Functional Analysis (Part II)

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Some References

- John B. Conway "A course in functional analysis"
- Ronald G. Douglas "Banach algebra techniques in operator theory"

Lecture 1

15/4/2021

Let H be a vector space over $\mathbb{F}$ $\qquad \mathbb{F} = \mathbb{R}$ REAL NUMBER

$\mathbb{C}$ COMPLEX NUMBER

Def (SEMI-INNER PRODUCT)

$u: H \times H \longrightarrow \mathbb{F}$ SATISFYING THE FOLLOWING PROPERTIES:

a) $u(\alpha x + \beta y, z) = \alpha u(x,z) + \beta u(y,z)$

$\forall x,y,z \in H \quad \forall \alpha, \beta \in \mathbb{F}$

b) $u(x, \alpha y + \beta z) = \bar{\alpha} u(x,y) + \bar{\beta} u(x,z) \quad \forall \alpha,\beta \in \mathbb{F} \; \forall x,y,z \in H$

NOTATION

$\alpha \in \mathbb{F}$: if $\mathbb{F} = \mathbb{R}$ $\quad \bar{\alpha} = \alpha$

if $\mathbb{F} = \mathbb{C}$ $\quad \bar{\alpha}$ is complex conjugate

c) $\boxed{u(x,x) \geq 0}$

THIS MEANS $\left(u(x,x) \in \mathbb{R},\ \geq 0 \right)$

d) $u(x,y) = \overline{u(y,x)}$

$\Rightarrow (u(x,x) \in \mathbb{R})$

Remark • $u(0,y) = u(\alpha \cdot 0, y) = \alpha u(0,y) \quad \forall \alpha \in \mathbb{F}$

$\Rightarrow u(0,y) = 0 \quad \forall y \in H$

• $u(x,0) = 0 \quad \forall x \in H$ (similarly)

$\Rightarrow$ • $\boxed{u(0,0) = 0}$

From conditions to be a semi-inner product it follows

$\Rightarrow u(0,0) = 0$ Are there other $x \in H$ s.t. $u(x,x)=0$?

Def (inner product) $u: H \times H \to \mathbb{F}$ is an inner product

if it is a semi-inner product and

$$\boxed{u(x,x) = 0 \iff x = 0}$$ (non-degeneration)

NOTATION We denote an inner product by $\langle \cdot, \cdot \rangle$

Examples:

1 Euclidean space $H = \mathbb{F}^d$ ($\mathbb{R}^d$ or $\mathbb{C}^d$)

vector space over $\mathbb{F}$

STANDARD INNER PRODUCT $z, w \in \mathbb{F}^d$

$z = (z_1 \dots z_d)$ $w = (w_1 \dots w_d)$

$$\langle z, w \rangle := \sum_{i=1}^{d} z_i \overline{w_i}$$

complex conjugation

EXERCISE 1: CHECK THAT IT IS AN INNER PRODUCT ON $\mathbb{F}^d$

② $H = \left\{ \{x_n\}_{n\in\mathbb{N}} \text{ st all but finitely many elements are } \neq 0 \right\}$

sequence in $\mathbb{F}$
$(x_n \in \mathbb{F}\ \forall n)$

IDEA: WE ARE CONSIDERING THE 'UNION' OF ALL $\mathbb{F}^d$

$$H := \bigcup_{d=1}^{\infty} \mathbb{F}^d$$

- H is a vector space over $\mathbb{F}$

$$\{x_n\}_n + \{y_n\}_n = \{x_n + y_n\}_n$$

$$\alpha\{x_n\}_n = \{\alpha x_n\}_n$$

$\alpha \in \mathbb{F}$

- Define $\langle \{x_n\}_n, \{y_n\}_n \rangle := \sum_{n=1}^{\infty} x_n \overline{y_n}$

well-defined
(THIS IS A FINITE SUM)

EXERCISE 2: SHOW THAT IT IS AN INNER PRODUCT ON H

- Define $u(\{x_n\}, \{y_n\}) := \sum_{n=1}^{\infty} x_{2n} \overline{y_{2n}}$

EXERCISE 3: THIS IS A SEMI-INNER PRODUCT ON H BUT NOT AN INNER PRODUCT

③ $\ell^2 = \left\{ \{x_m\}_m : \sum_{m=1}^{\infty} |x_m|^2 < \infty \right\}$

sequences in $\mathbb{F}$

modulus of a complex number (or absolute value of a real number)

EXERCISE 4

- SHOW THAT ℓ^2 is a vector space over $\mathbb{F}$

Define:

- $\langle \{x_m\}, \{y_m\} \rangle := \sum_{m=1}^{\infty} x_m \overline{y_m}$

check that this defines an inner product on ℓ^2

④ (X, Ω, μ) measure space

Ω: 6-algebra

μ: countably additive measure

$$L^2(X, \Omega, \mu) = \left\{ f : X \to \mathbb{F} \quad \Omega\text{-measurable st } \int_X |f|^2 d\mu < \infty \right\} \Big/ \sim$$

vector space $\leftarrow$ CHECK IT!

QUOTIENT SPACE w.r.t THE EQUIVAL. RELATION:

$f \sim g$ if

$f - g = 0 \quad \mu$.a.e

$$\langle f, g \rangle := \int_X f \overline{g} \, d\mu$$

It defines an inner product on $L^2(X, \Omega, \mu)$

$$\left(\langle f, f \rangle = \int_X |f|^2 d\mu \quad \begin{aligned} &\geq 0 \\ &= 0 \iff f = 0 \text{ a.e} \end{aligned} \right)$$

Remark In example (4):

Take $X = \mathbb{N}$ $\quad \Omega = \mathcal{P}(\mathbb{N})$ ← set of subsets of $\mathbb{N}$

$\mu =$ counting measure

$$\mu(E) = \begin{cases} +\infty \\ |E| \end{cases}$$

CARDINALITY OF E is ∞

← CARDINALITY OF E IF IT IS FINITE

$$\boxed{L^2(\mathbb{N}, \Omega, \mu) = \ell^2}$$

ℓ^2 introduced before in example (3)

Let us discuss some properties of the inner product:

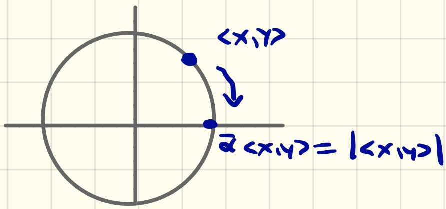
PROPOSITION (CAUCHY–SCHWARZ INEQUALITY)

Let H be a vector space and $\langle , \rangle$ a (semi)-inner product

$$\Rightarrow \quad |\langle x, y\rangle|^2 \leq \langle x, x\rangle \cdot \langle y, y\rangle \qquad \forall x, y \in H$$

PROOF. $x, y \in H$ $\qquad \langle x, y\rangle \in \mathbb{F}$

choose $\alpha \in \mathbb{F}$ $\ |\alpha| = 1$ st $\ \bar{\alpha}\langle x, y\rangle = |\langle x, y\rangle|$



Let $x, y \in H$ and α as before $\left(\bar{\alpha} \langle x,y \rangle = |\langle x,y \rangle| \right)$

$|\alpha| = 1$

$\forall r \in \mathbb{R}$

PROPERTIES (a, b)

$0 \le \langle x - r\alpha y, x - r\alpha y \rangle = \langle x,x \rangle - r\alpha \langle y,x \rangle$

$- r \underbrace{\bar{\alpha} \langle x,y \rangle}_{= |\langle x,y \rangle|} + r^2 \underbrace{\alpha \bar{\alpha}}_{= |\alpha|^2 = 1} \langle y,y \rangle$

PROPERTY (c) OF SEMI-INNER PRODUCT

$= r^2 \langle y,y \rangle - 2r |\langle x,y \rangle| + \langle x,x \rangle$

$\alpha \langle y,x \rangle =$

$\overset{(d)}{=} \alpha \overline{\langle x,y \rangle} =$

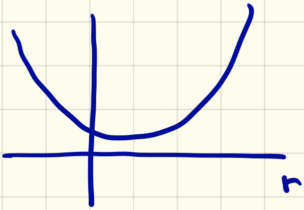
$= \overline{\left(\bar{\alpha} \langle x,y \rangle \right)}$

$= \overline{|\langle x,y \rangle|} =$

$= |\langle x,y \rangle|$

$\forall r \in \mathbb{R}$ $\quad \underbrace{r^2 \langle y,y \rangle - 2r |\langle x,y \rangle| + \langle x,x \rangle \ge 0}$

IT IS A PARABOLA



$\Delta = \boxed{|\langle x,y \rangle|^2 - \langle x,x \rangle \langle y,y \rangle \le 0}$

$\Updownarrow$

$|\langle x,y \rangle|^2 \le \langle x,x \rangle \langle y,y \rangle$ □

$f(x) = ax^2 + bx + c$

$\Rightarrow \Delta = b^2 - 4ac$

IF $f(x) \ge 0 \ \forall x$

$\Rightarrow \Delta \le 0$

QUESTION: IS IT POSSIBLE THAT

$$|\langle x,y\rangle|^2 = \langle x,x\rangle\cdot\langle y,y\rangle \quad \text{FOR SOME } x,y\in H?$$

PROP. Let $\langle\, ,\rangle$ BE AN INNER PRODUCT.
Equality holds in Cauchy–Schwarz inequality

$\iff$ x and y are collinear

$$x = \alpha y \quad \text{for some } \alpha\in\mathbb{F}$$

proof $(\Leftarrow)$ $$|\langle x,y\rangle|^2 = |\langle \alpha y, y\rangle|^2 =$$

$$= |\alpha|^2 |\langle y,y\rangle|^2$$

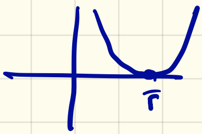
THIS IMPLICATION IS TRUE ALSO FOR SEMI-INNER PRODUCTS

$$\langle x,x\rangle = \langle \alpha y, \alpha y\rangle = \overset{|\alpha|^2}{\overset{\shortparallel}{\alpha\bar{\alpha}}}\langle y,y\rangle$$

$$\langle x,x\rangle\cdot\langle y,y\rangle = |\alpha|^2\langle y,y\rangle$$

$(\Rightarrow)$ if equality holds then $\Delta = 0$ (SEE PROOF OF CAUCHY–SCHWARZ INEQ.)

$\Rightarrow \exists\, \bar{r}\in\mathbb{R}$ st



$$\bar{r}^2\langle y,y\rangle - 2\bar{r}|\langle x,y\rangle| + \langle x,x\rangle = 0$$

$$\Downarrow$$

$$\langle x-\bar{r}\alpha y, x-\bar{r}\alpha y\rangle = 0 \xRightarrow{\text{BEING AN INNER PRODUCT}} x-\bar{r}\alpha y = 0$$

$$\Rightarrow x = \bar{r}\alpha y \qquad \square$$

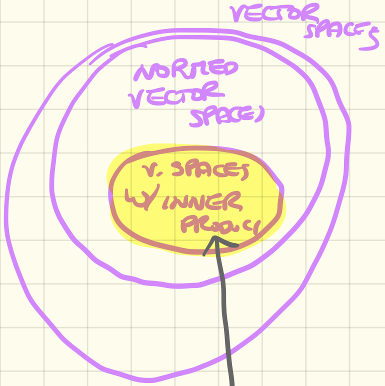
$(H, \langle \cdot, \cdot \rangle)$

$\uparrow$ inner product

We can define a **NORM** on $(H, \langle \cdot, \cdot \rangle)$

$\|x\|^2 = \langle x, x \rangle$ $\leftarrow$ norm induced by the inner product

$\downarrow$



WE'LL DISCUSS THAT THEY FORM A PROPER SUBSET

Prop $\|\cdot\|$ SO DEFINED IS A NORM

proof

- $\|x\| \geq 0$

  $\|x\| = \sqrt{\langle x, x \rangle}$

- $\|x\| = 0 \implies x = 0$

  $\|x\| = \sqrt{\langle x, x \rangle}$

  $\uparrow$ INNER PRODUCT $\langle x, x \rangle = 0 \iff x = 0$

- $\|\alpha x\| = |\alpha| \, \|x\| \quad \forall \alpha \in \mathbb{F}$

  $$\|\alpha x\| = \sqrt{\langle \alpha x, \alpha x \rangle} = \sqrt{\alpha \bar{\alpha} \langle x, x \rangle} = \sqrt{|\alpha|^2 \langle x, x \rangle}$$

  $$= |\alpha| \, \|x\|$$

- TRIANGLE INEQUALITY HOLDS

  $\|x + y\| \leq \|x\| + \|y\|$ $\longrightarrow$ next page

Let's prove triangle inequality

$$\|x+y\|^2 = \langle x+y, x+y \rangle$$

$$= \underbrace{\langle x,x\rangle}_{\|x\|^2} + \underbrace{\langle x,y\rangle + \langle y,x\rangle}_{} + \underbrace{\langle y,y\rangle}_{\|y\|^2}$$

$$\Downarrow$$

$$\langle x,y\rangle + \overline{\langle x,y\rangle}$$

$$= 2\,\mathrm{Re}(\langle x,y\rangle)$$

$$\leq 2\,|\langle x,y\rangle| \leq 2\,\underset{\sqrt{\langle x,x\rangle}}{\|x\|}\,\underset{\sqrt{\langle y,y\rangle}}{\|y\|}$$

CAUCHY-SCHWARZ INEQ

$$\Rightarrow \|x+y\|^2 \leq \|x\|^2 + 2\|x\|\,\|y\| + \|y\|^2$$

$$= (\|x\| + \|y\|)^2$$

$$\Rightarrow \|x+y\| \leq \|x\| + \|y\| \quad \square$$

(Question: when does = hold in the triangle inequality?)

EXERCISE 5