}^3$ orthogonal to T_pS. Given $v \in T_pS$ of length 1, let H_v be the plane passing through p and parallel to v and $N(p)$. Then the intersection $H_v \cap S$ is, at least in a neighborhood of p, the support of a regular curve.*

PROOF. The plane H_v has equation $\langle x - p, v \wedge N(p)\rangle = 0$. So if $\varphi\colon U \to S$ is a local parametrization centered at p, a point $\varphi(y) \in \varphi(U)$ belongs to $H_v \cap S$ if and only if $y \in U$ satisfies the equation $f(y) = 0$, where

$$f(y) = \langle \varphi(y) - p, v \wedge N(p)\rangle \ .$$

If we prove that $C = \{y \in U \mid f(y) = 0\}$ is the support of a regular curve σ near O, we are done, as $H_v \cap \varphi(U) = \varphi(C)$ is in this case the support of the regular curve $\varphi \circ \sigma$ near p.

Now,

$$\frac{\partial f}{\partial y_i}(O) = \langle \partial_i|_p, v \wedge N(p)\rangle \ ;$$

so if O were a critical point of f, the vector $v \wedge N(p)$ would be orthogonal to both $\partial_1|_p$ and $\partial_2|_p$, and hence orthogonal to T_pS, that is, parallel to $N(p)$, whereas it is not. So O is not a critical point of f, and by Proposition 1.2 we know that C is a graph in a neighborhood of O. □

DEFINITION 3.17. Let S be a surface. Given $p \in S$, choose a versor $N(p) \in \mathbb{R}^3$ orthogonal to T_pS. Take $v \in T_pS$ of length one, and let H_v be the plane through p and parallel to v and $N(p)$. The regular curve σ, parametrized by arc length, with $\sigma(0) = p$ whose support is the intersection $H_v \cap S$ in a neighborhood of p is the *normal section* of S at p along v (see Fig. 4). Since $\mathrm{Span}\{v, N(p)\} \cap T_pS = \mathbb{R}v$, the tangent versor of the normal section at p has to be $\pm v$; we shall orient the normal section curve so that $\dot\sigma(0) = v$. In particular, σ is uniquely defined in a neighborhood of 0 (why?).

The normal section is a curve that only depends on the geometry of the surface S in the direction of the tangent versor v; so we may try and use it to give a geometric definition of the curvature of a surface.

DEFINITION 3.18. Let S be a surface, $p \in S$ and let $N(p) \in \mathbb{R}^3$ be a versor orthogonal to T_pS. Given $v \in T_pS$ of length 1, orient the plane H_v by choosing $\{v, N(p)\}$ as positive basis. The *normal curvature* of S at p along v is the oriented curvature at p of the normal section of S at p along v (considerered as a plane curve contained in H_v) .

REMARK 3.11. Clearly, the normal section curve does not depend on the choice of the particular versor $N(p)$ orthogonal to T_pS. The normal curvature, on the other hand, does: if we substitute $-N(p)$ for $N(p)$, the normal curvature changes sign (why?).

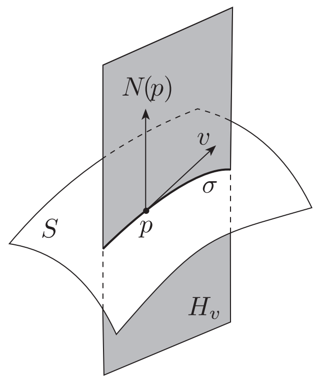


FIGURE 4. Normal section

It is straightforward to verify (do it!) that the normal curvature of a right circular cylinder with radius $r > 0$ is actually zero along the directions tangent to the generatrices, and equals $\pm 1/r$ along the directions tangent to the parallels (which are normal sections); computing the curvature along other directions is however more complicated. For the cylinder the other normal sections are ellipses, so it can somehow be done; but for an arbitrary surface the problem becomes harder, as normal section curves are defined only implicitly (as the intersection of a plane and a surface), and so computing their oriented curvature might not be easy.

To solve this problem (and, as you will see, we shall solve it and obtain simple explicit formulas to compute normal curvatures), let us introduce a second way to study the curvature of a surface. In a sense, the curvature of a curve is a measure of the variation of its tangent line; the curvature of a surface might then be a measure of the variation of its tangent plane. Now, the tangent line to a curve is determined by the tangent versor, that is, by a vector-valued map, uniquely defined up to sign, so measuring the variation of the tangent line is equivalent to differentiating this map. Instead, at first glance we might think that to determine the tangent plane might be necessary to choose a basis, and this choice is anything but unique. But, since we are talking about surfaces in $\mathbb{R}^3$, the tangent plane actually is also determined by the normal versor, which is unique up to sign; so we may try to measure the variation of the tangent plane by differentiating the normal versor.

Let us now try and make this argument formal and rigorous. As we shall see, we shall actually obtain an effective way of computing the normal curvature; but to get there we shall need a bit of work.

We begin with a crucial definition.

DEFINITION 3.19. Let $S \subset \mathbb{R}^3$ be an oriented surface. The *Gauss map* of S is the normal versor field $N: S \to S^2$ that identifies the given orientation.

REMARK 3.12. Even if for the sake of simplicity we shall often work only with oriented surfaces, much of what we are going to say in this chapter holds for every surface. Indeed, every surface is locally orientable: if $\varphi: U \to S$ is a local parametrization at a point p, then $N = \partial_1 \wedge \partial_2 / \|\partial_1 \wedge \partial_2\|$ is a Gauss map of $\varphi(U)$. Therefore every result of a local nature we shall prove by using Gauss maps and that does not change by substituting $-N$ for N actually holds for an arbitrary surface.

The Gauss map determines uniquely the tangent planes to the surface, since T_pS is orthogonal to $N(p)$; so the variation of N measures how the tangent planes change, that is, how far the surface is from being a plane (see also Exercise 3.18).

The argument presented above suggests that the curvature of a surface might then be related to the differential of the Gauss map, just like the curvature of a curve was related to the derivative of the tangent versor. To verify the correctness of this guess, let us study some examples.

EXAMPLE 3.16. In a plane parametrized as in Example 2.2 we have

$$N \equiv \frac{\vec{v}_1 \wedge \vec{v}_2}{\|\vec{v}_1 \wedge \vec{v}_2\|} ,$$

so N is constant and $\mathrm{d}N \equiv O$.

EXAMPLE 3.17. Let $S = S^2$. By using any of the parametrizations described in Example 3.4 we find $N(p) = p$, a result consistent with Example 2.22. So the Gauss map of the unit sphere is the identity map, and in particular we have $\mathrm{d}N_p = \mathrm{id}$ for all $p \in S^2$.

EXAMPLE 3.18. Let $S \subset \mathbb{R}^3$ be a right circular cylinder of equation $x_1^2+x_2^2 = 1$. Corollary 3.1 tells us that a Gauss map of S is given by

$$N(p) = \begin{vmatrix} p_1 \\ p_2 \\ 0 \end{vmatrix}$$

for all $p = (p_1, p_2, p_3) \in S$. In particular,

$$T_pS = N(p)^\perp = \{v \in \mathbb{R}^3 \mid v_1p_1 + v_2p_2 = 0\} .$$

Moreover, as N is the restriction to S of a linear map of $\mathbb{R}^3$ in itself, we get (why?) $\mathrm{d}N_p(v) = (v_1, v_2, 0)$ for all $v = (v_1, v_2, v_3) \in T_pS$.

In particular, $\mathrm{d}N_p(T_pS) \subseteq T_pS$, and as an endomorphism of T_pS the differential of the Gauss map has an eigenvalue equal to zero and one equal to 1. The eigenvector corresponding to the zero eigenvalue is $(0, 0, 1)$, that is, the direction along which we already know the cylinder has zero normal curvature; the eigenvector corresponding to the eigenvalue 1 is tangent to the parallels of the cylinder, so it is exactly the direction along which the cylinder has normal curvature 1. As we shall see, this is not a coincidence.

EXAMPLE 3.19. Let $\Gamma_h \subset \mathbb{R}^3$ be the graph of a function $h\colon U \to \mathbb{R}$, where $U \subset \mathbb{R}^2$ is open, and let $\varphi\colon U \to \Gamma_h$ be the usual parametrization $\varphi(x) = \big(x, h(x)\big)$ of Γ_h. Example 2.23 tells us that a Gauss map $N\colon \Gamma_h \to S^2$ of Γ_h is given by

$$N \circ \varphi = \frac{\partial_1 \wedge \partial_2}{\|\partial_1 \wedge \partial_2\|} = \frac{1}{\sqrt{1 + \|\nabla h\|^2}} \begin{vmatrix} -\partial h/\partial x_1 \\ -\partial h/\partial x_2 \\ 1 \end{vmatrix} .$$

Let us compute how the differential of N acts on the tangent planes of Γ_h.

Choose $p = \varphi(x) \in \Gamma_h$; recalling Remark 2.24 we get

$$\begin{aligned}
\mathrm{d}N_p(\partial_j) &= \frac{\partial(N\circ\varphi)}{\partial x_j}(x) \\
&= \frac{1}{(1+\|\nabla h\|^2)^{3/2}}\left\{\left[\frac{\partial h}{\partial x_1}\frac{\partial h}{\partial x_2}\frac{\partial^2 h}{\partial x_j \partial x_2} - \left(1+\left(\frac{\partial h}{\partial x_2}\right)^2\right)\frac{\partial^2 h}{\partial x_j \partial x_1}\right]\partial_1\right. \\
&\qquad\left. + \left[\frac{\partial h}{\partial x_1}\frac{\partial h}{\partial x_2}\frac{\partial^2 h}{\partial x_j \partial x_1} - \left(1+\left(\frac{\partial h}{\partial x_1}\right)^2\right)\frac{\partial^2 h}{\partial x_j \partial x_2}\right]\partial_2\right\};
\end{aligned}$$

in particular, $\mathrm{d}N_p(T_p\Gamma_h) \subseteq T_p\Gamma_h$ for all $p \in \Gamma_h$.

EXAMPLE 3.20. Let S be a helicoid, parametrized as in Example 3.5. Then

$$(N\circ\varphi)(x,y) = \frac{1}{\sqrt{a^2+y^2}}\begin{vmatrix} -a\sin x \\ a\cos x \\ -y \end{vmatrix}.$$

Let now $p = \varphi(x_0,y_0) \in S$, and take $v = v_1\partial_1 + v_2\partial_2 \in T_pS$. Arguing as in the previous example we find

$$\begin{aligned}
\mathrm{d}N_p(v) &= v_1\frac{\partial(N\circ\varphi)}{\partial x}(x_0,y_0) + v_2\frac{\partial(N\circ\varphi)}{\partial y}(x_0,y_0) \\
&= -\frac{a}{(a^2+y_0^2)^{3/2}}\,v_2\partial_1 - \frac{a}{(a^2+y_0^2)^{1/2}}\,v_1\partial_2 .
\end{aligned}$$

In particular, $\mathrm{d}N_p(T_pS) \subseteq T_pS$ in this case too.

EXAMPLE 3.21. Let $S \subset \mathbb{R}^3$ be a catenoid, parametrized as in Example 3.6. Then

$$(N\circ\psi)(x,y) = \frac{1}{\cosh x}\begin{vmatrix} -\cos y \\ -\sin y \\ \sinh x \end{vmatrix}.$$

Let now $p = \psi(x_0,y_0) \in S$, and take $w = w_1\partial_1 + w_2\partial_2 \in T_pS$. Then we get

$$\mathrm{d}N_p(w) = \frac{w_1}{a\cosh^2 x_0}\,\partial_1 - \frac{w_2}{a\cosh^2 x_0}\,\partial_2 .$$

In particular, $\mathrm{d}N_p(T_pS) \subseteq T_pS$, once again.

EXAMPLE 3.22. Let $S \subset \mathbb{R}^3$ be a surface of revolution, oriented by the Gauss map $N\colon S \to S^2$ we computed in Example 3.14. Then

$$\begin{aligned}
\mathrm{d}N_p\left(\left.\frac{\partial}{\partial t}\right|_p\right) &= \frac{\beta'\alpha'' - \alpha'\beta''}{\big((\alpha')^2+(\beta')^2\big)^{3/2}}\left.\frac{\partial}{\partial t}\right|_p , \\
\mathrm{d}N_p\left(\left.\frac{\partial}{\partial \theta}\right|_p\right) &= \frac{-\beta'/\alpha}{\sqrt{(\alpha')^2+(\beta')^2}}\left.\frac{\partial}{\partial \theta}\right|_p ,
\end{aligned}$$

and again $\mathrm{d}N_p(T_pS) \subseteq T_pS$ for all $p \in S$.

In all previous examples the differential of the Gauss map maps the tangent plane of the surface in itself; this is not a coincidence. By definition, $\mathrm{d}N_p$ maps T_pS in $T_{N(p)}S^2$. But, as already remarked (Example 2.22), the tangent plane to the sphere in a point is orthogonal to that point; so $T_{N(p)}S^2$ is orthogonal to $N(p)$, and thus it coincides with T_pS. Summing up, *we may consider the differential of the*

Gauss map at a point $p \in S$ as an endomorphism of T_pS. And it is not just any endomorphism: it is symmetric. To prove this, we need a result from Differential Calculus (see [**3**, Theorem 3.3, p. 92]):

THEOREM 3.4 (Schwarz). *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $f \in C^2(\Omega)$. Then*

$$\forall i, j = 1, \dots, n \qquad \frac{\partial^2 f}{\partial x_i \partial x_j} \equiv \frac{\partial^2 f}{\partial x_j \partial x_i} .$$

Hence:

PROPOSITION 3.5. *Let $S \subset \mathbb{R}^3$ be an oriented surface, and let $N \colon S \to S^2$ its Gauss map. Then $\mathrm{d}N_p$ is an endomorphism of T_pS, symmetric with respect to the scalar product $\langle \cdot \, , \cdot \rangle_p$ for all $p \in S$.*

PROOF. Choose a local parametrization φ centered at p, and let $\{\partial_1, \partial_2\}$ be the basis of T_pS induced by φ. It suffices (why?) to prove that $\mathrm{d}N_p$ is symmetric on the basis, that is, that

$$\langle \mathrm{d}N_p(\partial_1), \partial_2 \rangle_p = \langle \partial_1, \mathrm{d}N_p(\partial_2) \rangle_p . \tag{34}$$

Now, by definition $\langle N \circ \varphi, \partial_2 \rangle \equiv 0$. Differentiating with respect to x_1 and recalling Remark 2.24 we get

$$\begin{aligned} 0 &= \frac{\partial}{\partial x_1} \langle N \circ \varphi, \partial_2 \rangle (O) = \left\langle \frac{\partial (N \circ \varphi)}{\partial x_1}(O), \frac{\partial \varphi}{\partial x_2}(O) \right\rangle + \left\langle N(p), \frac{\partial^2 \varphi}{\partial x_1 \partial x_2}(O) \right\rangle \\ &= \langle \mathrm{d}N_p(\partial_1), \partial_2 \rangle_p + \left\langle N(p), \frac{\partial^2 \varphi}{\partial x_1 \partial x_2}(O) \right\rangle . \end{aligned}$$

Analogously, by differentiating $\langle N \circ \varphi, \partial_1 \rangle \equiv 0$ with respect to x_2 we get

$$0 = \langle \mathrm{d}N_p(\partial_2), \partial_1 \rangle_p + \left\langle N(p), \frac{\partial^2 \varphi}{\partial x_1 \partial x_2}(O) \right\rangle ,$$

and (34) follows from Theorem 3.4. □

We have a scalar product and a symmetric endomorphism; Linear Algebra suggests us to mix them together.

DEFINITION 3.20. Let $S \subset \mathbb{R}^3$ be an oriented surface, and denote by $N \colon S \to S^2$ its Gauss map. The *second fundamental form* of S is then the quadratic form $Q_p \colon T_pS \to \mathbb{R}$ given by

$$\forall v \in T_pS \qquad Q_p(v) = -\langle \mathrm{d}N_p(v), v \rangle_p .$$

REMARK 3.13. The minus sign in the previous definition will be necessary for equation (35) to hold.

REMARK 3.14. By changing the orientation of S the Gauss map changes sign, and so the second fundamental form changes sign too.

EXAMPLE 3.23. Of course, the second fundamental form of a plane is zero everywhere.

EXAMPLE 3.24. The second fundamental form of a cylinder oriented by the Gauss map given in Example 3.18 is $Q_p(v) = -v_1^2 - v_2^2$.

EXAMPLE 3.25. The second fundamental form of the sphere oriented by the Gauss map of Example 3.17 is the opposite of the first fundamental form: $Q_p = -I_p$.

EXAMPLE 3.26. Let $\Gamma_h \subset \mathbb{R}^3$ be the graph of a function $h\colon U \to \mathbb{R}$, with $U \subseteq \mathbb{R}^2$ open, oriented by the Gauss map of Example 3.19. Recalling the Example 3.2 we find

$$\begin{aligned} Q_p(v) &= -\langle \mathrm{d}N_p(\partial_1), \partial_1\rangle_p v_1^2 - 2\langle \mathrm{d}N_p(\partial_1), \partial_2\rangle_p v_1 v_2 - \langle \mathrm{d}N_p(\partial_2), \partial_2\rangle_p v_2^2 \\ &= \frac{1}{\sqrt{1+\|\nabla h(x)\|^2}}\left[\frac{\partial^2 h}{\partial x_1^2}(x)v_1^2 + 2\frac{\partial^2 h}{\partial x_1 \partial x_2}(x)v_1 v_2 + \frac{\partial^2 h}{\partial x_2^2}(x)v_2^2\right] \end{aligned}$$

for all $p = \big(x, h(x)\big) \in \Gamma_h$ and all $v = v_1\partial_1 + v_2\partial_2 \in T_p\Gamma_h$. In other words, the matrix representing the second fundamental form with respect to the basis $\{\partial_1, \partial_2\}$ is $(1+\|\nabla h\|^2)^{-1/2}\mathrm{Hess}(h)$, where $\mathrm{Hess}(h)$ is the Hessian matrix of h.

EXAMPLE 3.27. Let $S \subset \mathbb{R}^3$ be a helicoid, oriented by the Gauss map of Example 3.20. Then, recalling Problem 2.2 and Example 3.5, we get

$$\begin{aligned} Q_p(v) &= \frac{a}{(a^2+y_0^2)^{1/2}}\left[F(x_0,y_0)\left(v_1^2 + \frac{v_2^2}{a^2+y_0^2}\right) + 2G(x_0,y_0)v_1v_2\right] \\ &= \frac{2a}{(a^2+y_0^2)^{1/2}}\, v_1 v_2 \end{aligned}$$

for all $p = \varphi(x_0, y_0) \in S$ and $v = v_1\partial_1 + v_2\partial_2 \in T_pS$.

EXAMPLE 3.28. Let $S \subset \mathbb{R}^3$ be a catenoid, oriented by the Gauss map of Example 3.21. Then

$$Q_p(w) = -\frac{E(x_0,y_0)}{a\cosh^2 x_0}w_1^2 + \frac{G(x_0,y_0)}{a\cosh^2 x_0}w_2^2 = -aw_1^2 + aw_2^2$$

for all $p = \psi(x_0, y_0) \in S$ and $w = w_1\partial_1 + w_2\partial_2 \in T_pS$.

EXAMPLE 3.29. Let $S \subset \mathbb{R}^3$ be a surface of revolution, oriented by the Gauss map of Example 3.22. Then

$$Q_p(v) = \frac{\alpha'\beta'' - \alpha''\beta'}{\sqrt{(\alpha')^2+(\beta')^2}}v_1^2 + \frac{\alpha\beta'}{\sqrt{(\alpha')^2+(\beta')^2}}v_2^2$$

for all $p = \big(\alpha(t)\cos\theta, \alpha(t)\sin\theta, \beta(t)\big) \in S$ and $v = v_1\partial/\partial t + v_2\partial/\partial\theta \in T_pS$.

The second fundamental form, just like the normal curvature, allows us to associate a number with each tangent versor to a surface; moreover, the second fundamental form, like the normal curvature, has to do with how much a surface curves. The second fundamental form, however, has an obvious advantage: as we have seen in the previous examples, it is very easy to compute starting from a local parametrization. This is important because we shall now show that *the normal curvature coincides with the second fundamental form.* To prove this, take an arbitrary curve $\sigma\colon (-\varepsilon, \varepsilon) \to S$ in S, parametrized by arc length, and set $\sigma(0) = p \in S$ and $\dot\sigma(0) = v \in T_pS$. Set $N(s) = N\big(\sigma(s)\big)$; clearly, $\langle\dot\sigma(s), N(s)\rangle \equiv 0$. By differentiating, we find

$$\langle\ddot\sigma(s), N(s)\rangle \equiv -\langle\dot\sigma(s), \dot N(s)\rangle\ .$$

But $\dot N(0) = \mathrm{d}N_p(v)$; so

$$Q_p(v) = -\langle \mathrm{d}N_p(v), \dot\sigma(0)\rangle = \langle\ddot\sigma(0), N(p)\rangle\ . \tag{35}$$

Moreover, if σ is biregular we have $\ddot\sigma = \kappa\vec{n}$, where κ is the curvature of σ, and $\vec{n}$ is the normal versor of σ, and so we have

$$Q_p(v) = \kappa(0)\langle\vec{n}(0), N(p)\rangle\ .$$

This formulas suggest the following

DEFINITION 3.21. Let $\sigma\colon I \to S$ be a curve parametrized by arc length contained in an oriented surface S. The *normal curvature* $\kappa_n\colon I \to \mathbb{R}$ of σ is the function given by

$$\kappa_n = \langle \ddot{\sigma}, N \circ \sigma \rangle = \kappa \langle \vec{n}, N \circ \sigma \rangle \ ,$$

where the second equality holds when σ is biregular. In other words, the normal curvature of σ is the (signed) length of the projection of the acceleration vector $\ddot{\sigma}$ along the direction orthogonal to the surface. Moreover, by (35) we know that

$$\kappa_n(s) = Q_{\sigma(s)}\bigl(\dot{\sigma}(s)\bigr) \ . \tag{36}$$

REMARK 3.15. If the orientation of S is inverted, the normal curvature function changes sign.

If σ is the normal section of S at p along v, its normal versor at p is (why?) exactly $N(p)$, so the normal curvature of S at p along v is the normal curvature of σ at p. Hence we are at last able to prove that the second fundamental form gives the normal curvature of the surface:

PROPOSITION 3.6 (Meusnier). *Let $S \subset \mathbb{R}^3$ be an oriented surface with Gauss map $N\colon S \to S^2$, and $p \in S$. Then:*

(i) *two curves in S passing through p tangent to the same direction have the same normal curvature at p;*
(ii) *the normal curvature of S at p along a vector $v \in T_pS$ of length 1 is given by $Q_p(v)$.*

PROOF. (i) Indeed, if σ_1 and σ_2 are curves in S with $\sigma_1(0) = \sigma_2(0) = p$ and $\dot{\sigma}_1(0) = \dot{\sigma}_2(0) = v$ then (36) tells us that the normal curvature at 0 of both curves is given by $Q_p(v)$.

(ii) If σ is the normal section of S at p along v we have already remarked that $\ddot{\sigma}(0) = \tilde{\kappa}(0)N(p)$, where $\tilde{\kappa}$ is the oriented curvature of σ, and the assertion follows from (35). □

3.5. Principal, Gaussian and mean curvatures

We have now proved that the normal curvatures of a surface are exactly the values of the second fundamental form on the tangent versors. This suggests that a more in-depth study of normal curvatures by using the properties of the differential of the Gauss map should be possible (and useful). As we shall see, the basic fact is that $\mathrm{d}N_p$ is a symmetric endomorphism, and so (by the spectral theorem) it is diagonalizable.

DEFINITION 3.22. Let $S \subset \mathbb{R}^3$ be an oriented surface, denote by $N\colon S \to S^2$ its Gauss map, and take $p \in S$. A *principal direction* of S at p is an eigenvector of $\mathrm{d}N_p$ of length one, and the corresponding eigenvalue with the sign changed is a *principal curvature*.

If $v \in T_pS$ is a principal direction with principal curvature k, we have

$$Q_p(v) = -\langle \mathrm{d}N_p(v), v \rangle_p = -\langle -kv, v \rangle_p = k \ ,$$

and so the principal curvatures are normal curvatures. To be precise, they are the smallest and the largest normal curvatures at the point:

Proposition 3.7. *Let $S \subset \mathbb{R}^3$ be an oriented surface and denote by $N\colon S \to S^2$ its Gauss map. Take $p \in S$. Then we may find principal directions v_1, $v_2 \in T_pS$ with corresponding principal curvatures k_1, $k_2 \in \mathbb{R}$, with $k_1 \le k_2$ and such that:*

(i) *$\{v_1, v_2\}$ is an orthonormal basis of T_pS;*
(ii) *given a versor $v \in T_pS$, let $\theta \in (-\pi, \pi]$ be a determination of the angle between v_1 and v, so that $\cos\theta = \langle v_1, v\rangle_p$ and $\sin\theta = \langle v_2, v\rangle_p$. Then*

$$Q_p(v) = k_1 \cos^2\theta + k_2 \sin^2\theta \tag{37}$$

(Euler's formula)*;*
(iii) *k_1 is the smallest normal curvature at p, and k_2 is the largest normal curvature at p. More precisely, the set of possible normal curvatures of S at p is the interval $[k_1, k_2]$, that is,*

$$\{Q_p(v) \mid v \in T_pS,\ I_p(v) = 1\} = [k_1, k_2]\ .$$

Proof. Since $\mathrm{d}N_p$ is a symmetric endomorphism of T_pS, the spectral theorem (see [**2**, Theorem 13.5.5, p. 100], or [**4**, Theorem 22.2, p. 311]) provides us with an orthonormal basis consisting of eigenvectors $\{v_1, v_2\}$ that satisfies (i).

Given $v \in T_pS$ of length one, we may write $v = \cos\theta\, v_1 + \sin\theta\, v_2$, and so we get

$$\begin{aligned} Q_p(v) &= -\langle \mathrm{d}N_p(v), v\rangle_p = \langle k_1 \cos\theta\, v_1 + k_2 \sin\theta\, v_2, \cos\theta\, v_1 + \sin\theta\, v_2\rangle_p \\ &= k_1 \cos^2\theta + k_2 \sin^2\theta\ . \end{aligned}$$

Finally, if $k_1 = k_2$ then $\mathrm{d}N_p$ is a multiple of the identity, all normal curvatures are equal and (iii) is trivial. If, on the other hand, $k_1 < k_2$ then (37) tells us that

$$Q_p(v) = k_1 + (k_2 - k_1)\sin^2\theta\ .$$

So the normal curvature has a maximum (respectively, a minimum) for $\theta = \pm\pi/2$ (respectively, $\theta = 0, \pi$), that is, for $v = \pm v_2$ (respectively, $v = \pm v_1$), and this maximum (respectively, minimum) is exactly k_2 (respectively, k_1). Moreover, for $\theta \in (-\pi, \pi]$ the normal curvature takes all possible values between k_1 and k_2, and (iii) is proved. □

When you learned about linear endomorphisms you certainly saw that two fundamental quantities for describing their behavior are the trace (given by the sum of the eigenvalues) and the determinant (given by the product of the eigenvalues). You will then not be surprised in learning that the trace and (even more so) the determinant of $\mathrm{d}N_p$ are going to play a crucial role when studying surfaces.

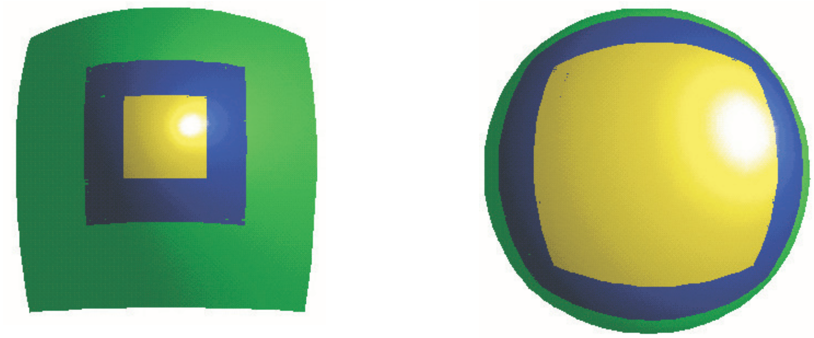
Definition 3.23. Let $S \subset \mathbb{R}^3$ be an oriented surface, and denote by $N\colon S \to S^2$ its Gauss map. The *Gaussian curvature* of S is the function $K\colon S \to \mathbb{R}$ given by

$$\forall p \in S \qquad K(p) = \det(\mathrm{d}N_p)\ ,$$

while the *mean curvature* of S is the function $H\colon S \to \mathbb{R}$ given by

$$\forall p \in S \qquad H(p) = -\frac{1}{2}\mathrm{tr}(\mathrm{d}N_p)\ .$$

Remark 3.16. If k_1 and k_2 are the principal curvatures of S at $p \in S$, then $K(p) = k_1k_2$ and $H(p) = (k_1 + k_2)/2$.



$\varphi(B_\delta)$ $N \circ \varphi(B_\delta)$

FIGURE 5

REMARK 3.17. If we change the orientation on S the Gauss map N changes sign, and so both the principal curvatures and the mean curvature change sign; the Gaussian curvature K, on the other hand, does not change. So we may define the Gaussian curvature for non-orientable surfaces too: if p is a point of an arbitrary surface S, the Gaussian curvature of S at p is the Gaussian curvature at p of the image of an arbitrary local parametrization of S centered at p (remember Remark 3.12 too). Analogously, the absolute value of the mean curvature is well defined on non-orientable surfaces too.

REMARK 3.18. The Gaussian curvature admits an interesting interpretation in terms of ratios of areas. Let $\varphi\colon U \to \mathbb{R}^3$ be a local parametrization of a surface $S \subset \mathbb{R}^3$ centered at $p \in S$, and denote by $B_\delta \subset \mathbb{R}^2$ the open disk with center in the origin and radius $\delta > 0$. Then if $K(p) \neq 0$ we have (see Fig 5)

$$|K(p)| = \lim_{\delta \to 0} \frac{\operatorname{Area}\bigl(N \circ \varphi(B_\delta)\bigr)}{\operatorname{Area}\bigl(\varphi(B_\delta)\bigr)} .$$

To prove this, note first that $K(p) = \det \mathrm{d}N_p \neq 0$ implies that $N \circ \varphi|_{B_\delta}$ is a local parametrization of the sphere for $\delta > 0$ small enough. Then Theorem 3.3 and Lemma 3.1 imply

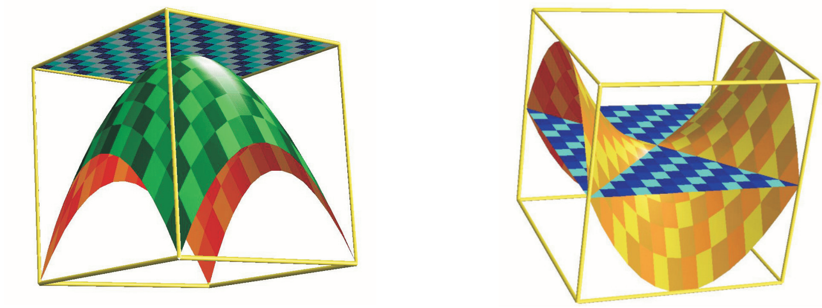
$$\begin{aligned}\operatorname{Area}\bigl(N \circ \varphi(B_\delta)\bigr) &= \int_{B_\delta} \left\| \frac{\partial (N \circ \varphi)}{\partial x_1} \wedge \frac{\partial (N \circ \varphi)}{\partial x_2} \right\| \mathrm{d}x_1\, \mathrm{d}x_2 \\ &= \int_{B_\delta} |K|\, \|\partial_1 \wedge \partial_2\|\, \mathrm{d}x_1\, \mathrm{d}x_2 ,\end{aligned}$$

and

$$\operatorname{Area}\bigl(\varphi(B_\delta)\bigr) = \int_{B_\delta} \|\partial_1 \wedge \partial_2\|\, \mathrm{d}x_1\, \mathrm{d}x_2 .$$

Hence,

$$\begin{aligned}\lim_{\delta \to 0} \frac{\operatorname{Area}\bigl(N \circ \varphi(B_\delta)\bigr)}{\operatorname{Area}\bigl(\varphi(B_\delta)\bigr)} &= \frac{\lim\limits_{\delta \to 0} (\pi\delta^2)^{-1} \int_{B_\delta} |K|\, \|\partial_1 \wedge \partial_2\|\, \mathrm{d}x_1\, \mathrm{d}x_2}{\lim\limits_{\delta \to 0} (\pi\delta^2)^{-1} \int_{B_\delta} \|\partial_1 \wedge \partial_2\|\, \mathrm{d}x_1\, \mathrm{d}x_2} \\ &= \frac{K(p) \|\partial_1|_p \wedge \partial_2|_p\|}{\|\partial_1|_p \wedge \partial_2|_p\|} = K(p) ,\end{aligned}$$



(a) (b)

FIGURE 6

using the mean value theorem for multiple integrals (see [**3**, Problem 6, p. 190] for a sketch of the proof).

REMARK 3.19. The sign of the Gaussian curvature may give an idea of how a surface looks like. If $p \in S$ is a point with $K(p) > 0$, all normal curvatures at p have the same sign. Intuitively, this means that all normal sections of S at p curve on the same side (why?) with respect to T_pS, and so in proximity of p the surface lies all on a single side of the tangent plane: see Fig. 6.(a). On the other hand, if $K(p) < 0$, we have normal curvatures of both signs at p; this means that the normal sections may curve on opposite sides with respect to T_pS, and so in a neighborhood of p the surface has sections on both sides of the tangent plane: see Fig. 6.(b). Nothing can be said a priori when $K(p) = 0$. Problem 3.17 and Exercises 3.53 and 3.48 will formalize these intuitive ideas.

The previous remark suggests a classification of the points of S according to the sign of the Gaussian curvature.

DEFINITION 3.24. Let $S \subset \mathbb{R}^3$ be an oriented surface, and denote by $N\colon S \to S^2$ its Gauss map. A point $p \in S$ is *elliptic* if $K(p) > 0$ (and so all normal curvatures at p have the same sign); *hyperbolic* if $K(p) < 0$ (and so there are normal curvatures at p with opposite signs); *parabolic* if $K(p) = 0$ but $\mathrm{d}N_p \neq O$; and *planar* if $\mathrm{d}N_p = O$.

The rest of this section will be devoted to finding an effective procedure for computing the various kind of curvatures (principal, Gaussian and mean) we introduced. Let us begin by studying how to express the second fundamental form in local coordinates.

Fix a local parametrization $\varphi\colon U \to S$ at $p \in S$ of an oriented surface $S \subset \mathbb{R}^3$ with Gauss map $N\colon S \to S^2$. If $v = v_1\partial_1 + v_2\partial_2 \in T_pS$, then

$$Q_p(v) = Q_p(\partial_1)v_1^2 - 2\langle \mathrm{d}N_p(\partial_1), \partial_2\rangle_p v_1 v_2 + Q_p(\partial_2)v_2^2 \,. \tag{38}$$

So it is natural to give the following

DEFINITION 3.25. Let $\varphi\colon U \to S$ be a local parametrization of a surface S. The *form coefficients* of S with respect to φ are the three functions e, f, and $g\colon U \to \mathbb{R}$

defined by

$$
\begin{aligned}
e(x) &= Q_{\varphi(x)}(\partial_1) = -\langle \mathrm{d}N_{\varphi(x)}(\partial_1), \partial_1 \rangle_{\varphi(x)} \;, \\
f(x) &= -\langle \mathrm{d}N_{\varphi(x)}(\partial_1), \partial_2 \rangle_{\varphi(x)} \;, \\
g(x) &= Q_{\varphi(x)}(\partial_2) = -\langle \mathrm{d}N_{\varphi(x)}(\partial_2), \partial_2 \rangle_{\varphi(x)}
\end{aligned}
\tag{39}
$$

for all $x \in U$, where $N = \partial_1 \wedge \partial_2 / \|\partial_1 \wedge \partial_2\|$, as usual.

Remark 3.20. Again, this is Gauss' notation. We shall sometimes also use the more modern notation $e = h_{11}$, $f = h_{12} = h_{21}$, and $g = h_{22}$.

Remark 3.21. By differentiating the identities $\langle N \circ \varphi, \partial_j \rangle \equiv 0$ for $j = 1$, 2, it is straightforward to get the following expressions for form coefficients:

$$
e = \left\langle N \circ \varphi, \frac{\partial^2 \varphi}{\partial x_1^2} \right\rangle \;, \; f = \left\langle N \circ \varphi, \frac{\partial^2 \varphi}{\partial x_1 \partial x_2} \right\rangle \;, \; g = \left\langle N \circ \varphi, \frac{\partial^2 \varphi}{\partial x_2^2} \right\rangle \;. \tag{40}
$$

Remark 3.22. We have introduced e, f and g as functions defined on U. However, it will sometimes be more convenient to consider them as functions defined on $\varphi(U)$, that is, to replace them with $e \circ \varphi^{-1}$, $f \circ \varphi^{-1}$ and $g \circ \varphi^{-1}$, respectively. Finally, form coefficients also significantly depend on the local parametrization we have chosen, as it is easy to verify (see Example 3.32).

Remark 3.23. Metric and form coefficients depend on the chosen local parametrization, whereas the Gaussian curvature and the absolute value of the mean curvature do not, since they are defined directly from the Gauss map, without using local parametrizations.

Clearly, the form coefficients are (why?) functions of class C^∞ on U that completely determine the second fundamental form: indeed, from (38) we get

$$
Q_p(v_1 \partial_1 + v_2 \partial_2) = e(x) v_1^2 + 2 f(x) v_1 v_2 + g(x) v_2^2
$$

for all $p = \varphi(x) \in \varphi(U)$ and $v_1 \partial_1 + v_2 \partial_2 \in T_p S$.

Furthermore, (40) can be used to explicitly compute the form coefficients (as we shall momentarily verify on our usual examples). So, to get an effective way for computing principal, Gaussian and mean curvatures it will suffice to express them in terms of metric and form coefficients. Remember that principal, Gaussian and mean curvatures are defined from the eigenvalues of $\mathrm{d}N_p$; so it may be helpful to try and write the matrix $A \in M_{2,2}(\mathbb{R})$ representing $\mathrm{d}N_p$ with respect to the basis $\{\partial_1, \partial_2\}$ using the functions E, F, G, e, f and g. Now, for all $v = v_1 \partial_1 + v_2 \partial_2$, $w = w_1 \partial_1 + w_2 \partial_2 \in T_p S$, we have

$$
\begin{vmatrix} w_1 & w_2 \end{vmatrix} \begin{vmatrix} e & f \\ f & g \end{vmatrix} \begin{vmatrix} v_1 \\ v_2 \end{vmatrix} = -\langle \mathrm{d}N_p(v), w \rangle_p = - \begin{vmatrix} w_1 & w_2 \end{vmatrix} \begin{vmatrix} E & F \\ F & G \end{vmatrix} A \begin{vmatrix} v_1 \\ v_2 \end{vmatrix} \;;
$$

from this it follows (why?) that

$$
\begin{vmatrix} e & f \\ f & g \end{vmatrix} = - \begin{vmatrix} E & F \\ F & G \end{vmatrix} A \;.
$$

Now, $\begin{vmatrix} E & F \\ F & G \end{vmatrix}$ is the matrix that represents a positive definite scalar product with respect to a basis; in particular, it is invertible and has positive determinant $EG - F^2$. So we have proved the following

Proposition 3.8. *Let $\varphi\colon U \to S$ be a local parametrization of a surface $S \subset \mathbb{R}^3$, and set $N = \partial_1 \wedge \partial_2 / \|\partial_1 \wedge \partial_2\|$. Then the matrix $A \in M_{2,2}(\mathbb{R})$ representing the endomorphism* $\mathrm{d}N$ *with respect to the basis $\{\partial_1, \partial_2\}$ is given by*

$$\begin{aligned} A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} &= -\begin{vmatrix} E & F \\ F & G \end{vmatrix}^{-1} \begin{vmatrix} e & f \\ f & g \end{vmatrix} \\ &= -\frac{1}{EG - F^2} \begin{vmatrix} eG - fF & fG - gF \\ fE - eF & gE - fF \end{vmatrix} \,. \end{aligned} \tag{41}$$

In particular, the Gaussian curvature is given by

$$K = \det(A) = \frac{eg - f^2}{EG - F^2} \,, \tag{42}$$

the mean curvature is given by

$$H = -\frac{1}{2}\mathrm{tr}(A) = \frac{1}{2}\frac{eG - 2fF + gE}{EG - F^2} \,, \tag{43}$$

and the principal curvatures by

$$k_{1,2} = H \pm \sqrt{H^2 - K} \,. \tag{44}$$

Remark 3.24. If $\varphi\colon U \to S$ is a local parametrization with $F \equiv f \equiv 0$ the previous formulas become simpler:

$$K = \frac{eg}{EG} \,, \quad H = \frac{1}{2}\left(\frac{e}{E} + \frac{g}{G}\right) \,, \quad k_1 = \frac{e}{E} \,, \quad k_2 = \frac{g}{G} \,.$$

We may now compute the various curvatures for our usual examples.

Example 3.30. In the plane we have $e \equiv f \equiv g \equiv 0$, no matter which parametrization we are using, since the second fundamental form is zero everywhere. In particular, the principal, Gaussian and mean curvatures are all zero everywhere.

Example 3.31. For the right circular cylinder with the parametrization of Example 3.3 we have $e \equiv -1$ and $f \equiv g \equiv 0$, so $K \equiv 0$, $H \equiv -1/2$, $k_1 = -1$, and $k_2 = 0$.

Example 3.32. We have seen in Example 3.25 that on the sphere oriented as in Example 3.17 we have $Q_p = -I_p$. This means that for any parametrization the form coefficients have the same absolute value and opposite sign as the corresponding metric coefficients. In particular, $K \equiv 1$, $H \equiv -1$ and $k_1 \equiv k_2 \equiv -1$.

Example 3.33. Let $U \subseteq \mathbb{R}^2$ be an open set, $h \in C^\infty(U)$, and $\varphi\colon U \to \mathbb{R}^3$ the local parametrization of the graph Γ_h given by $\varphi(x) = \big(x, h(x)\big)$. Recalling Examples 3.2 and 3.19 we get

$$e = \frac{1}{\sqrt{1 + \|\nabla h\|^2}}\frac{\partial^2 h}{\partial x_1^2} \,, \; f = \frac{1}{\sqrt{1 + \|\nabla h\|^2}}\frac{\partial^2 h}{\partial x_1 \partial x_2} \,, \; g = \frac{1}{\sqrt{1 + \|\nabla h\|^2}}\frac{\partial^2 h}{\partial x_2^2} \,,$$

hence,

$$\begin{aligned} K &= \frac{1}{(1 + \|\nabla h\|^2)^2} \det \mathrm{Hess}(h) \,, \\ H &= \frac{1}{2(1 + \|\nabla h\|^2)^{3/2}} \left[\frac{\partial^2 h}{\partial x_1^2}\left(1 + \left|\frac{\partial h}{\partial x_2}\right|^2\right) + \frac{\partial^2 h}{\partial x_2^2}\left(1 + \left|\frac{\partial h}{\partial x_1}\right|^2\right) \right. \\ &\qquad \left. -2\frac{\partial^2 h}{\partial x_1 \partial x_2}\frac{\partial h}{\partial x_1}\frac{\partial h}{\partial x_2} \right] \,. \end{aligned}$$

EXAMPLE 3.34. For a helicoid parametrized as in Example 3.5, we easily find $f = a/\sqrt{a^2+y^2}$ and $e \equiv g \equiv 0$, so

$$K = -\frac{a^2}{(a^2+y^2)^2}\,, \qquad H \equiv 0\,, \qquad k_{1,2} = \pm\frac{a}{a^2+y^2}\,.$$

EXAMPLE 3.35. For a catenoid parametrized as in Example 3.6, we find $e \equiv -a$, $f \equiv 0$ and $g \equiv a$, so

$$K = -\frac{1}{a^2\cosh^4 x}\,, \qquad H \equiv 0\,, \qquad k_{1,2} = \pm\frac{1}{a\cosh^2 x}\,.$$

EXAMPLE 3.36. Let S be a surface of revolution, parametrized as in Example 3.7. The form coefficients are then given by

$$e = \frac{\alpha'\beta'' - \beta'\alpha''}{\sqrt{(\alpha')^2+(\beta')^2}}, \quad f \equiv 0, \quad g = \frac{\alpha\beta'}{\sqrt{(\alpha')^2+(\beta')^2}}\,.$$

Recalling Remark 3.24, we get

$$K = \frac{\beta'(\alpha'\beta''-\beta'\alpha'')}{\alpha\big((\alpha')^2+(\beta')^2\big)^2}\,, \quad H = \frac{\alpha(\alpha'\beta''-\beta'\alpha'')+\beta'\big((\alpha')^2+(\beta')^2\big)}{2\alpha\big((\alpha')^2+(\beta')^2\big)^{3/2}}\,,$$
$$k_1 = \frac{\alpha'\beta''-\beta'\alpha''}{\big((\alpha')^2+(\beta')^2\big)^{3/2}}\,, \quad k_2 = \frac{\beta'}{\alpha\big((\alpha')^2+(\beta')^2\big)^{1/2}}\,.$$

If the generatrix of S is parametrized by arc length, these formulas become quite simpler: by differentiating $\dot\alpha^2 + \dot\beta^2 \equiv 1$ we get $\dot\alpha\ddot\alpha + \dot\beta\ddot\beta \equiv 0$, and so

$$K = -\frac{\ddot\alpha}{\alpha}\,, \quad H = \frac{\dot\beta + \alpha(\dot\alpha\ddot\beta - \dot\beta\ddot\alpha)}{2\alpha}\,, \quad k_1 = \dot\alpha\ddot\beta - \dot\beta\ddot\alpha\,, \quad k_2 = \frac{\dot\beta}{\alpha}\,.$$

EXAMPLE 3.37. Let $\sigma\colon (\pi/2,\pi) \to \mathbb{R}^3$ be the upper half of the tractrix given by

$$\sigma(t) = \left(\sin t, 0, \cos t + \log\tan\frac{t}{2}\right)\,;$$

see Problem 1.3. The surface of revolution S obtained by rotating the tractrix around the z-axis is called *pseudosphere*; see Fig. 7 (and Exercise 2.18). By using the previous example, it is easy to find (see also Problem 3.8) that the pseudosphere has constant Gaussian curvature equal to -1.

REMARK 3.25. The plane is an example of surface with constant Gaussian curvature equal to zero, and spheres are examples of surfaces with positive constant Gaussian curvature (Exercise 3.27). Other examples of surfaces with zero constant Gaussian curvature are cylinders (Exercise 3.26). The pseudosphere, on the other hand, is an example of a surface with negative constant Gaussian curvature but, unlike planes, cylinders, and spheres, it is not a closed surface in $\mathbb{R}^3$. This is not a coincidence: it is possible to prove that closed surfaces in $\mathbb{R}^3$ with negative constant Gaussian curvature *do not exist* (*Hilbert's Theorem*). Moreover, it is also possible to prove that spheres are the only closed surfaces with positive constant Gaussian curvature, and that planes and cylinder are the only closed surfaces with zero constant Gaussian curvature.

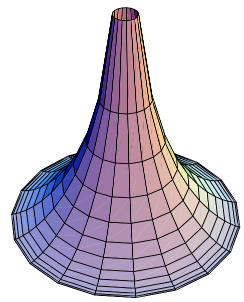


Figure 7. The pseudosphere

3.6. Gauss' Theorema egregium

The goal of this section is to prove that the Gaussian curvature is an intrinsic property of a surface: it only depends on the first fundamental form, and not on the way the surface is immersed in $\mathbb{R}^3$. As you can imagine, it is an highly unexpected result; the definition of K directly involves the Gauss map, which is very strongly related to the embedding of the surface in $\mathbb{R}^3$. Nevertheless, the Gaussian curvature can be measured staying within the surface, forgetting the ambient space. In particular, two isometric surfaces have the same Gaussian curvature; and this will give us a necessary condition a surface has to satisfy for the existence of a similitude with an open subset of the plane.

The road to get to this result is almost as important as the result itself. The idea is to proceed as we did to get Frenet-Serret formulas for curves. The Frenet frame allows us to associate with each point of the curve a basis of $\mathbb{R}^3$; hence it is possible to express the derivatives of the Frenet frame as a linear combination of the frame itself, and the coefficients turn out to be fundamental geometric quantities for studying the curve.

Let us see how to adapt such an argument to surfaces. Let $\varphi\colon U \to S$ be a local parametrization of a surface $S \subset \mathbb{R}^3$, and let $N\colon \varphi(U) \to S^2$ be the Gauss map of $\varphi(U)$ given by $N = \partial_1 \wedge \partial_2 / \|\partial_1 \wedge \partial_2\|$, as usual. The triple $\{\partial_1, \partial_2, N\}$ is a basis of $\mathbb{R}^3$ everywhere, and so we may express any vector of $\mathbb{R}^3$ as a linear combination of these vectors. In particular, there must exist functions Γ_{ij}^h, h_{ij}, $a_{ij} \in C^\infty(U)$ such that

$$\frac{\partial^2 \varphi}{\partial x_i \partial x_j} = \Gamma_{ij}^1 \partial_1 + \Gamma_{ij}^2 \partial_2 + h_{ij} N \,, \tag{45}$$

$$\frac{\partial (N \circ \varphi)}{\partial x_j} = a_{1j} \partial_1 + a_{2j} \partial_2 \,, \tag{46}$$

for i, $j = 1$, 2, where in last formula there are no terms proportional to N because $\|N\| \equiv 1$ implies that all partial derivatives of $N \circ \varphi$ are orthogonal to N. Note further that, by Theorem 3.4, the terms Γ_{ij}^r and h_{ij} are symmetric with respect to their lower indices, that is, $\Gamma_{ji}^r = \Gamma_{ij}^r$ and $h_{ji} = h_{ij}$ for all i, j, $r = 1$, 2.

We already know some of the functions appearing in (46). For instance, since $\partial(N \circ \varphi)/\partial x_i = \mathrm{d}N_p(\partial_i)$, the terms a_{ij} are just the components of the matrix A that represents $\mathrm{d}N_p$ with respect to the basis $\{\partial_1, \partial_2\}$, and so they are given by (41). The terms h_{ij} are known too: by (40) we know that they are exactly the form coefficients (thus the notation is consistent with Remark 3.20). So the only quantities that are still unknown are the coefficients Γ_{ij}^r.

Definition 3.26. The functions Γ_{ij}^r are the *Christoffel symbols* of the local parametrization φ.

We proceed now to compute Christoffel symbols. Taking the scalar product of (45) with ∂_1 and ∂_2 $(i = j = 1)$ yields

$$\begin{cases} E\Gamma_{11}^1 + F\Gamma_{11}^2 = \left\langle \dfrac{\partial^2\varphi}{\partial x_1^2}, \partial_1 \right\rangle = \dfrac{1}{2}\dfrac{\partial}{\partial x_1}\langle \partial_1, \partial_1 \rangle = \dfrac{1}{2}\dfrac{\partial E}{\partial x_1}\,, \\ F\Gamma_{11}^1 + G\Gamma_{11}^2 = \left\langle \dfrac{\partial^2\varphi}{\partial x_1^2}, \partial_2 \right\rangle = \dfrac{\partial}{\partial x_1}\langle \partial_1, \partial_2 \rangle - \left\langle \partial_1, \dfrac{\partial^2\varphi}{\partial x_1 \partial x_2} \right\rangle = \dfrac{\partial F}{\partial x_1} - \dfrac{1}{2}\dfrac{\partial E}{\partial x_2}\,. \end{cases} \tag{47}$$

Analogously, we find

$$\begin{cases} E\Gamma_{12}^1 + F\Gamma_{12}^2 = \dfrac{1}{2}\dfrac{\partial E}{\partial x_2}\,, \\ F\Gamma_{12}^1 + G\Gamma_{12}^2 = \dfrac{1}{2}\dfrac{\partial G}{\partial x_1}\,, \end{cases} \tag{48}$$

and

$$\begin{cases} E\Gamma_{22}^1 + F\Gamma_{22}^2 = \dfrac{\partial F}{\partial x_2} - \dfrac{1}{2}\dfrac{\partial G}{\partial x_1}\,, \\ F\Gamma_{22}^1 + G\Gamma_{22}^2 = \dfrac{1}{2}\dfrac{\partial G}{\partial x_2}\,. \end{cases} \tag{49}$$

These are three square linear systems whose matrix of coefficients has determinant $EG - F^2$, which is always positive; so they have a unique solution, and it can be expressed in terms of metric coefficients and of their derivatives (see Exercise 3.57).

Remark 3.26. Note that, in particular, *the Christoffel symbols only depend on the first fundamental form of S*, and so they are intrinsic. As a consequence, *any quantity that can be written in terms of Christoffel symbols is intrinsic*: it only depends on the metric structure of the surface, and not on the way the surface is immersed in $\mathbb{R}^3$.

Remark 3.27. We explicitly remark, since it will be useful later on, that if the local parametrization is orthogonal (that is, if $F \equiv 0$) the Christoffel symbols have a particularly simple expression:

$$\begin{cases} \Gamma_{11}^1 = \dfrac{1}{2E}\dfrac{\partial E}{\partial x_1}\,, \quad \Gamma_{12}^1 = \Gamma_{21}^1 = \dfrac{1}{2E}\dfrac{\partial E}{\partial x_2}\,, \quad \Gamma_{22}^1 = -\dfrac{1}{2E}\dfrac{\partial G}{\partial x_1}\,, \\ \Gamma_{11}^2 = -\dfrac{1}{2G}\dfrac{\partial E}{\partial x_2}\,, \quad \Gamma_{12}^2 = \Gamma_{21}^2 = \dfrac{1}{2G}\dfrac{\partial G}{\partial x_1}\,, \quad \Gamma_{22}^2 = \dfrac{1}{2G}\dfrac{\partial G}{\partial x_2}\,. \end{cases} \tag{50}$$

Let us see now the value of the Christoffel symbols in our canonical examples.

Example 3.38. By Example 3.1, we know that the Christoffel symbols of the plane are zero everywhere.

Example 3.39. The Christoffel symbols of the right circular cylinder parametrized as in Example 3.3 are identically zero too.

Example 3.40. The Christoffel symbols of the local parametrization of the sphere $\varphi(x,y)=(x,y,\sqrt{1-x^2-y^2})$ are

$$\begin{cases} \Gamma_{11}^1=\dfrac{x}{1-x^2-y^2}\;, & \Gamma_{12}^1=\Gamma_{21}^1=\dfrac{x^2y}{1-x^2-y^2}\;,\\ \Gamma_{22}^1=\dfrac{x(1-x^2)}{1-x^2-y^2}\;, & \Gamma_{11}^2=\dfrac{y(1-y^2)}{1-x^2-y^2}\;,\\ \Gamma_{12}^2=\Gamma_{21}^2=\dfrac{xy^2}{1-x^2-y^2}\;, & \Gamma_{22}^2=\dfrac{y(1-x^2)}{1-x^2-y^2}\;. \end{cases}$$

On the other hand, the Christoffel symbols of the other local parametrization of the sphere $\psi(\theta,\psi)=(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta)$ given in Example 3.4 are

$$\begin{cases} \Gamma_{11}^1\equiv 0\;, \quad \Gamma_{12}^1=\Gamma_{21}^1\equiv 0\;, \quad \Gamma_{22}^1=-\sin\theta\cos\theta\;,\\ \Gamma_{11}^2\equiv 0\;, \quad \Gamma_{12}^2=\Gamma_{21}^2=\dfrac{\cos\theta}{\sin\theta}\;, \quad \Gamma_{22}^2\equiv 0\;. \end{cases}$$

Example 3.41. Let $U\subseteq\mathbb{R}^2$ be an open set, $h\in C^\infty(U)$, and $\varphi\colon U\to\mathbb{R}^3$ the local parametrization of the graph Γ_h given by $\varphi(x)=\big(x,h(x)\big)$. Recalling Example 3.2 we get

$$\begin{cases} \Gamma_{11}^1=\dfrac{(\partial h/\partial x_1)(\partial^2h/\partial x_1^2)}{1+\|\nabla h\|^2}\;, & \Gamma_{12}^1=\Gamma_{21}^1=\dfrac{(\partial h/\partial x_1)(\partial^2h/\partial x_1\partial x_2)}{1+\|\nabla h\|^2}\;,\\ \Gamma_{22}^1=\dfrac{(\partial h/\partial x_1)(\partial^2h/\partial x_2^2)}{1+\|\nabla h\|^2}\;, & \Gamma_{11}^2=\dfrac{(\partial h/\partial x_2)(\partial^2h/\partial x_1^2)}{1+\|\nabla h\|^2}\;,\\ \Gamma_{12}^2=\Gamma_{21}^2=\dfrac{(\partial h/\partial x_2)(\partial^2h/\partial x_1\partial x_2)}{1+\|\nabla h\|^2}\;, & \Gamma_{22}^2=\dfrac{(\partial h/\partial x_2)(\partial^2h/\partial x_2^2)}{1+\|\nabla h\|^2}\;. \end{cases}$$

Example 3.42. The Christoffel symbols of the helicoid parametrized as in Example 3.5 are

$$\begin{cases} \Gamma_{11}^1\equiv 0\;, \quad \Gamma_{12}^1=\Gamma_{21}^1=\dfrac{y}{a^2+y^2}\;, \quad \Gamma_{22}^1\equiv 0\;,\\ \Gamma_{11}^2=-y\;, \quad \Gamma_{12}^2=\Gamma_{21}^2\equiv 0\;, \quad \Gamma_{22}^2\equiv 0\;. \end{cases}$$

Example 3.43. The Christoffel symbols of the catenoid parametrized as in Example 3.6 are

$$\begin{cases} \Gamma_{11}^1=\dfrac{\sinh x}{\cosh x}\;, \quad \Gamma_{12}^1=\Gamma_{21}^1\equiv 0\;, \quad \Gamma_{22}^1=-\dfrac{\sinh x}{\cosh x}\;,\\ \Gamma_{11}^2\equiv 0\;, \quad \Gamma_{12}^2=\Gamma_{21}^2=\dfrac{\sinh x}{\cosh x}\;, \quad \Gamma_{22}^2\equiv 0\;. \end{cases}$$

Example 3.44. We conclude with the Christoffel symbols of a surface of revolution parametrized as in Example 3.7:

$$\begin{cases} \Gamma_{11}^1=\dfrac{\alpha'\alpha''+\beta'\beta''}{(\alpha')^2+(\beta')^2}\;, \quad \Gamma_{12}^1=\Gamma_{21}^1\equiv 0\;, \quad \Gamma_{22}^1=-\dfrac{\alpha\alpha'}{(\alpha')^2+(\beta')^2}\;,\\ \Gamma_{11}^2\equiv 0\;, \quad \Gamma_{12}^2=\Gamma_{21}^2=\dfrac{\alpha'}{\alpha}\;, \quad \Gamma_{22}^2\equiv 0\;. \end{cases} \tag{51}$$

Now, unlike what happened for curvature and torsion, the Christoffel symbols cannot be chosen arbitrarily; they must satisfy some compatibility conditions. To find them, let us compute the third derivatives of the parametrization.

As for the second derivatives, there exist functions A^r_{ijk}, $B_{ijk} \in C^\infty(U)$ such that

$$\frac{\partial^3\varphi}{\partial x_i \partial x_j \partial x_k} = A^1_{ijk}\partial_1 + A^2_{ijk}\partial_2 + B_{ijk}N \ .$$

Again by Theorem 3.4 we are sure that the functions A^r_{ijk} and B_{ijk} are symmetric in the lower indices. In particular,

$$A^r_{ijk} = A^r_{jik} = A^r_{ikj} \qquad \text{and} \qquad B_{ijk} = B_{jik} = B_{ikj} \tag{52}$$

for all i, j, k, $r = 1$, 2.

To compute the expression of A^r_{ijk} and B_{ijk}, we differentiate (45) and then insert (45) and (46) in what we find. We get

$$A^r_{ijk} = \frac{\partial \Gamma^r_{jk}}{\partial x_i} + \Gamma^1_{jk}\Gamma^r_{i1} + \Gamma^2_{jk}\Gamma^r_{i2} + h_{jk}a_{ri} \ , \quad B_{ijk} = \Gamma^1_{jk}h_{i1} + \Gamma^2_{jk}h_{i2} + \frac{\partial h_{jk}}{\partial x_i} \ .$$

Recalling that $A^r_{ijk} - A^r_{jik} = 0$, we find, for all i, j, k, $r = 1$, 2 the fundamental *Gauss' equations*:

$$\frac{\partial \Gamma^r_{jk}}{\partial x_i} - \frac{\partial \Gamma^r_{ik}}{\partial x_j} + \sum_{s=1}^{2}\left(\Gamma^s_{jk}\Gamma^r_{is} - \Gamma^s_{ik}\Gamma^r_{js}\right) = -(h_{jk}a_{ri} - h_{ik}a_{rj}) \ . \tag{53}$$

Before examining what can be deduced from the symmetry of B_{ijk}, note an important consequence of Gauss' equations. If we write (53) for $i = r = 1$ and $j = k = 2$ (see Exercise 3.58 for the other cases), we get

$$\begin{aligned} \frac{\partial \Gamma^1_{22}}{\partial x_1} - \frac{\partial \Gamma^1_{12}}{\partial x_2} + \sum_{s=1}^{2}\left(\Gamma^s_{22}\Gamma^1_{1s} - \Gamma^s_{12}\Gamma^1_{2s}\right) &= -(h_{22}a_{11} - h_{12}a_{12}) \\ &= \frac{(eg - f^2)G}{EG - F^2} = GK \ . \end{aligned}$$

Since, as already remarked, the Christoffel symbols only depend on the first fundamental form, we have proved the very famous *Gauss' Theorema Egregium*:

THEOREM 3.5 (Gauss' Theorema Egregium). *The Gaussian curvature K of a surface is given by the formula*

$$K = \frac{1}{G}\left[\frac{\partial \Gamma^1_{22}}{\partial x_1} - \frac{\partial \Gamma^1_{12}}{\partial x_2} + \sum_{s=1}^{2}\left(\Gamma^s_{22}\Gamma^1_{1s} - \Gamma^s_{12}\Gamma^1_{2s}\right)\right] \ . \tag{54}$$

In particular, the Gaussian curvature of a surface is an intrinsic property, that is, it only depends on the first fundamental form.

As a consequence, two locally isometric surfaces must have the same Gaussian curvature:

COROLLARY 3.2. *Let $F\colon S \to \tilde{S}$ be a local isometry between two surfaces. Then $\tilde{K} \circ F = K$, where K is the Gaussian curvature of S and $\tilde{K}$ is the Gaussian curvature of $\tilde{S}$. More generally, if F is a similitude with scale factor $r > 0$ then $\tilde{K} \circ F = r^{-2}K$.*

PROOF. It immediately follows from Theorem 3.5, Proposition 3.1, the definition of similitude, and Exercise 3.11. $\square$

REMARK 3.28. Warning: there exist maps $F\colon S \to \tilde{S}$ that satisfy $\tilde{K} \circ F = K$ but are not local isometries; see Exercise 3.41.

As a consequence of Corollary 3.2, *if a surface S is locally isometric to (or, more in general, has a similitude with) a portion of a plane, then the Gaussian curvature of S is zero everywhere.* Hence, there is no local isometry between a portion of a sphere and a portion of a plane, because the sphere has Gaussian curvature positive everywhere while the plane has zero Gaussian curvature: anguished cartographers have to accept that it is not possible to draw a geographical map that preserves distances, not even scaled by some factor.

One last consequence of Theorem 3.5 is another explicit formula for computing the Gaussian curvature:

LEMMA 3.3. *Let $\varphi\colon U \to S$ be an orthogonal local parametrization of a surface S. Then*

$$K = -\frac{1}{2\sqrt{EG}}\left\{\frac{\partial}{\partial x_2}\left(\frac{1}{\sqrt{EG}}\frac{\partial E}{\partial x_2}\right) + \frac{\partial}{\partial x_1}\left(\frac{1}{\sqrt{EG}}\frac{\partial G}{\partial x_1}\right)\right\}.$$

PROOF. If we substitute (50) in (54), we get

$$\begin{aligned}
K &= \frac{1}{G}\left[-\frac{\partial}{\partial x_1}\left(\frac{1}{2E}\frac{\partial G}{\partial x_1}\right) - \frac{\partial}{\partial x_2}\left(\frac{1}{2E}\frac{\partial E}{\partial x_2}\right) - \frac{1}{4E^2}\frac{\partial G}{\partial x_1}\frac{\partial E}{\partial x_1}\right.\\
&\qquad\left. + \frac{1}{4EG}\frac{\partial G}{\partial x_2}\frac{\partial E}{\partial x_2} - \frac{1}{4E^2}\left(\frac{\partial E}{\partial x_2}\right)^2 + \frac{1}{4EG}\left(\frac{\partial G}{\partial x_1}\right)^2\right]\\
&= \frac{1}{4E^2G^2}\left(E\frac{\partial G}{\partial x_2} + G\frac{\partial E}{\partial x_2}\right)\frac{\partial E}{\partial x_2} - \frac{1}{2EG}\frac{\partial^2 E}{\partial x_2^2}\\
&\qquad + \frac{1}{4E^2G^2}\left(G\frac{\partial E}{\partial x_1} + E\frac{\partial G}{\partial x_1}\right)\frac{\partial G}{\partial x_1} - \frac{1}{2EG}\frac{\partial^2 G}{\partial x_1^2}\\
&= -\frac{1}{2\sqrt{EG}}\left\{\frac{\partial}{\partial x_2}\left(\frac{1}{\sqrt{EG}}\frac{\partial E}{\partial x_2}\right) + \frac{\partial}{\partial x_1}\left(\frac{1}{\sqrt{EG}}\frac{\partial G}{\partial x_1}\right)\right\}.
\end{aligned}$$

□

We close this chapter by completing the discussion of (52). The condition $B_{ijk} - B_{jik} = 0$ yields, for all i, j, $k = 1, 2$ the *Codazzi-Mainardi equations*:

$$\sum_{s=1}^{2}\left(\Gamma_{jk}^{s}h_{is} - \Gamma_{ik}^{s}h_{js}\right) = \frac{\partial h_{ik}}{\partial x_j} - \frac{\partial h_{jk}}{\partial x_i}. \tag{55}$$

Though less important than Gauss' equations, the Codazzi-Mainardi equations can be nonetheless very useful when studying surfaces.

Summing up, if φ is a local parametrization of a regular surface the coordinates of φ have to satisfy the systems of partial differential equations (45)–(46), whose coefficients depend on the metric and form coefficients E, F, G, e, f and g, which in turn satisfy the compatibility conditions (53) and (55). Conversely, it is possible to prove the *fundamental theorem of the local theory of surfaces* (also known as *Bonnet's theorem*), which basically says that functions E, F, G, e, f and g with E, G, $EG - F^2 > 0$ and satisfying (53) and (55) are locally the metric and form coefficients of a regular surface, unique up to a rigid motion of $\mathbb{R}^3$.

We conclude with two definitions which will be useful in the exercises of this chapter.

Definition 3.27. Let $S \subset \mathbb{R}^3$ be an oriented surface, and denote by $N\colon S \to S^2$ its Gauss map. A *line of curvature* of the surface S is a curve σ in S such that $\dot\sigma$ is always a principal direction.

Definition 3.28. Let $S \subset \mathbb{R}^3$ be an oriented surface, and denote by $N\colon S \to S^2$ its Gauss map. An *asymptotic direction* at $p \in S$ is a versor $v \in T_pS$ such that $Q_p(v) = 0$. An *asymptotic curve* of the surface S is a curve σ in S such that $\dot\sigma$ is always an asymptotic direction.

Remark 3.29. Since by exchanging orientations the second fundamental form just changes sign, and since each surface is locally orientable, the notions of principal direction, asymptotic direction, line of curvature, and asymptotic curve are well defined for every surface, not just orientable ones.

Guided problems

Notation. From this section onwards, we shall use the following convention for writing partial derivatives: if $\varphi\colon U \to \mathbb{R}$ is a function of class C^k (with $k \geq 2$) defined in an open set $U \subset \mathbb{R}^2$ with coordinates (u, v), we shall denote the partial derivatives of φ by

$$\begin{cases} \varphi_u = \dfrac{\partial \varphi}{\partial u}\ ,\ \varphi_v = \dfrac{\partial \varphi}{\partial v}\ , & \text{for first order derivatives;} \\ \varphi_{uu} = \dfrac{\partial^2 \varphi}{\partial u^2}\ ,\ \varphi_{uv} = \dfrac{\partial^2 \varphi}{\partial u \partial v}\ ,\ \varphi_{vv} = \dfrac{\partial^2 \varphi}{\partial v^2}\ , & \text{for second order derivatives.} \end{cases}$$

An analogous notation will sometimes be used for partial derivatives of functions of more than 2 variables, or for higher-order derivatives.

Definition 3.P.1. If $\varphi\colon U \to S$ is a local parametrization of a surface S, and we denote by (u, v) the coordinates in U, then a *u-curve* (respectively, a *v-curve*) is a coordinate curve of the form $u \mapsto \varphi(u, v_0)$ (respectively, $v \mapsto \varphi(u_0, v)$).

Problem 3.1. Let $S \subset \mathbb{R}^3$ be the surface of equation $z = xy^2$.

(i) Determine the first fundamental form of S and its metric coefficients.
(ii) Determine the second fundamental form Q of S.
(iii) Prove that $K \leq 0$ everywhere, and that $K = 0$ only for the points of S with $y = 0$.
(iv) Prove that $(0, 0, 0)$ is a planar point of S.
(v) Determine the principal directions in the points of S with zero Gaussian curvature.
(vi) Prove that the curves σ_1, $\sigma_2\colon \mathbb{R} \to S$ given by

$$\sigma_1(t) = (x_0 + t, y_0, z_0 + t\, y_0^2) \quad \text{and} \quad \sigma_2(t) = (\mathrm{e}^t x_0, \mathrm{e}^{-2t}\, y_0, \mathrm{e}^{-3t}\, z_0)$$

are asymptotic curves passing through $(x_0, y_0, z_0) \in S$ for all x_0, $y_0 \in \mathbb{R}$.

Solution. (i) Let $\varphi\colon \mathbb{R}^2 \to S$ be the parametrization $\varphi(u, v) = (u, v, uv^2)$ of S seen as a graph. Then

$$\partial_1 = \varphi_u = (1, 0, v^2)\ , \qquad \partial_2 = \varphi_v = (0, 1, 2uv)\ ,$$

and so the metric coefficients are given by

$$E = \langle \partial_1, \partial_1 \rangle = 1 + v^4\ , \quad F = \langle \partial_1, \partial_2 \rangle = 2uv^3\ , \quad G = \langle \partial_2, \partial_2 \rangle = 1 + 4u^2v^2\ .$$

In particular, $EG - F^2 = 1 + v^4 + 4u^2v^2$; moreover, the first fundamental form is

$$\begin{aligned} I_{\varphi(u,v)}(v_1\partial_1 + v_2\partial_2) &= Ev_1^2 + 2Fv_1v_2 + Gv_2^2 \\ &= (1 + v^4)v_1^2 + 4uv^3v_1v_2 + (1 + 4u^2v^2)v_2^2\ . \end{aligned}$$

(ii) To determine the form coefficients e, f and g, we shall use (40). First of all, note that

$$\begin{cases} \varphi_u \wedge \varphi_v = (-v^2, -2uv, 1)\ , \\ N = \dfrac{\varphi_u \wedge \varphi_v}{\|\varphi_u \wedge \varphi_v\|} = \dfrac{1}{\sqrt{1 + v^4 + 4u^2v^2}}(-v^2, -2uv, 1)\ ; \end{cases}$$

moreover, the second-order partial derivatives are

$$\varphi_{uu} = (0, 0, 0)\ , \quad \varphi_{uv} = (0, 0, 2v)\ , \quad \varphi_{vv} = (0, 0, 2u)\ .$$

Then,

$$\begin{aligned} e &= \langle N, \varphi_{uu} \rangle = 0 \;, \\ f &= \langle N, \varphi_{uv} \rangle = \frac{2v}{\sqrt{1+v^4+4u^2v^2}} \;, \\ g &= \langle N, \varphi_{vv} \rangle = \frac{2u}{\sqrt{1+v^4+4u^2v^2}} \;. \end{aligned}$$

In particular, $eg - f^2 = -4v^2/(1+v^4+4u^2v^2)$, and the second fundamental form is given by

$$\begin{aligned} Q_{\varphi(u,v)}(v_1\partial_1 + v_2\partial_2) &= e\,v_1^2 + 2f\,v_1v_2 + g\,v_2^2 \\ &= \frac{4v}{\sqrt{1+v^4+4u^2v^2}}\,v_1v_2 + \frac{2u}{\sqrt{1+v^4+4u^2v^2}}\,v_2^2 \;. \end{aligned}$$

(iii) The previous computations yield

$$K = \frac{eg-f^2}{EG-F^2} = \frac{-4v^2}{(1+v^4+4u^2v^2)^2} \;.$$

So K is always nonpositive, and is zero if and only if $v=0$, which is equivalent to $y=0$.

(iv) Since $(0,0,0) = \varphi(0,0)$, and $e=f=g=0$ in the origin, it follows that $(0,0,0)$ is a planar point.

(v) Recall that the matrix A representing the differential of the Gauss map with respect to the basis $\{\varphi_u, \varphi_v\}$ is given by

$$A = \frac{-1}{EG-F^2} \begin{vmatrix} G & -F \\ -F & E \end{vmatrix} \begin{vmatrix} e & f \\ f & g \end{vmatrix} \;.$$

So in this case we have

$$A = \frac{-2}{(1+v^4+4u^2v^2)^{3/2}} \begin{vmatrix} -2uv^4 & v+2u^2v^3 \\ v+v^5 & u-uv^4 \end{vmatrix} \;.$$

In particular, when $v=y=0$ we get

$$A = \begin{vmatrix} 0 & 0 \\ 0 & -2u \end{vmatrix} \;,$$

and so the principal directions coincide with the coordinate directions.

(vi) First of all,

$$\sigma_1(t) = \varphi(x_0+t, y_0) \quad \text{and} \quad \sigma_2(t) = \varphi(\mathrm{e}^t x_0, \mathrm{e}^{-2t} y_0) \;,$$

so they actually are curves in S. Differentiating, we get

$$\begin{aligned} \sigma_1'(t) &= (1, 0, y_0^2) = \varphi_u(x_0+t, y_0) \;, \\ \sigma_2'(t) &= (\mathrm{e}^t x_0, -2\mathrm{e}^{-2t} y_0, -3\mathrm{e}^{-3t} z_0) \\ &= \mathrm{e}^t x_0 \varphi_u(\mathrm{e}^t x_0, \mathrm{e}^{-2t} y_0) - 2\mathrm{e}^{-2t} y_0 \varphi_v(\mathrm{e}^t x_0, \mathrm{e}^{-2t} y_0) \;. \end{aligned}$$

Recalling the expression we found for the second fundamental form, we obtain $Q\big(\sigma_1'(t)\big) \equiv 0$ and $Q\big(\sigma_2'(t)\big) \equiv 0$, and so σ_1 and σ_2 are asymptotic curves. □

Problem 3.2. Let $S \subset \mathbb{R}^3$ be the regular surface with global parametrization $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^3$ given by $\varphi(u,v) = (u, v, u^2 - v^2)$.

(i) Determine the metric coefficients of S with respect to φ.
(ii) Determine a Gauss map for S.

(iii) Compute the second fundamental form and the Gaussian curvature of S.
(iv) Let $\sigma\colon I \to S$ be a curve with $\sigma(0) = O \in S$. Prove that the normal curvature of σ at the origin belongs to the interval $[-2, 2]$.

SOLUTION. (i) Differentiating we find

$$\partial_1 = \varphi_u = (1, 0, 2u) , \qquad \partial_2 = \varphi_v = (0, 1, -2v) ;$$

so the metric coefficients of φ are given by

$$E = 1 + 4u^2 , \quad F = -4uv , \quad G = 1 + 4v^2 .$$

(ii) It is enough to consider

$$N = \frac{\varphi_u \wedge \varphi_v}{\|\varphi_u \wedge \varphi_v\|} = \frac{1}{\sqrt{4u^2 + 4v^2 + 1}}(-2u, 2v, 1) .$$

(iii) The second-order partial derivatives of φ are

$$\varphi_{uu} = (0, 0, 2) , \quad \varphi_{uv} = (0, 0, 0) \quad \text{and} \quad \varphi_{vv} = (0, 0, -2) ;$$

so the form coefficients of φ are

$$e = \frac{2}{\sqrt{4u^2 + 4v^2 + 1}} , \quad f \equiv 0 , \quad g = \frac{-2}{\sqrt{4u^2 + 4v^2 + 1}} ,$$

and the second fundamental form is given by

$$Q_{\varphi(u,v)}(v_1\partial_1 + v_2\partial_2) = e\, v_1^2 + 2f\, v_1 v_2 + g\, v_2^2 = \frac{2(v_1^2 - v_2^2)}{\sqrt{4u^2 + 4v^2 + 1}} .$$

Moreover,

$$K = \frac{eg - f^2}{EG - F^2} = \frac{-4}{(4u^2 + 4v^2 + 1)^2}$$

is always negative, and so all the points of S are hyperbolic.

(iv) We know that the normal curvature of σ is given by the second fundamental form computed in the tangent versor of σ, and that the second fundamental form of S at the origin $O = \varphi(0, 0)$ is given by $Q_O(v_1\partial_1 + v_2\partial_2) = 2(v_1^2 - v_2^2)$. Moreover, if $\dot\sigma(0) = v_1\partial_1 + v_2\partial_2$ then

$$1 = \|\dot\sigma(0)\|^2 = E(0,0)v_1^2 + 2F(0,0)v_1v_2 + G(0,0)v_2^2 = v_1^2 + v_2^2 .$$

In particular, we may write $v_1 = \cos\theta$ and $v_2 = \sin\theta$ for a suitable $\theta \in \mathbb{R}$; hence,

$$\kappa_n(0) = Q_O\bigl(\dot\sigma(0)\bigr) = 2(\cos^2\theta - \sin^2\theta) = 2\cos(2\theta) \in [-2, 2] ,$$

as claimed. □

DEFINITION 3.P.2. A point p of a surface S is called *umbilical* if $\mathrm{d}N_p$ is a multiple of the identity map on T_pS. In other words, p is umbilical if the two principal curvatures in p coincide.

PROBLEM 3.3. Prove that an oriented surface S consisting entirely of umbilical points is necessarily contained in a sphere or in a plane (and these are surfaces consisting only of umbilical points; see Examples 3.16 and 3.17).

SOLUTION. By assumption, there exists a function $\lambda\colon S \to \mathbb{R}$ such that we have $\mathrm{d}N_p(v) = \lambda(p)v$ for all $v \in T_pS$ and $p \in S$, where $N\colon S \to S^2$ is the Gauss map of S. In particular, if φ is a local parametrization we have

$$\frac{\partial(N \circ \varphi)}{\partial x_1} = \mathrm{d}N(\partial_1) = (\lambda \circ \varphi)\partial_1 , \quad \text{and} \quad \frac{\partial(N \circ \varphi)}{\partial x_2} = \mathrm{d}N(\partial_2) = (\lambda \circ \varphi)\partial_2 .$$

Differentiating again we get

$$\begin{aligned}\frac{\partial^2(N\circ\varphi)}{\partial x_2\partial x_1} &= \frac{\partial(\lambda\circ\varphi)}{\partial x_2}\,\partial_1 + (\lambda\circ\varphi)\frac{\partial^2\varphi}{\partial x_2\partial x_1}\,,\\ \frac{\partial^2(N\circ\varphi)}{\partial x_1\partial x_2} &= \frac{\partial(\lambda\circ\varphi)}{\partial x_1}\,\partial_2 + (\lambda\circ\varphi)\frac{\partial^2\varphi}{\partial x_1\partial x_2}\,,\end{aligned}$$

and so

$$\frac{\partial(\lambda\circ\varphi)}{\partial x_2}\,\partial_1 - \frac{\partial(\lambda\circ\varphi)}{\partial x_1}\,\partial_2 \equiv O\,.$$

But ∂_1 and ∂_2 are linearly independent; therefore this implies

$$\frac{\partial(\lambda\circ\varphi)}{\partial x_2} \equiv \frac{\partial(\lambda\circ\varphi)}{\partial x_1} \equiv 0\,,$$

that is $\lambda\circ\varphi$ is constant.

So we have proved that λ is locally constant: being S is connected, λ is constant on all S. Indeed, choose $p_0 \in S$ and put $R = \{p \in S \mid \lambda(p) = \lambda(p_0)\}$. This set is not empty ($p_0 \in R$), it is closed since λ is continuous, and is open because λ is locally constant; so by the connectedness of S we have $R = S$, that is, λ is globally constant.

If $\lambda \equiv 0$, the differential of the Gauss map is zero everywhere, that is, N is everywhere equal to a vector $N_0 \in S^2$. Choose $p_0 \in S$, and define $h\colon S \to \mathbb{R}$ by setting $h(q) = \langle q - p_0, N_0\rangle$. If $\varphi\colon U \to S$ is an arbitrary local parametrization of S, we have

$$\frac{\partial(h\circ\varphi)}{\partial x_j} = \langle\partial_j, N_0\rangle \equiv 0$$

for $j = 1, 2$. It follows that h is locally constant, and so it is constant by the same argument as above. Since $h(p_0) = 0$, we get $h \equiv 0$, which means exactly that S is contained in the plane through p_0 and orthogonal to N_0.

If instead $\lambda \equiv \lambda_0 \neq 0$, let $q\colon S \to \mathbb{R}^3$ be given by $q(p) = p - \lambda_0^{-1}N(p)$. Then

$$\mathrm{d}q_p = \mathrm{id} - \frac{1}{\lambda_0}\mathrm{d}N_p = \mathrm{id} - \frac{1}{\lambda_0}\lambda_0\,\mathrm{id} \equiv O\,,$$

therefore q is (locally constant and thus) constant; denote by q_0 the value of q, that is, $q \equiv q_0$. Hence $p - q_0 \equiv \lambda_0^{-1}N(p)$, and so

$$\forall p \in S \qquad\qquad \|p - q_0\|^2 = \frac{1}{\lambda_0^2}\,.$$

In other words, S is contained in the sphere of center q_0 and radius $1/|\lambda_0|$, and we are done. □

Problem 3.4. **When are the coordinate lines lines of curvature?** Let $\varphi\colon U \to S \subset \mathbb{R}^3$ be a local parametrization of a regular surface S, and assume that no point of $\varphi(U)$ is umbilical. Prove that all the coordinate curves are lines of curvature if and only if $F \equiv f \equiv 0$.

Solution. Saying that the coordinate curves are always lines of curvature is equivalent to saying that the coordinate directions are always principal directions, and this in turn is equivalent to saying that the matrix A representing the differential of the Gauss map in the basis $\{\varphi_u, \varphi_v\}$ is always diagonal.

Now, recalling (41), we immediately see that if $F \equiv f \equiv 0$ then A is diagonal; so, in this case, the coordinate curves are always curvature lines (even when there are umbilical points).

Conversely, assume that the coordinate lines are lines of curvature. This means that the vectors φ_u and φ_v are principal directions; in particular, since no point is umbilical, φ_u and φ_v are orthogonal, and so $F \equiv 0$. Now, the off-diagonal entries of A are $-f/G$ and $-f/E$; since they have to be zero, we get $f \equiv 0$, as claimed. Notice that in umbilical points all directions are principal, and so coordinate curves are always lines of curvature at umbilical points. $\square$

PROBLEM 3.5. Let S be an oriented surface, and $N\colon S \to S^2$ its Gauss map. Prove that a curve $\sigma\colon I \to S$ is a line of curvature if and only if, having set $N(t) = N\big(\sigma(t)\big)$, we have $N'(t) = \lambda(t)\sigma'(t)$ for a suitable function $\lambda\colon I \to \mathbb{R}$ of class C^∞. In this case, $-\lambda(t)$ is the (principal) curvature of S along $\sigma'(t)$.

SOLUTION. It suffices to remark that

$$\mathrm{d}N_{\sigma(t)}\big(\sigma'(t)\big) = \frac{\mathrm{d}(N \circ \sigma)}{\mathrm{d}t}(t) = N'(t) ,$$

so $\sigma'(t)$ is an eigenvector of $dN_{\sigma(t)}$ if and only if $N'(t) = \lambda(t)\sigma'(t)$ for some $\lambda(t) \in \mathbb{R}$. $\square$

PROBLEM 3.6. **Characterization of the lines of curvature.** Let $S \subset \mathbb{R}^3$ be an oriented surface, $\varphi\colon U \to S$ a local parametrization, and let $\sigma\colon I \to \varphi(U)$ be a regular curve with support contained in $\varphi(U)$, so we can write $\sigma(t) = \varphi\big(u(t), v(t)\big)$. Prove that σ is a line of curvature if and only if

$$(fE - eF)(u')^2 + (gE - eG)u'v' + (gF - fG)(v')^2 \equiv 0 .$$

SOLUTION. By definition, we know that σ is a line of curvature if and only if $\mathrm{d}N_{\sigma(t)}\big(\sigma'(t)\big) = \lambda(t)\sigma'(t)$ for a suitable function λ of class C^∞. Now it suffices to use Proposition 3.8 for expressing $\mathrm{d}N_{\sigma(t)}\big(\sigma'(t)\big)$, and eliminate λ from the system of equations given by $\mathrm{d}N_{\sigma(t)}\big(\sigma'(t)\big) = \lambda(t)\sigma'(t)$, recalling that $\sigma'(t) \neq O$ always because σ is regular. $\square$

PROBLEM 3.7. **Characterization of asymptotic curves.** Let $\varphi\colon U \to S$ be a local parametrization of an oriented surface, and let $\sigma\colon I \to \varphi(U)$ be a regular curve with support contained in $\varphi(U)$, so we can write $\sigma(t) = \varphi\big(u(t), v(t)\big)$. Prove that σ is an asymptotic curve if and only if

$$e(u')^2 + 2fu'v' + g(v')^2 \equiv 0 .$$

In particular, deduce that the coordinate curves are asymptotic curves (necessarily in a neighborhood of a hyperbolic point) if and only if $e = g = 0$.

SOLUTION. By definition, a curve σ is an asymptotic curve if and only if $Q_{\sigma(t)}\big(\sigma'(t)\big) \equiv 0$, where Q_p is the second fundamental form at $p \in S$. Since $\sigma'(t) = u'\,\varphi_u + v'\,\varphi_v$, where $\varphi_u = \partial_1$ and $\varphi_v = \partial_2$ are computed in $\sigma(t)$, the assertions immediately follows recalling that the form coefficients e, f, g represent the second fundamental form in the basis $\{\partial_1, \partial_2\}$. $\square$

PROBLEM 3.8. Let $S \subset \mathbb{R}^3$ the (upper half of the) pseudosphere obtained by rotating around the z-axis the (upper half of the) tractrix $\sigma\colon (\pi/2, \pi) \to \mathbb{R}^3$ given by

$$\sigma(t) = \big(\sin t, 0, \cos t + \log\tan(t/2)\big) ;$$

see Exercise 2.18 and Example 3.37. In particular, S is the support of the immersed surface $\varphi\colon (\pi/2, \pi) \times \mathbb{R} \to \mathbb{R}^3$ given by

$$\varphi(t, \theta) = \left(\sin t \cos\theta, \sin t \sin\theta, \cos t + \log\tan\frac{t}{2}\right) .$$

(i) Determine the Gauss map $N\colon S \to S^2$ induced by φ.
(ii) Determine the differential $\mathrm{d}N$ of the Gauss map and the Gaussian curvature of S.
(iii) Determine the mean curvature of S.

Solution. (i) To determine the Gauss map of S we compute, as usual, the partial derivatives of the parametrization:

$$\partial_1 = \varphi_t = \cos t(\cos\theta, \sin\theta, \mathrm{cotan}\, t) , \quad \partial_2 = \varphi_\theta = \sin t(-\sin\theta, \cos\theta, 0) .$$

Hence, $\varphi_t \wedge \varphi_\theta = \cos t(-\cos t\, \cos\theta, -\cos t\, \sin\theta, \sin t)$ and $\|\varphi_t \wedge \varphi_\theta\| = \cos t$, and so

$$N\big(\varphi(t, \theta)\big) = (-\cos t \cos\theta, -\cos t \sin\theta, \sin t) .$$

(ii) To determine the differential $\mathrm{d}N_p$ at $p = \varphi(t, \theta) \in S$, we use the fact that $\mathrm{d}N_p(\varphi_t) = \partial(N \circ \varphi)/\partial t$ and $\mathrm{d}N_p(\varphi_\theta) = \partial(N \circ \varphi)/\partial\theta$. We find that

$$\mathrm{d}N_p(\varphi_t) = \sin t\,(\cos\theta, \sin\theta, \mathrm{cotan}\, t) = (\tan t)\,\varphi_t \ , \ \mathrm{d}N_p(\varphi_\theta) = -(\mathrm{cotan}\, t)\varphi_\theta .$$

So the matrix representing $\mathrm{d}N_p$ with respect to the basis $\{\varphi_t, \varphi_\theta\}$ is

$$A = \begin{vmatrix} \tan t & 0 \\ 0 & -\mathrm{cotan}\, t \end{vmatrix} ;$$

in particular, $K = \det(A) \equiv -1$, as claimed in Example 3.37. Incidentally, S is called "pseudosphere" exactly because it has constant — even if negative — Gaussian curvature, like the usual sphere.

(iii) It suffices to notice that the mean curvature is given by

$$H = -\frac{1}{2}\mathrm{tr}(A) = -\frac{1}{2}(\tan t + \mathrm{cotan} t) = -\frac{1}{\sin 2t} .$$

□

Problem 3.9. Compute the Gaussian curvature and the mean curvature of the ellipsoid $S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + 4y^2 + 9z^2 = 1\}$ without using local parametrizations.

Solution. Since S is the vanishing locus of the function $h\colon \mathbb{R}^3 \to \mathbb{R}$ given by $h(x, y, z) = x^2 + 4y^2 + 9z^2 - 1$, Corollary 3.1 tells us that a Gauss map $N\colon S \to S^2$ of S is

$$N(x, y, z) = \alpha(x, y, z)(x, 4y, 9z) ,$$

where $\alpha\colon S \to \mathbb{R}$ is the function $\alpha(x, y, z) = (x^2 + 16y^2 + 81z^2)^{-1/2}$. Moreover, the tangent plane at $p = (x_0, y_0, z_0)$ is given by

$$T_pS = \{(v_1, v_2, v_3) \in \mathbb{R}^3 \mid x_0v_1 + 4y_0v_2 + 9z_0v_3 = 0\} .$$

The Jacobian matrix of N seen as a map from $\mathbb{R}^3 \setminus \{O\}$ to $\mathbb{R}^3$ is

$$\mathbf{J} = \begin{vmatrix} \alpha + x\alpha_x & x\alpha_y & x\alpha_z \\ 4y\alpha_x & 4\alpha + 4y\alpha_y & 4y\alpha_z \\ 9z\alpha_x & 9z\alpha_y & 9\alpha + 9z\alpha_z \end{vmatrix} ;$$

so

$$\mathrm{d}N_p(v_1, v_2, v_3) = \mathbf{J} \begin{vmatrix} v_1 \\ v_2 \\ v_3 \end{vmatrix} = \alpha(p) \begin{vmatrix} v_1 \\ 4v_2 \\ 9v_3 \end{vmatrix} - \alpha^3(x_0 v_1 + 16 y_0 v_2 + 81 z_0 v_3) \begin{vmatrix} x_0 \\ 4y_0 \\ 9z_0 \end{vmatrix} .$$

Suppose that $x_0 \neq 0$. A basis $\mathcal{B} = \{w_1, w_2\}$ of the tangent vectors to S at p is then given by

$$w_1 = (-9z_0, 0, x_0) \quad \text{and} \quad w_2 = (-4y_0, x_0, 0) .$$

Computing explicitly $\mathrm{d}N_p(w_1)$ and $\mathrm{d}N_p(w_2)$, and writing them as linear combinations with respect to the basis $\mathcal{B}$, we get that the matrix representing $\mathrm{d}N_p$ with respect to $\mathcal{B}$ is

$$A = \alpha \begin{vmatrix} 9(1 - 72\alpha^2 z_0^2) & -108\alpha^2 y_0 z_0 \\ -288\alpha^2 y_0 z_0 & 4(1 - 12\alpha^2 y_0^2) \end{vmatrix} .$$

So, if $x_0 \neq 0$ the Gaussian curvature and the mean curvature are

$$K = \frac{36}{(x_0^2 + 16y_0^2 + 81z_0^2)^2} , \quad H = \frac{36(x_0^2 + y_0^2 + z_0^2 - 1) - 13}{2(x_0^2 + 16y_0^2 + 81z_0^2)^{3/2}} ;$$

note that K is always positive.

We have now found Gaussian and mean curvatures in all points $p = (x, y, z) \in S$ with $x \neq 0$. But $S \cap \{x = 0\}$ is an ellipse C, and $S \setminus C$ is an open set dense in S. Since the Gaussian and mean curvatures are continuous, and the expressions we have found are defined and continuous on all S, they give the values of K and H on all S. □

Problem 3.10. Let $S \subset \mathbb{R}^3$ be a regular surface with a global parametrization $\varphi\colon \mathbb{R} \times \mathbb{R}^+ \to S$ whose metric coefficients satisfy $E(u, v) = G(u, v) = v$ e $F(u, v) \equiv 0$. Prove that S is not locally isometric to a sphere.

Solution. Since the parametrization is orthogonal, we may use (50) to compute the Christoffel symbols of φ. We obtain $\Gamma_{11}^1 = 0$, $\Gamma_{11}^2 = -\frac{1}{2v}$, $\Gamma_{12}^1 = \frac{1}{2v}$, $\Gamma_{12}^2 = 0$, $\Gamma_{22}^1 = 0$, $\Gamma_{22}^2 = \frac{1}{2v}$. Gauss' Theorema Egregium 3.5 then implies

$$K = \frac{1}{G} \left[\frac{\partial \Gamma_{22}^1}{\partial u} - \frac{\partial \Gamma_{12}^1}{\partial v} + \sum_{s=1}^{2} \left(\Gamma_{22}^s \Gamma_{1s}^1 - \Gamma_{12}^s \Gamma_{2s}^1 \right) \right] = \frac{1}{2v^3} .$$

Hence, K is not constant in any open set of S, and so (Corollary 3.2) S cannot be locally isometric to a sphere. □

Problem 3.11. Let $\sigma\colon (a, b) \to R^3$ be a biregular curve whose support is contained in the sphere S^2 with radius 1 and center in the origin of $\mathbb{R}^3$. Show that if the curvature of σ is constant then the support of σ is contained in a circle.

Solution. We may assume that σ is parametrized by arc length; moreover, remember that if we orient S^2 as in Example 3.4 we have $\sigma = N \circ \sigma$. Since the support of σ is contained in S^2, the derivative $\dot{\sigma}$ is tangent to S^2, and so it is orthogonal to σ. Further, by Proposition 3.6 (Meusnier) and Example 3.25, the normal curvature of σ is equal to -1 everywhere, once more because σ takes values in S^2. Hence, (35) and (36) imply $\langle \sigma, \vec{n} \rangle \equiv -1/\kappa$.

Now, $1 = \|\sigma\|^2 = |\langle \sigma, \dot{\sigma} \rangle|^2 + |\langle \sigma, \vec{n} \rangle|^2 + |\langle \sigma, \vec{b} \rangle|^2$; since $\langle \sigma, \dot{\sigma} \rangle \equiv 0$ and $\langle \sigma, \vec{n} \rangle$ is a non zero constant, we deduce that $\langle \sigma, \vec{b} \rangle$ is a constant too. Hence,

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} \langle \sigma, \vec{b} \rangle = -\tau \langle \sigma, \vec{n} \rangle = \frac{\tau}{\kappa} ,$$

so $\tau \equiv 0$ and σ is plane. But the support of a plane regular curve with constant curvature is contained in a circle, and we are done. $\square$

PROBLEM 3.12. Put $U = (0,1) \times (0,\pi)$ and let $\varphi\colon U \to \mathbb{R}^3$ be the map defined by $\varphi(u,v) = \big(u \cos v, u \sin v, \phi(v)\big)$, where $\phi \in C^\infty\big((0,\pi)\big)$ is a homeomorphism with its image.

(i) Show that the image S of φ is a regular surface.
(ii) Compute the Gaussian curvature in every point of S and check whether there exists an open subset of S that is locally isometric to a plane.
(iii) Give conditions for a point of S to be an umbilical point.

SOLUTION. (i) Note that

$$\partial_1 = \varphi_u = (\cos v, \sin v, 0) \quad \partial_2 = \varphi_v = \big(-u \sin v, u \cos v, \phi'(v)\big)\ ;$$

hence,

$$\varphi_u \wedge \varphi_v = (\phi'(v)\, \sin v, -\phi'(v) \cos v, u)$$

is never zero, because its third component is never zero. So the differential of φ is injective in every point. Moreover, φ is injective, and $\psi\colon S \to U$ given by $\psi(x,y,z) = (\sqrt{x^2+y^2}, \phi^{-1}(z))$ is a continuous inverse of φ.

(ii) The metric coefficients are $E = 1$, $F = 0$ and $G = u^2 + \phi'(v)^2$, while $\|\varphi_u \wedge \varphi_v\| = \sqrt{u^2+\phi'(v)^2}$. To determine the form coefficients, we compute

$$\varphi_{uu} = (0,0,0)\ ,\ \varphi_{uv} = (-\sin v, \cos v, 0)\ ,\ \varphi_{vv} = \big(-u \cos v, -u \sin v, \phi''(v)\big)\ ,$$

and so

$$e \equiv 0\ , \quad f = \frac{-\phi'(v)}{\sqrt{u^2+\phi'(v)^2}}\ , \quad g = \frac{u\phi''(v)}{\sqrt{u^2+\phi'(v)^2}}\ .$$

In particular,

$$K = -\frac{\phi'(v)^2}{(u^2+\phi'(v)^2)^2}\ .$$

Since ϕ is injective, ϕ' cannot be zero on an interval, so K cannot be zero in an open set. Consequently, no open subset of S can be locally isometric to a plane.

(iii) Recalling (44), which gives the principal curvatures in terms of the mean and Gaussian curvatures, the relation that characterizes the umbilical points is $H^2 - K = 0$. Using (43), we find that the mean curvature of S is given by

$$H = \frac{1}{2}\frac{u\phi''(v)}{(u^2+\phi'(v)^2)^{3/2}}\ ,$$

and so

$$H^2 - K = \frac{1}{4}\frac{u^2\phi''(v)^2}{(u^2+\phi'(v)^2)^3} + \frac{\phi'(v)^2}{(u^2+\phi'(v)^2)^2}\ .$$

Hence we have $H^2 - K = 0$ if and only if $\phi'(v) = \phi''(v) = 0$, and so the umbilical points of S are exactly the points of the form $\big(u \cos v_0, u \sin v_0, \phi(v_0)\big)$, where $v_0 \in (0,\pi)$ satisfies $\phi'(v_0) = \phi''(v_0) = 0$. $\square$

PROBLEM 3.13. Let $\Sigma = \{(x,y,z) \in \mathbb{R}^3 \mid xyz = 1\}$.

(i) Determine the largest subset S of Σ such that S is a regular surface.
(ii) Prove that the points $(x,y,z) \in S$ such that $|x| = |y| = |z| = 1$ are umbilical points of S.

SOLUTION. (i) Consider the function $f\colon \mathbb{R}^3 \to \mathbb{R}$ given by $f(x,y,z) = xyz$. Since $\nabla f = (yz, xz, xy)$, we find that 1 is a regular value for f, and so $\Sigma = S$ is a regular surface.

(ii) Let $p = (x,y,z) \in \Sigma$. In a neighborhood of p, the surface Σ is the graph of the function $g\colon \mathbb{R}^* \times \mathbb{R}^* \to \mathbb{R}$ given by $g(x,y) = 1/xy$. So we may take as parametrization of Σ near p the parametrization of the graph of g given by $\varphi(u,v) = \big(u, v, g(u,v)\big)$. Then, proceeding in the usual way, we find

$$\partial_1 = \left(1, 0, -\frac{1}{u^2 v}\right),\ \partial_2 = \left(0, 1, -\frac{1}{uv^2}\right),\ N = \frac{1}{\sqrt{u^2+v^2+u^4v^4}}(v, u, u^2v^2)\ ,$$

$$E = 1 + \frac{1}{u^4v^2}\ ,\ F = \frac{1}{u^3v^3}\ ,\ G = 1 + \frac{1}{u^2v^4}\ ,$$

$$e = \frac{2v}{u\sqrt{u^2+v^2+u^4v^4}}\ ,\ f = \frac{1}{\sqrt{u^2+v^2+u^4v^4}}\ ,\ g = \frac{2u}{v\sqrt{u^2+v^2+u^4v^4}}\ ,$$

$$K = \frac{3u^4v^4}{(u^2+v^2+u^4v^4)^2}\ ,\ H = \frac{uv(1+u^2v^4+u^4v^2)}{(u^2+v^2+u^4v^4)^{3/2}}\ ,$$

$$H^2 - K = \frac{u^2v^2(1+u^8v^4+u^4v^8-u^2v^4-u^6v^6-u^4v^2)}{(u^2+v^2+u^4v^4)^3}\ .$$

In particular, all points of the form $\varphi(u,v)$ with $|u| = |v| = 1$, that is, all points $p \in S$ with $|x| = |y| = |z| = 1$, are umbilical points. □

PROBLEM 3.14. Let S be an oriented surface in $\mathbb{R}^3$ and $\sigma\colon \mathbb{R} \to S$ a biregular curve of class C^∞ that is an asymptotic curve of S. Prove that $T_{\sigma(s)}S$ is the osculating plane to σ at $\sigma(s)$ for all $s \in \mathbb{R}$.

SOLUTION. We may assume that σ is parametrized by arc length. Using the usual notation, we have to show that the versors $\vec{t}(s)$ and $\vec{n}(s)$ span the plane $T_{\sigma(s)}S$ tangent to S at $\sigma(s)$; in other words, we have to prove that $\vec{t}(s)$ and $\vec{n}(s)$ are orthogonal to the normal versor $N\big(\sigma(s)\big)$. Since $\vec{t}(s) \in T_{\sigma(s)}S$, by definition of a tangent plane to a surface, it suffices to show that $\vec{n}(s)$ and $N\big(\sigma(s)\big)$ are orthogonal. But by the biregularity of σ we know that

$$\big\langle \vec{n}(s), N\big(\sigma(s)\big)\big\rangle = \frac{1}{\kappa(s)} Q_{\sigma(s)}\big(\dot\sigma(s)\big) = 0\ ,$$

since σ is an asymptotic curve. □

DEFINITION 3.P.3. Let $S \subset \mathbb{R}^3$ be a surface oriented by an atlas $\mathcal{A}$. Then the atlas $\mathcal{A}^-$ obtained by exchanging coordinates in all the parametrizations of $\mathcal{A}$, that is, $\varphi \in \mathcal{A}^-$ if and only if $\varphi \circ \chi \in \mathcal{A}$ where $\chi(x,y) = (y,x)$, is called *opposite* of $\mathcal{A}$.

PROBLEM 3.15. Let S be a surface oriented by an atlas $\mathcal{A}$, and take another local parametrization $\varphi\colon U \to S$ of S, with U connected. Prove that either φ has the same orientation as all local parametrizations of $\mathcal{A}$, or has the same orientation as all local parametrizations of $\mathcal{A}^-$.

SOLUTION. Let N be the normal versor field determining the given orientation, and $\{\partial_1, \partial_2\}$ the basis induced by φ. Exactly as in the proof of Proposition 3.4, we find that $\partial_1 \wedge \partial_2 / \|\partial_1 \wedge \partial_2\| \equiv \pm N$ on $\varphi(U)$, with constant sign since U is connected. So (33) implies that if the sign is positive then φ determines the same orientation of all elements of $\mathcal{A}$, while if the sign is negative it determines the opposite orientation. □

PROBLEM 3.16. Let $S \subset \mathbb{R}^3$ be a surface in which the absolute value of the mean curvature is never zero. Prove that S is orientable.

SOLUTION. Let $\mathcal{A} = \{\varphi_\alpha\}$ be an atlas on S such that the domain U_α of each φ_α is connected. Using the usual Gauss map N_α induced by φ_α, we may define a mean curvature on $\varphi_\alpha(U_\alpha)$ with a well defined sign, since its absolute value is never zero and U_α is connected. Up to exchanging coordinates in U_α, we may then assume that the mean curvature induced by N_α is always positive.

Define now $N\colon S \to S^2$ by setting $N(p) = N_\alpha(p)$ for all $p \in \varphi_\alpha(U_\alpha)$. To conclude, it suffices to verify that N is well defined, that is it does not depend on α. Take $p \in \varphi_\alpha(U_\alpha) \cap \varphi_\beta(U_\beta)$. If we had $N_\alpha(p) = -N_\beta(p)$, then we would have $N_\alpha \equiv -N_\beta$ in a whole neighborhood of p; so the mean curvature induced by N_α and the mean curvature induced by N_β would have opposite sign in a neighborhood of p, against our assumptions. □

PROBLEM 3.17. Let $p \in S$ be a point of a surface $S \subset \mathbb{R}^3$. Prove that if p is elliptic then there exists a neighborhood V of p in S such that $V \setminus \{p\}$ is contained in one of the two open half-spaces bounded by the affine tangent plane $p + T_pS$. Prove that if, on the other hand, p is hyperbolic then every neighborhood of p in S intersects both the open half-spaces bounded by the plane $p + T_pS$.

SOLUTION. Let $\varphi\colon U \to S$ be a local parametrization centered at p, and define the function $d\colon U \to \mathbb{R}$ by setting $d(x) = \big\langle \varphi(x) - p, N(p) \big\rangle$, where N is the Gauss map induced by φ. Clearly, $\varphi(x) \in p + T_pS$ if and only if $d(x) = 0$, and $\varphi(x)$ belongs to one or the other of the half-spaces bounded by $p + T_pS$ depending on the sign of $d(x)$. Expanding d as a Taylor series around the origin, we get

$$\begin{aligned} d(x) &= d(O) + \sum_{j=1}^{2} \frac{\partial d}{\partial x_j}(O)x_j + \frac{1}{2}\sum_{i,j=1}^{2} \frac{\partial^2 d}{\partial x_i \partial x_j}(O)x_i x_j + o(\|x\|^2) \\ &= e(p)x_1^2 + 2f(p)x_1x_2 + g(p)x_2^2 + o(\|x\|^2) \qquad (56) \\ &= Q_p(x_1\partial_1 + x_2\partial_2) + o(\|x\|^2)\ . \end{aligned}$$

Now, if p is elliptic then the two principal curvatures at p have the same sign and are different from zero; in particular, Q_p is positive (or negative) definite. But then (56) implies that $d(x)$ has constant sign in a punctured neighborhood of the origin, and so there exists a neighborhood $V \subset S$ of p such that all points of $V \setminus \{p\}$ belong to one of the two open half-spaces bounded by $p + T_pS$.

If, on the other hand, p is hyperbolic, the two principal curvatures in p have opposite signs and are different from zero; in particular, Q_p is indefinite. Hence $d(x)$ changes sign in every neighborhood of the origin, and so every neighborhood of p in S intersects both the open half-spaces bounded by $p + T_pS$. □

PROBLEM 3.18. **Osculating quadric to a level surface.** Given a function $f \in C^\infty(\Omega)$ admitting 0 as regular value, where $\Omega \subset \mathbb{R}^3$ is an open set, let $p_0 = (x_1^o, x_2^o, x_3^o) \in S = f^{-1}(0)$ be a point of the level surface of f.

(i) Determine a quadric $\mathcal{Q}$ passing through p_0 and such that S and $\mathcal{Q}$ have the same tangent plane at p_0 and the same second fundamental form. The quadric $\mathcal{Q}$ is called *osculating quadric*.

(ii) Show that p_0 is elliptic, hyperbolic or parabolic for S if and only if it is for $\mathcal{Q}$.

(iii) Let S be the surface of R^3 of equation $x_1 + x_1^3 + x_2^2 + x_3^3 = 0$. Using the osculating quadric, show that the point $p_0 = (-1, 1, 1)$ is hyperbolic.

Solution. Developing f in Taylor series around p_0, we find

$$f(x) = \sum_{j=1}^{3} \frac{\partial f}{\partial x_j}(p_0)(x_j - x_j^o) + \frac{1}{2}\sum_{i,j=1}^{3} \frac{\partial^2 f}{\partial x_i \partial x_j}(p_0)(x_i - x_i^o)(x_j - x_j^o) + o(\|x\|^2)\ .$$

Choose as $\mathcal{Q}$ the quadric determined by the polynomial

$$P(x) = \sum_{j=1}^{3} \frac{\partial f}{\partial x_j}(p_0)(x_j - x_j^o) + \frac{1}{2}\sum_{i,j=1}^{3} \frac{\partial^2 f}{\partial x_i \partial x_j}(p_0)(x_i - x_i^o)(x_j - x_j^o)\ .$$

So f and P have the same first and second derivatives in p_0. Since the tangent plane to S (respectively, to $\mathcal{Q}$) at p_0 is orthogonal to the gradient of f (respectively, P) at p_0, and $\nabla f(p_0) = \nabla P(p_0)$, we immediately find that $T_{p_0}S = T_{p_0}\mathcal{Q}$. Moreover, the differential of the Gauss map of S at p_0 only depends on the first derivatives of ∇f at p_0, that is, on the second derivatives of f at p_0; since P has the same (first and) second derivatives at p_0 as f, it follows that the differential of the Gauss map for S acts on $T_{p_0}S = T_{p_0}\mathcal{Q}$ like the differential of the Gauss map for $\mathcal{Q}$, and as a consequence S and $\mathcal{Q}$ have the same second fundamental form at p_0, and p_0 is elliptic (hyperbolic, parabolic) for S if and only if it is for $\mathcal{Q}$.

In case (iii), the polynomial P is

$$P(x) = 4(x_1 + 1) + 2(x_2 - 1) + 3(x_3 - 1) - 3(x_1 + 1)^2 + (x_2 - 1)^2 + 3(x_3 - 1)^2\ .$$

The theorem of metric classification for quadrics (see [**2**, Vol. II, p. 163]) tells us that the quadric $\mathcal{Q}$ is obtained by a rigid motion from a one-sheeted hyperboloid. Since all points of $\mathcal{Q}$ are hyperbolic (see Exercise 3.54), p_0 is hyperbolic for S too. □

Definition 3.P.4. A surface $S \subset \mathbb{R}$ is *ruled* if there exists a family $\{r_\lambda\}_{\lambda \in \mathbb{R}}$ of disjoint open line segments (or whole straight lines) whose union is S. The lines r_λ are called *generators* (or *rulings*) of S. A *cone* is a ruled surface whose generators all pass through a common point.

Problem 3.19. Let S be a ruled (regular) surface. Show that S does not contain elliptic points, and as a consequence $K \leq 0$ in each point of S.

Solution. By definition, for each point $p \in S$, there is a line segment contained in S and passing through p. A line segment within a surface always has zero normal curvature; so every point $p \in S$ has an asymptotic direction, which necessarily implies $K(p) \leq 0$. □

Problem 3.20. **Tangent surface to a curve.** Let $\sigma\colon I \to \mathbb{R}^3$ be a regular curve of class C^∞, with $I \subseteq \mathbb{R}$ an open interval. The map $\tilde{\varphi}\colon I \times \mathbb{R} \to \mathbb{R}^3$, defined by $\tilde{\varphi}(t, v) = \sigma(t) + v\,\sigma'(t)$, is called *tangent surface* to σ. Every affine tangent line to σ is called a *generator* of the tangent surface.

(i) Show that $\tilde{\varphi}$ is not an immersed surface.
(ii) Show that if the curvature κ of σ is nowhere zero then the restriction $\varphi = \tilde{\varphi}|_U\colon U \to \mathbb{R}^3$ of $\tilde{\varphi}$ to the subset $U = \{(t, v) \in I \times \mathbb{R} \mid v > 0\}$ is an immersed surface.

(iii) Show that the tangent plane along a generator of the tangent surface is constant in $S = \varphi(U)$.

SOLUTION. (i) It suffices to prove that the differential of $\tilde{\varphi}$ is not injective somewhere. Since $\varphi_t = \sigma' + v\,\sigma''$ and $\varphi_v = \sigma'$, we have $\varphi_t \wedge \varphi_v = v\,\sigma'' \wedge \sigma'$. So, using the expression (13) of the curvature of a curve in an arbitrary parameter, we find

$$\|\varphi_t \wedge \varphi_v\| = |v|\,\|\sigma'\|^3 \kappa\,.$$

In particular, the differential of φ is not injective when $v = 0$.

(ii) More precisely, we have proved that the differential of φ is injective in (t, v) if and only if $v \neq 0$ and $\kappa(t) \neq 0$, and so φ is an immersed surface.

(iii) It is sufficient to remark that the direction of the vector $\varphi_t \wedge \varphi_v$ is orthogonal to the tangent plane to S at the required point. Since this direction does not depend on v, the tangent plane is constant along a generator of S. □

Exercises

FIRST FUNDAMENTAL FORM

3.1. Determine the metric coefficients and the first fundamental form for the regular surface with global parametrization $\varphi(u, v) = (u, v, u^4 + v^4)$.

3.2. Let $S \subset \mathbb{R}^3$ be the surface with global parametrization $\varphi\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^3$ given by $\varphi(u, v) = (u \cos v, u \sin v, u)$. Prove that the coordinate curves of φ are orthogonal to each other in every point.

3.3. Let $S \subset \mathbb{R}^3$ be the catenoid, parametrized as in Problem 2.1. Given $r \in \mathbb{R}$, let $\sigma\colon \mathbb{R} \to S$ be the curve contained in the catenoid defined by $\sigma(t) = \varphi(t, rt)$. Compute the length of σ between $t = 0$ and $t = t_0$, using the first fundamental form of S.

3.4. Let $\varphi\colon \mathbb{R}^+ \times (0, 2\pi) \to \mathbb{R}^3$ be the local parametrization of the one-sheeted cone $S \subset \mathbb{R}^3$ given by $\varphi(u, v) = (u \cos v, u \sin v, u)$. Given $\beta \in \mathbb{R}$, determine the length of the curve $\sigma\colon [0, \pi] \to S$ expressed by $\sigma(t) = \varphi(\mathrm{e}^{t \operatorname{cotan}(\beta)/\sqrt{2}}, t)$.

3.5. Let $S \subset \mathbb{R}^3$ be a regular surface with local parametrization $\varphi(u, v)$ whose metric coefficients satisfy $E \equiv 1$ and $F \equiv 0$. Show that the coordinate v-curves cut each u-curve in segments of equal length.

3.6. Determine the metric coefficients of the unit sphere $S^2 \subset \mathbb{R}^3$ with respect to the parametrization found by using the stereographic projection (see Exercise 2.4).

3.7. Determine the first fundamental form of the xy-plane minus the origin, parametrized by polar coordinates.

ISOMETRIES AND SIMILITUDES

3.8. Find two surfaces S_1 and S_2 such that S_1 is locally isometric to S_2 but S_2 is not locally isometric to S_1.

3.9. Determine for which values of a, $b \in \mathbb{R}$ the surface

$$S_{a,b} = \{(x, y, z) \in \mathbb{R}^3 \mid z = a\,x^2 + b\,y^2\}$$

is locally isometric to a plane.

3.10. Let $\sigma = (\sigma_1, \sigma_2)\colon \mathbb{R} \to \mathbb{R}^2$ be a regular plane curve parametrized by arc length. Let $S \subset \mathbb{R}^3$ be the right cylinder on σ parametrized by

$$\varphi(u, v) = \big(\sigma_1(u), \sigma_2(u), v\big)\ .$$

Prove that S is locally isometric to the cylinder of equation $x^2 + y^2 + 2x = 0$.

3.11. Let $H\colon S \to \tilde{S}$ be a similitude with scale factor $r > 0$. Given a local parametrization $\varphi\colon U \to S$, put $\tilde{\varphi} = H \circ \varphi$ and let E, F, G (respectively, $\tilde{E}$, $\tilde{F}$, $\tilde{G}$) be the metric coefficients with respect to φ (respectively, $\tilde{\varphi}$). Prove that $\tilde{E} = r^2 E$, $\tilde{F} = r^2 F$ and $\tilde{G} = r^2 G$.

ORIENTABLE SURFACES

3.12. Let σ, $\tau\colon \mathbb{R} \to \mathbb{R}^3$ be the trajectories, parametrized by arc length, of two points that are moving subject to the following conditions:

(a) σ starts at $\sigma(0) = (0, 0, 0)$, and moves along the x-axis in the positive direction;
(b) τ starts at $\tau(0) = (0, a, 0)$, where $a \neq 0$, and moves parallel to the positive direction of the z-axis.

Denote by $S \subset \mathbb{R}^3$ the union of the straight lines passing through $\sigma(t)$ and $\tau(t)$ as t varies in $\mathbb{R}$.

(i) Prove that S is a regular surface.
(ii) Find, for every point $p \in S$, a basis of the tangent plane T_pS.
(iii) Prove that S is orientable.

3.13. Let $S \subset \mathbb{R}^3$ be a surface oriented by an atlas $\mathcal{A} = \{\varphi_\alpha\}$. Given $p \in S$ and a basis $\{v_1, v_2\}$ of T_pS, prove that $\{v_1, v_2\}$ is a positive basis of T_pS if and only if it determines on T_pS the same orientation as the basis $\{\partial_{1,\alpha}|_p, \partial_{2,\alpha}|_p\}$ for all $\varphi_\alpha \in \mathcal{A}$ such that p belongs to the image of φ_α.

3.14. How many orientations does an orientable surface admit?

3.15. Determine a normal versor field for the surface S in $\mathbb{R}^3$ with global parametrization $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^3$ given by $\varphi(u, v) = (\mathrm{e}^u, u + v, u)$, and compute the angle between the coordinate curves.

3.16. Determine a normal versor field for the surface S in $\mathbb{R}^3$ of equation $z = \mathrm{e}^{xy}$. Find for which values of λ, $\mu \in \mathbb{R}$ the vector $(\lambda, 0, \mu)$ is tangent to S at $p_0 = (0, 0, 1)$.

3.17. Let $S \subset \mathbb{R}^3$ be a surface oriented by an atlas $\mathcal{A}$, and let $\mathcal{A}^-$ be the opposite of $\mathcal{A}$. Prove that $\mathcal{A}^-$ is also oriented, and that all $\varphi \in \mathcal{A}$ and $\varphi^- \in \mathcal{A}^-$ with intersecting images determine opposite orientations.

SECOND FUNDAMENTAL FORM

3.18. Prove that if S is an oriented surface with $\mathrm{d}N \equiv O$ then S is contained in a plane.

3.19. Let S be a regular level surface defined by $F(x, y, z) = 0$, with $F \in C^\infty(U)$ and $U \subset \mathbb{R}^3$ open. Show that, for all $p \in S$, the second fundamental form Q_p is the restriction to T_pS of the quadratic form on $\mathbb{R}^3$ induced by the Hessian matrix $\mathrm{Hess}(F)(p)$.

3.20. Consider the surface in $\mathbb{R}^3$ parametrized by $\varphi(u, v) = (u, v, u^2 + v^2)$. Determine the normal curvature of the curve $t \mapsto \varphi(t^2, t)$ contained in it.

3.21. Determine the normal curvature of a regular curve σ whose support is contained in a sphere of radius 3.

PRINCIPAL, GAUSSIAN AND MEAN CURVATURES

3.22. Let $\sigma\colon I \to \mathbb{R}^3$ be a biregular curve parametrized by arc length, and assume there is $M > 0$ such that $\kappa(s) \leq M$ for all $s \in I$. For all $\varepsilon > 0$ let $\varphi^\varepsilon\colon I \times (0, 2\pi) \to \mathbb{R}^3$ be given by

$$\varphi^\varepsilon(s, \theta) = \sigma(s) + \varepsilon \cos\theta\, \mathbf{n}(s) + \varepsilon \sin\theta\, \mathbf{b}(s).$$

(i) Prove that if $\varepsilon < 1/M$ then $\mathrm{d}\varphi^\varepsilon_x$ is injective for all $x \in I \times (0, 2\pi)$.
(ii) Assume that there exists $\varepsilon > 0$ such that φ^ε is globally injective and a homeomorphism with its image, so that it is a local parametrization of a surface $S^\varepsilon = \varphi^\varepsilon\big(I \times (0, 2\pi)\big)$. Find a normal versor field on S^ε, and compute the Gaussian and mean curvatures of S^ε.
(iii) Prove that for any interval $[a, b] \subset I$ there exists $\varepsilon > 0$ such that the restriction $\varphi^\varepsilon|_{(a,b)\times(0,2\pi)}$ is globally injective and a homeomorphism with its image.

3.23. Let $\rho\colon \mathbb{R} \to \mathbb{R}$ be a C^∞ function, and let $\varphi\colon \mathbb{R} \times (0, 2\pi) \to \mathbb{R}^3$ be given by

$$\varphi(z, \theta) = \big(\rho(z)\cos\theta, \rho(z)\sin\theta, z\big).$$

(i) Prove that φ parametrizes a regular surface S if and only if ρ is nowhere zero.
(ii) When S is a surface, write using ρ the first fundamental form with respect to the parametrization φ, and compute the Gaussian curvature of S.

3.24. Let $S \subset \mathbb{R}^3$ be the paraboloid of revolution of equation $z = x^2 + y^2$.

(i) Compute the Gaussian and mean curvatures of S at each point.
(ii) Compute the principal directions of S at the points of the support of the curve $\sigma\colon \mathbb{R} \to S$ given by

$$\sigma(t) = (2\cos t, 2\sin t, 4)\ .$$

3.25. Prove that $H^2 \geq K$ always on an orientable surface S. For which points $p \in S$ does equality hold?

3.26. Prove that cylinders have Gaussian curvature equal to zero everywhere.

3.27. Prove that the Gaussian curvature of a sphere with radius $R > 0$ is $K \equiv 1/R^2$, while its mean curvature (with respect to the usual orientation) is $H \equiv -1/R$.

3.28. Let $\sigma = (\sigma_1, \sigma_2)\colon \mathbb{R} \to \mathbb{R}^2$ be a regular plane curve parametrized by arc length. Let $S \subset \mathbb{R}^3$ be the right cylinder on σ parametrized by

$$\varphi(u,v) = \bigl(\sigma_1(u), \sigma_2(u), v\bigr) .$$

Find the curvatures and the principal directions of S as functions of the curvature κ of σ.

3.29. Denote by $S \subset \mathbb{R}^3$ the subset

$$S = \{(x,y,z) \in \mathbb{R}^3 \mid (1+|x|)^2 - y^2 - z^2 = 0\} .$$

(i) Prove that $T = S \cap \{(x,y,z) \in \mathbb{R}^3 \mid x > 0\}$ is a regular surface.
(ii) Prove that S is not a regular surface.
(iii) Compute the Gaussian curvature and the mean curvature of T.

3.30. Let $S \subset \mathbb{R}^3$ be a surface, and $H \subset \mathbb{R}^3$ a plane such that $C = H \cap S$ is the support of a regular curve. Assume moreover that H is tangent to S at every point of C. Prove that the Gaussian curvature of S is zero at each point of C.

3.31. Let $S \subset \mathbb{R}^3$ be an orientable surface, and let N be a normal versor field on S. Consider the map $F\colon S \times \mathbb{R} \to \mathbb{R}^3$ defined by $F(p,t) = p + tN_p$.

(i) Show that F is smooth.
(ii) Show that the differential $\mathrm{d}F$ is singular at the point (p,t) if and only if $-1/t$ is one of the principal curvatures of S at p.

3.32. Prove that a surface with Gaussian curvature positive everywhere is necessarily orientable.

3.33. Let $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$\varphi(u,v) = (\mathrm{e}^u \cos v, \mathrm{e}^v \cos u, v) .$$

(i) Find the largest $c > 0$ such that φ restricted to $\mathbb{R} \times (-c,c)$ is a local parametrization of a regular surface $S \subset \mathbb{R}^3$.
(ii) Prove that the Gaussian curvature of S is nowhere positive.

(*Hint:* use the well-known formula $K = (eg - f^2)/(EG - F^2)$, without explicitly computing $eg - f^2$.)

3.34. Let $\varphi\colon U \to S$ be a local parametrization of a surface $S \subset \mathbb{R}^3$, and let N be the Gauss map induced by φ. Show that we have $N_u \wedge N_v = K(\varphi_u \wedge \varphi_v)$, where K is the Gaussian curvature.

3.35. Let $\sigma\colon [a,b] \to \mathbb{R}^3$ be a regular closed curve of class C^∞. Assume that the support of σ is contained in the ball with center in the origin and radius r. Show that there exists at least one point where σ has curvature at least $1/r$.

3.36. Let $\sigma\colon \mathbb{R} \to \mathbb{R}^3$ be a regular curve of class C^∞. Assume that the curvature of σ is greater than $1/r$ at every point. Is it true that the support of σ is contained in a ball of radius r?

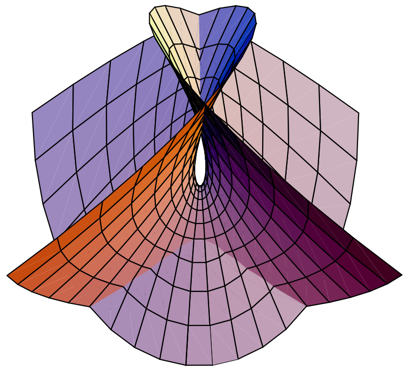


FIGURE 8. Enneper's surface

LINES OF CURVATURE

3.37. Let $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^3$ be the immersed surface (*Enneper's surface*; see Fig. 8) given by

$$\varphi(u,v) = \left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2 \right) .$$

(i) Prove that a connected component S of $\varphi(\mathbb{R}^2) \setminus (\{x=0\} \cup \{y=0\})$ is a regular surface.
(ii) Show that the metric coefficients of S are $F \equiv 0$, $E = G = (1+u^2+v^2)^2$.
(iii) Prove that the form coefficients of S are $e = 2$, $g = -2$, $f = 0$.
(iv) Compute the principal curvatures of S at each point.
(v) Determine the lines of curvature of S.

3.38. Let $S \subset \mathbb{R}^3$ be an oriented surface with Gauss map $N\colon S \to S^2$, and take $p \in S$.

(i) Prove that a vector $v \in T_pS$ is a principal direction if and only if

$$\big\langle \mathrm{d}N_p(v) \wedge v, N(p) \big\rangle = 0.$$

(ii) If $S = f^{-1}(a)$ is a level surface at a regular value $a \in \mathbb{R}$ for some function $f \in C^\infty(\mathbb{R}^3)$ prove that a vector $v \in T_pS$ is a principal direction if and only if

$$\det \begin{vmatrix} \frac{\partial f}{\partial x_1}(p) & \sum\limits_{i=1}^3 v_i \frac{\partial^2 f}{\partial x_i \partial x_1}(p) & v^1 \\ \frac{\partial f}{\partial x_2}(p) & \sum\limits_{i=1}^3 v_i \frac{\partial^2 f}{\partial x_i \partial x_2}(p) & v_2 \\ \frac{\partial f}{\partial x_3}(p) & \sum\limits_{i=1}^3 v_i \frac{\partial^2 f}{\partial x_i \partial x_3}(p) & v_3 \end{vmatrix} = 0 .$$

3.39. Assume that two surfaces in $\mathbb{R}^3$ intersect along a curve σ in such a way that the tangent planes form a constant angle. Show that if σ is a line of curvature in one of the two surfaces it is a line of curvature in the other surface too.

ISOMETRIES, AGAIN

3.40. Let $\varphi\colon (0,2\pi)\times(0,2\pi)\to\mathbb{R}^3$ be the global parametrization of the surface $S\subset\mathbb{R}^3$ given by

$$\varphi(u,v)=\big((2+\cos u)\cos v,(2+\cos u)\sin v,\sin u\big)\ .$$

(i) Determine the metric and form coefficients.
(ii) Determine the principal curvatures and the lines of curvature.
(iii) Determine whether S is locally isometric to a plane.

3.41. Let φ, $\tilde{\varphi}\colon \mathbb{R}^+\times(0,2\pi)\to\mathbb{R}^3$ be given by

$$\varphi(u,v)=(u\cos v,u\sin v,\log u),\qquad \tilde{\varphi}(u,v)=(u\cos v,u\sin v,v)\ ;$$

the image S of φ is the surface of revolution generated by the curve $(t,\log t)$, while the image $\tilde{S}$ of $\tilde{\varphi}$ is a portion of an helicoid. Prove that $K\circ\varphi\equiv\tilde{K}\circ\tilde{\varphi}$, where K (respectively, $\tilde{K}$) is the Gaussian curvature of S (respectively, $\tilde{S}$), but that $\tilde{\varphi}\circ\varphi^{-1}$ is not an isometry. Prove next that S and $\tilde{S}$ are not locally isometric. (*Hint:* in these parametrizations K depends on a single parameter, which can be determined in the same way in both surfaces. Assuming the existence of a local isometry, write the action on the coefficients of the first fundamental form: since the conditions imposed by the equality on these coefficients cannot be satisfied, the local isometry cannot exists.)

ASYMPTOTIC CURVES

3.42. Let $S\subset\mathbb{R}^3$ be the surface with global parametrization $\varphi\colon\mathbb{R}^2\to\mathbb{R}^3$ given by $\varphi(u,v)=(u,v,uv)$.

(i) Determine the asymptotic curves of S.
(ii) Determine the values the curvature of the normal sections of S takes at the origin.

3.43. Let σ be a regular curve of class C^∞ on a surface S in $\mathbb{R}^3$. Show that if σ is an asymptotic curve then the normal to σ is always tangent to S.

3.44. Let S be a regular surface in $\mathbb{R}^3$.

(i) Show that if ℓ is a line segment contained in S then ℓ is an asymptotic curve for S.
(ii) Show that if S contains three distinct line segments passing through a given point $p\in S$ then the second fundamental form of S at p is zero, that is, p is a planar point.

3.45. Determine the asymptotic curves of the regular surface (see also Exercise 3.1) with global parametrization $\varphi(u,v)=(u,v,u^4+v^4)$.

3.46. Let p be a point of a regular surface $S\subset\mathbb{R}^3$. Assume that at p there are exactly two distinct asymptotic directions. Show that there exist a neighborhood U of p in S and two maps X, $Y\colon U\to\mathbb{R}^3$ of class C^∞ such that for all $q\in U$ the vectors $X(q)$ and $Y(q)$ are linearly independent and asymptotic tangent vectors to S at q.

ELLIPTIC, HYPERBOLIC, PARABOLIC, PLANAR AND UMBILICAL POINTS

3.47. **Characterization of umbilical points.** Show that a point p of a regular surface S is umbilical if and only if, in a local parametrization of S, the first and the second fundamental form are equal. In particular, show that, when the form coefficients are different from zero, the point p is umbilical if and only if

$$\frac{E}{e} \equiv \frac{F}{f} \equiv \frac{G}{g} \ ,$$

and that in this case the normal curvature equals $\kappa_n = E/e$.

3.48. Let S be the graph of the function $f(x, y) = x^4 + y^4$. Prove that the point $O \in S$ is planar and that S lies within one of the two closed half-spaces bounded by the plane $T_O S$.

3.49. Find the umbilical points of the two-sheeted hyperboloid of equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1.$$

3.50. Find the umbilical points of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \ .$$

3.51. Let S be a connected regular surface in which every point is planar. Show that S is contained in a plane.

3.52. Let S be a closed, connected regular surface in $\mathbb{R}^3$. Show that S is a plane if and only if through every point p of S (at least) three distinct straight lines lying entirely in S pass.

3.53. Let S be the graph of the function $f(x, y) = x^3 - 3y^2 x$ (this surface is sometimes called *monkey saddle*). Show that the point $O \in S$ is planar and that every neighborhood of O in S intersects both the open half-spaces bounded by the plane $T_O S$.

3.54. Let $\mathcal{Q}$ be a quadric in $\mathbb{R}^3$ such that $\mathcal{Q}$ is a regular surface but not a plane (see Problem 2.4).

(i) Show that $\mathcal{Q}$ has no parabolic points.
(ii) Show that if $\mathcal{Q}$ has a hyperbolic point then all points of $\mathcal{Q}$ are hyperbolic.
(iii) Conclude that if $\mathcal{Q}$ has an elliptic point, then all points are elliptic.
(iv) Determine which quadrics have only hyperbolic points and which quadrics have only elliptic points.

3.55. Determine whether the origin O is an elliptic, hyperbolic, parabolic or planar point in the surface of equation

(i) $z - xy = 0$;
(ii) $z - y^2 - x^4 = 0$;
(iii) $x + y + z - x^2 - y^2 - z^3 = 0$.

GAUSS' THEOREMA EGREGIUM

3.56. Compute the Christoffel symbols for the polar coordinates of the plane.

3.57. Prove that the Christoffel symbols can be computed using the following formula:

$$\Gamma_{ij}^k = \frac{1}{2}\sum_{l=1}^2 g^{kl}\left(\frac{\partial g_{il}}{\partial x_j} + \frac{\partial g_{lj}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_l}\right) ,$$

where $g_{11} = E$, $g_{12} = g_{21} = F$, $g_{22} = G$, and (g^{ij}) is the inverse matrix of matrix (g_{ij}).

3.58. Check that the equations (53) written for the other possible values of i, j, k and r are either trivially satisfied, or a consequence of the symmetry of the Christoffel symbols, or equivalent to (54).

3.59. Let E be the ellipsoid of equation

$$\frac{1}{4}x^2 + y^2 + \frac{1}{9}z^2 = 3 .$$

(i) Compute the Gaussian curvature K and the principal directions of E at the point $p = (2, 1, 3) \in E$.
(ii) Compute the integral of the Gaussian curvature K on the intersection of E with the octant

$$Q = \{(x, y, z) \mid x \geq 0, y \geq 0, z \geq 0\} .$$

3.60. Check that the compatibility conditions that are a consequence of the identity $\partial^2(N \circ \varphi)/\partial x_i \partial x_j \equiv \partial^2(N \circ \varphi)/\partial x_j \partial x_i$ are either always satisfied or equivalent to (55).

CONFORMAL MAPS

Definition 3.E.1. A map $H\colon S_1 \to S_2$ of class C^∞ between two surfaces in $\mathbb{R}^3$ is *conformal* if there exists a function $\lambda\colon S_1 \to \mathbb{R}^*$ of class C^∞ nowhere vanishing such that

$$\langle \mathrm{d}H_p(v_1), \mathrm{d}H_p(v_2)\rangle_{H(p)} = \lambda^2(p)\langle v_1, v_2\rangle_p$$

for all $p \in S_1$ and all v_1, $v_2 \in T_pS_1$. The map H is *locally conformal at* p if there are neighborhoods U_1 of p in S_1 and U_2 of $H(p)$ in S_2 such that the restriction of $H|_{U_1}\colon U_1 \to U_2$ is conformal. Two surfaces S_1 and S_2 are *conformally equivalent* if there exists a conformal diffeomorphism $H\colon S_1 \to S_2$. Finally, S_1 is *locally conformal* to S_2 if for all $p \in S_1$ there exist a point $q \in S_2$ and a conformal diffeomorphism between a neighborhood of p in S_1 and a neighborhood of q in S_2.

3.61. Show that the stereographic projection (see Exercise 2.4) is a conformal map.

3.62. Prove an analogue of Proposition 3.1 for conformal maps: Let S, $\tilde{S} \subset \mathbb{R}^3$ be two surfaces. Then S is locally conformal to $\tilde{S}$ if and only if for every point $p \in S$ there exist a point $\tilde{p} \in \tilde{S}$, an open set $U \subseteq \mathbb{R}^2$, a function $\lambda \in C^\infty(U)$ nowhere zero, a local parametrization $\varphi\colon U \to S$ of S centered at p, and a local parametrization $\tilde{\varphi}\colon U \to \tilde{S}$ of $\tilde{S}$ centered at $\tilde{p}$ such that $\tilde{E} \equiv \lambda^2 E$, $\tilde{F} \equiv \lambda^2 F$ and $\tilde{G} \equiv \lambda^2 G$, where E, F, G (respectively, $\tilde{E}$, $\tilde{F}$, $\tilde{G}$) are the metric coefficients of S with respect to φ (respectively, of $\tilde{S}$ with respect to $\tilde{\varphi}$).

Definition 3.E.2. A local parametrization of a surface S is called *isothermal* if $E \equiv G$ e $F \equiv 0$.

3.63. Prove that two surfaces both having an atlas consisting of isothermal local parametrizations are locally conformal. (*Remark*: It is possible to prove that every regular surface admits an atlas consisting of isothermal local parametrizations; as a consequence, two regular surfaces are always locally conformal.)

3.64. Let $\varphi\colon U \to S$ be a isothermal local parametrization. Prove that

$$\varphi_{uu} + \varphi_{vv} = 2EHN \ , \quad \text{and that} \quad K = -\frac{\Delta \log G}{G} \ ,$$

where Δ denotes the Laplacian.

RULED SURFACES

Definition 3.E.3. A *conoid* in $\mathbb{R}^3$ is a ruled surface in $\mathbb{R}^3$ whose rulings are parallel to a plane H and intersect a straight line ℓ. The conoid is said to be *right* if the line ℓ is orthogonal to the plane H. The line ℓ is called *axis* of the conoid.

3.65. Show that the right helicoid parametrized as in Problem 2.2 is a right conoid.

3.66. Let $S \subset \mathbb{R}^3$ be a right conoid having rulings parallel to the plane $z = 0$ and the z-axis as its axis. Prove that it is the image of a map $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^3$ of the form

$$\varphi(t, v) = \big(v \cos f(t), v \sin f(t), t\big) \ ,$$

where $f\colon \mathbb{R} \to \mathbb{R}$ is such that $f(t)$ is a determination of the angle between the ruling contained in $z = t$ and the plane $y = 0$. Prove that the map φ is an immersed surface if f is of class C^∞.

3.67. Prove that the cylinders introduced in Definition 2.P.3 are ruled surfaces.

3.68. Given a regular curve $\sigma\colon I \to \mathbb{R}^3$ of class C^∞, and a curve $\vec{v}\colon I \to S^2$ of class C^∞ on the sphere, let $\varphi\colon I \times \mathbb{R}^* \to \mathbb{R}^3$ be defined by

$$\varphi(t, v) = \sigma(t) + v\,\vec{v}(t) \ . \tag{57}$$

Prove that φ is an immersed surface if and only if $\vec{v}$ and $\sigma' + v\vec{v}'$ are everywhere linearly independent. In this case, φ is called a *parametrization in ruled form* of its support S, the curve σ is called *base curve* or *directrix*, and the lines $v \mapsto \varphi(t_0, v)$ are called *(rectilinear) generators* of S.

3.69. Let $S \subset \mathbb{R}^3$ be the hyperbolic paraboloid of equation $z = x^2 - y^2$.

(i) Find two parametric representations in ruled form (see Exercise 3.68) of S, corresponding to two different systems of generators.
(ii) Determine the generators of the two systems passing through the point $p = (1, 1, 0)$.

3.70. Prove that the tangent plane at the points of a generator of the (non-singular part of the) tangent surface (see Problem 3.20) to a biregular curve C coincides with the osculating plane to the curve C at the intersection point with the generator.

3.71. Let $\sigma\colon I \to \mathbb{R}^3$ be a regular plane curve of class C^∞, parametrized by arc length, with curvature $0 < \kappa < 1$. Let $\varphi\colon I \times (0, 2\pi) \to \mathbb{R}^3$ be the immersed surface given by $\varphi(t, v) = \sigma(t) + \cos v\,\vec{n}(t) + \vec{b}(t)$.

(i) Determine the Gaussian curvature and the mean curvature at every point of the support S of φ.
(ii) Determine the lines of curvature at each point of S.

3.72. Let $\sigma\colon I \to \mathbb{R}^3$ be a biregular curve of class C^∞, parametrized by arc length, and let $\varphi\colon I \times (-\varepsilon, \varepsilon) \to \mathbb{R}^3$ be the map given by $\varphi(s, \lambda) = \sigma(s) + \lambda \vec{n}(s)$.

(i) Show that, when ε is small enough, φ is a global parametrization of a surface S, called *normal surface* of σ.
(ii) Show that the tangent plane to S at a point of σ is the osculating plane to σ.

MINIMAL SURFACES

Definition 3.E.4. A surface $S \subset \mathbb{R}^3$ is *minimal* if its mean curvature vanishes everywhere.

3.73. Prove that there are no compact minimal surfaces.

3.74. Let $\varphi\colon U \to S$ be a global parametrization of a surface S. Given $h \in C^\infty(U)$, the *normal variation* of φ along h is the map $\varphi^h\colon U \times (-\varepsilon, \varepsilon) \to \mathbb{R}^3$ defined by

$$\varphi^h(x,t) = \varphi(x) + t\,h(x) N\bigl(\varphi(x)\bigr) ,$$

where $N\colon \varphi(U) \to S^2$ is the Gauss map induced by φ.

(i) Prove that for every open set $U_0 \subset U$ with compact closure in U there exists an $\varepsilon > 0$ such that $\varphi^h|_{U_0 \times (-\varepsilon,\varepsilon)}$ is an immersed surface.
(ii) Let $R \subset U$ be a regular region, and $A^h_R\colon (-\varepsilon,\varepsilon) \to \mathbb{R}$ the function defined by $A^h_R(t) = \mathrm{Area}\bigl(\varphi^h(R)\bigr)$. Prove that A^h_R is differentiable at zero and that

$$\frac{\mathrm{d}A^h_R}{\mathrm{d}t}(0) = -\int_{\varphi^h(R)} 2hH\,\mathrm{d}\nu .$$

(iii) Prove that $\varphi(U)$ is minimal if and only if

$$\frac{\mathrm{d}A^h_R}{\mathrm{d}t}(0) = 0$$

for every $h \in C^\infty(U)$ and every regular region $R \subset U$.

3.75. Prove that the catenoid is a minimal surface, and that no other surface of revolution is minimal.

3.76. Prove that the helicoid is a minimal surface. Conversely, prove that if $S \subset \mathbb{R}^3$ is a minimal ruled surface whose planar points are isolated then S is a helicoid. (*Hint:* Exercise 1.61 can help.)

3.77. Prove that Enneper's surface (see Exercise 3.37) is minimal where it is regular.

3.78. Let $S \subset \mathbb{R}^3$ be an oriented surface without umbilical points. Prove that S is a minimal surface if and only if the Gauss map $N\colon S \to S^2$ is a conformal map. Use this result to construct isothermal local parametrizations on minimal surfaces without umbilical points.

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