Lecture 6

3/5/2021

Rmk In general vector subspaces might not be closed

ex: $\mathcal{H} = L^2[0,1] = \{ f : [0,1] \to \mathbb{R} \text{ measurable } \int |f|^2 < \infty \} / \sim$

$\boxed{f \sim g \text{ iff } f - g = 0 \text{ a.e.} \quad f, g \in L^2}$

the $V = [C[0,1]]_\sim$

subspace but it is not closed in $L^2[0,1]$

$\Rightarrow$ it is dense!

In finite dim: all vector subspaces are closed

PROP Let $(\mathcal{H}, \langle , \rangle)$ be an Hilbert space

then, all finite dimensional vector subspaces are closed

proof By induction on the dim of vector subspace

Let V be a finite dim. subspace of $\mathcal{H}$

$m = \dim V \geq 0$

BASIS OF THE INDUCTION $m = 0$ $\quad V = \{0\}$

$x_m \equiv 0$

clearly closed $\{x_m\} \subset \underset{\{0\}}{\underset{\shortparallel}{V}}$ $\quad x_m \to \bar{x} \quad \Rightarrow \quad \bar{x} = 0$

Assume that the claim is true for $n = N$

and prove it for $n = N+1$

V finite dim vector space of dim N

$V = \mathcal{H} \longrightarrow$ in this case V is closed

$V \neq \mathcal{H} \longrightarrow \exists\ y \notin V$

consider $V \oplus \{y\} = \{ x + \lambda y : x \in V \ \lambda \in \mathbb{F} \}$ $\mathbb{F} = \mathbb{R}, \mathbb{C}$

↳ we want to show that $V \oplus \{y\}$ is closed

BEING CLOSED IS EQUIV. TO $\Longleftrightarrow$

$$\boxed{\begin{array}{l}\text{if } \{x_k + \lambda_k y\} \subseteq V \oplus \{y\} \text{ st } x_k + \lambda_k y \to \bar{z} \\ \text{then } \bar{z} \in V \oplus \{y\}\end{array}}$$

- $\exists M > 0 \quad \|x_k + \lambda_k y\| \leq M \quad \forall k \geq 0$

  every convergent sequence is bounded

- Claim: $\{\lambda_k\} \subseteq \mathbb{F}$ are bounded

  by contradiction, up to extract a subsequence,

  $|\lambda_k| \longrightarrow +\infty$

  $\lambda_k \neq 0$

$$0 \xleftarrow[k_j \to +\infty]{} \frac{\overbrace{x_{k_j} + \lambda_{k_j} y}^{\text{it is bounded.}}}{\lambda_{k_j}}$$

$$\left\| \frac{x_{k_j} + \lambda_{k_j} y}{\lambda_{k_j}} \right\| \le \frac{M}{|\lambda_{k_j}|} \xrightarrow[k_j \to \infty]{} 0$$

$$= \frac{x_{k_j}}{\lambda_{k_j}} + y$$

$$\Rightarrow \quad \underbrace{\frac{x_{k_j}}{\lambda_{k_j}}}_{\substack{\text{this is a sequence} \\ \text{in } V \leftarrow \text{closed}}} \longrightarrow -y \qquad \text{contradiction!}$$

so $\frac{x_{k_j}}{\lambda_{k_j}}$ must converge to an element of V but $y \notin V$

Hence $\{\lambda_k\}$ must be bounded

• $\Rightarrow \exists\ \lambda_{k_j} \longrightarrow \bar{\lambda} \in \mathbb{F}$

$$\underset{\text{by hyp.}}{\bar{z}} \longleftarrow x_{k_j} + \underbrace{\lambda_{k_j} y}_{\downarrow \ \bar{\lambda} y} \Rightarrow \{x_{k_j}\} \subseteq V \text{ is convergent}$$

$\Downarrow$ V is closed

$x_{k_j} \to \bar{x}$ in V

$$\bar{z} \longleftarrow x_{k_j} + \lambda_{k_j} y \longrightarrow \bar{x} + \bar{\lambda} y \Rightarrow \underbrace{\bar{z} = \bar{x} + \bar{\lambda} y}_{\text{uniqueness of the limit}} \qquad \square$$

Exercise

If V, W are closed vector subspaces, is it true that $V \oplus W$ is still closed?

(ANSWER: No)
find counterexample

Riesz Representation theorem

prop $L: \mathcal{H} \to \mathbb{F}$ linear functional (linear map from $\mathcal{H} \to \mathbb{F}$)

then the following properties are equivalent:

a) L is continuous

b) L is continuous at 0

c) L is continuous at some point of $\mathcal{H}$.

d) $\exists\, c > 0$ st $|L(h)| \leq c\,\|h\| \quad \forall\, h \in \mathcal{H}$ ← Boundedness

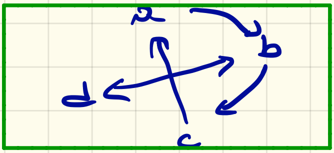
Proof (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c) obvious

(d) $\Longrightarrow$ (b) in fact (d) implies

$$\underbrace{|L(h) - L(0)|}_{L(0)=0} \le c \, \underbrace{\|h - 0\|}_{= \|h\|}$$

L is Lipschitz at 0 $\Longrightarrow$ continuity

<u>We need to prove:</u> (b) $\Longrightarrow$ (d) and (c) $\Longrightarrow$ (a)



Hence from this, we conclude all equivalences

$\boxed{c) \Longrightarrow a)}$

Assume by hp that L is continuous at some $h_0 \in H$

Let us show continuity at any other point $\bar{h} \in H$

If $h_n \to \bar{h}$ <u>claim:</u> $L(h_n) \to L(\bar{h})$ $\leftarrow$ it proves continuity at $\bar{h}$

$\hookrightarrow$ $\boxed{h_n - \bar{h} + h_0 \longrightarrow h_0}$ $\leftarrow$ new sequence

Continuity at $h_0 \Rightarrow$

$$L(h_m - \bar{h} + h_0) \xrightarrow{m\to\infty} L(h_0)$$

$\parallel \Leftarrow$ linearity

$$L(h_m) - L(\bar{h}) + L(h_0)$$

$$\Rightarrow \quad L(h_m) - L(\bar{h}) \longrightarrow 0 \quad m\to\infty$$

$$\Rightarrow \quad L(h_m) \to L(\bar{h}) \quad \text{as } m\to\infty$$

$\square$ (claim)

$\boxed{b) \Rightarrow d)}$ Assume L is continuous at 0

$\Leftrightarrow \forall \varepsilon > 0 \ \exists \delta = \delta(\varepsilon)$ st if $\|h\| < \delta$ then $|L(h)| < \varepsilon$

Take $\varepsilon = 1$

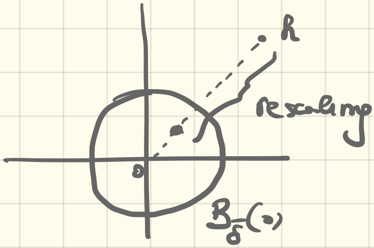
$\hookrightarrow \exists \bar{\delta} > 0$ st $\underbrace{\text{if } \|h\| < \bar{\delta} \text{ then } |L(h)| < 1}$

$$B_\delta(0) = \{\|h\| < \delta\}$$

then $L(B_\delta(0)) \subseteq \{ \underset{z \in \mathbb{F}}{|z| < 1} \}$

let $h \in \mathcal{H}$ and let $\lambda > 0$

then consider $\quad \dfrac{\bar\delta h}{\|h\|+\lambda} \in B_{\bar\delta}(0)$



then $\left| L\left(\dfrac{\bar\delta h}{\|h\|+\lambda} \right) \right| < 1$

$$\| $$

$$\frac{\bar\delta}{\|h\|+\lambda} |L(h)| < 1$$

$$\Rightarrow \quad |L(h)| < \frac{\|h\|+\lambda}{\bar\delta} \qquad \forall\, h \in \mathcal{H}$$

this reasoning is true $\forall \lambda > 0$, then

$$|L(h)| < \lim_{\lambda \to 0} \frac{\|h\|+\lambda}{\bar\delta} = \frac{\|h\|}{\bar\delta}$$

$\Rightarrow$ (d) holds with $c = \dfrac{1}{\bar\delta}$. $\square$

Def $L: \mathcal{H} \to \mathbb{F}$ is called a bounded linear functional

if L is linear and $\exists\, c>0$ st $|L(h)| \leq c\|h\|$ $\forall h \in \mathcal{H}$

$\Updownarrow$ it is equivalent to asking continuity

if $L: \mathcal{H} \to \mathbb{F}$ is bounded linear functional

then we define the norm of L:

$$\|L\| = \sup\{|L(h)| : \|h\| \leq 1\} < \infty$$

By def of bounded operator

EXERCISE L is a bounded linear functional

then

$$\|L\| = \sup\{|L(h)| : \|h\| = 1\}$$

$$= \sup\left\{\frac{|L(h)|}{\|h\|} : \forall h \neq 0\right\}$$

$$= \inf\{c>0 : |L(h)| \leq c\|h\| \quad \forall h \in \mathcal{H}\}$$

prove that they are equivalent