Lecture 5

29/4/2021

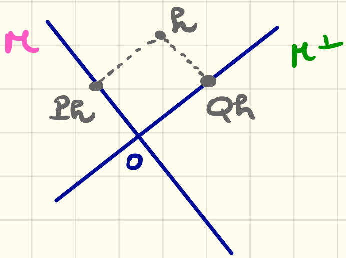
Recall: $(\mathcal{H}, \langle , \rangle)$ Hilbert space

orthogonality: $x, y \in \mathcal{H}$ $\quad x \perp y \iff \langle x, y \rangle = 0$

$A \subseteq \mathcal{H}$ $\quad A^{\perp} = \{ x \in \mathcal{H} : \langle x, a \rangle = 0 \ \forall a \in A \}$

↑ closed vector subspace

GOAL: Define orthogonal projections:



$h = Ph + Qh$

↑ unique way

Prop $K \subseteq \mathcal{H}$ closed convex set

$$\forall h \in \mathcal{H} \quad \exists!\ k_0 \in K \quad \|h - k_0\| = \underbrace{\inf\{ \|h - k\| : k \in K \}}_{=: d(h, K)}$$

Theorem let $M \subseteq H$ be a closed vector subspace

↑ it is convex

$h \in H$

(i) let $f_0 \in M$ be the unique element of M st

$$\|h - f_0\| = \text{dist}(h, M)$$

↑ it exists because of the previous proposition

Then: $h - f_0 \perp M$

ii) Conversely if $f_0 \in M$ st $h - f_0 \perp M$

$$\Rightarrow \|h - f_0\| = \text{dist}(h, M)$$

proof i) $f_0 \in M$ $\quad \|h - f_0\| = d(h, M) = \inf\{\|h - f\| : f \in M\}$

Take any $f \in M \Rightarrow f_0 + f \in M$ ← M is a vector subspace

$$\cancel{\|h - f_0\|^2} \leq \|h - \underbrace{(f_0 + f)}_{\in M}\|^2 = \|(h - f_0) - f\|^2 =$$

$$= \cancel{\|h - f_0\|^2} + \|f\|^2 - 2\,\text{Re}\langle h - f_0, f\rangle$$

↑ Polar identity

$$\Rightarrow \boxed{2\,\text{Re}\langle h - f_0, f\rangle \leq \|f\|^2} \quad \forall f \in M$$

Fix $f \in M$ and (*)

look at $\underbrace{\langle h-f_0, f\rangle}_{\in \mathbb{C}} = \rho e^{i\bar\theta}$ $\quad \rho \geq 0 \quad \bar\theta \in \mathbb{R}$

and apply previous inequality to $te^{i\bar\theta}$ $\quad t \in \mathbb{R}$

$$\underbrace{2\,\mathrm{Re}\langle h-f_0, te^{i\bar\theta} f\rangle}_{\|} \leq t^2 \|f\|^2$$

$$2\,\mathrm{Re}\, t e^{-i\bar\theta} \underbrace{\langle h-f_0, f\rangle}_{\rho e^{i\bar\theta}} = 2t\rho$$

$\forall t \in \mathbb{R}$. $\quad 2t\rho \leq t^2\|f\|^2$

if $t>0$ $\quad 2\rho \leq t\|f\|^2$

limit as $t \to 0^+$

$\Rightarrow \rho \leq 0$ but we know $\rho \geq 0$

$\Rightarrow \rho = 0$

$\Rightarrow$ look at (*) $\langle h-f_0, f\rangle = 0$

$\Rightarrow$ this is true $\forall f \in M$ $\Rightarrow h - f_0 \perp M$.

(ii) $f_0 \in M$ $\quad h - f_0 \perp M \quad (\Leftrightarrow \langle h - f_0, f \rangle = 0 \ \forall f \in M)$

Let $f \in M$

$$h - f_0 \perp \underbrace{f_0 - f}_{\in M} \leftarrow \text{vector subspace}$$

$$\|h - f\|^2 = \|(h - f_0) + (f_0 - f)\|^2 =$$

$$= \|h - f_0\|^2 + \underbrace{\|f_0 - f\|^2}_{\geq 0} \geq \|h - f_0\|^2$$

Pythagorean thm

$\forall f \in M \quad \|h - f_0\| \leq \|h - f\|$

$$\Rightarrow \|h - f_0\| \leq \operatorname{dist}(h, M)$$

$$\Rightarrow \|h - f_0\| = \operatorname{dist}(h, M) \qquad \square$$

$f_0 \in M$

Orthogonal projection

M closed vector subspace

THIS IS WELL-DEFINED BECAUSE f_0 IS UNIQUELY DETERMINED

previous thm

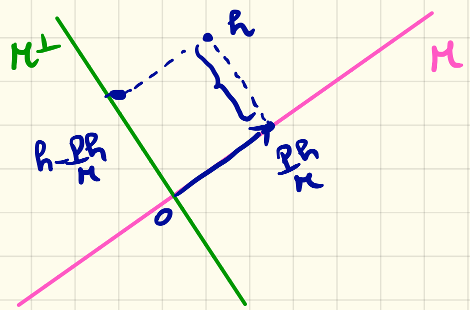
$P: H \to M$

$h \mapsto P_M(h) = f_0$

$\subset$ unique element of M realizing $\mathrm{dist}(\cdot h, M)$

orthogonal projection of h to M

From previous thm $h - P(h) \perp M$



direct sum of vector subspaces

$$H = M \oplus M^\perp$$

$$h = P_M(h) + (h - P_M(h))$$

Properties of the orthogonal projection

Thm: $\mathcal{M}$ closed linear subspace

$P : \mathcal{H} \to \mathcal{M}$ orthogonal projection

$h \longmapsto P(h) \leftarrow$ the unique element of $\mathcal{M}$ realizing $d(h, \mathcal{M})$

a) P is a linear map

b) $\|P(h)\| \leq \|h\|$ $\quad \forall h \in \mathcal{H}$

c) $P^2 = P$ $\quad (P^2 = P \circ P)$

d) $\operatorname{Ker} P = \mathcal{M}^\perp$ $\quad \operatorname{Range}(P) = \mathcal{M}$

$\quad P(h) = 0 \iff h \in \mathcal{M}^\perp$

Keep in mind

$h - Ph \perp \mathcal{M}$ $\quad (\ast\ast)$

$\|h - Ph\| = \operatorname{dist}(h, \mathcal{M})$ $\quad (\ast\ast\ast)$

proof

a) $h_1, h_2 \in \mathcal{H}$ $\quad \alpha_1, \alpha_2 \in \mathbb{F}$

Show that $P(\alpha_1 h_1 + \alpha_2 h_2) = \alpha_1 P(h_1) + \alpha_2 P(h_2)$

$f \in M$

$$< (\alpha_1 h_1 + \alpha_2 h_2) - \left(\alpha_1 P(h_1) + \alpha_2 P(h_2)\right), f >$$

linearity of $<,>$

$$= \alpha_1 < \underbrace{h_1 - P(h_1), f>}_{\overset{(**)}{=0}} + \alpha_2 < \underbrace{h_2 - P(h_2), f>}_{\overset{(**)}{=0}}$$

$$= 0$$

this means

$$< (\alpha_1 h_1 + \alpha_2 h_2) - \underbrace{\left(\alpha_1 P(h_1) + \alpha_2 P(h_2)\right)}_{= v}, f > \perp M$$

$\Rightarrow$ by our previous theorem (ii)

$$\|(\alpha_1 h_1 + \alpha_2 h_2) - v\| = \operatorname{dist}(\alpha_1 h_1 + \alpha_2 h_2, M)$$

$\Rightarrow$ by uniqueness v must be the projection:

$$P(\alpha_1 h_1 + \alpha_2 h_2) = \alpha_1 P(h_1) + \alpha_2 P(h_2) \quad \square$$

(b) take $h \in H = M \oplus M^\perp$

$$h = \underbrace{Ph}_{\in M} + \underbrace{(h - Ph)}_{\in M^\perp}$$

$$\|h\|^2 = \|Ph\|^2 + \underbrace{\|h-Ph\|^2}_{\geq 0} \geq \|Ph\|^2 \qquad \square$$

Pyth. thm

$Ph \perp (h - Ph)$

dist$(f, M) = 0 \Rightarrow P(f) = f$

c) Take $f \in M \Rightarrow P(f) = f$

Then $\forall h \in \mathcal{H}$ $\quad P(h) \in M$ by definition of projection

$$P^2(h) = P(\underbrace{P(h)}_{\in M}) = P(h) \qquad \forall h \in \mathcal{H}$$

$$\Rightarrow \boxed{P^2 = P}$$

d) $\ker P = M^\perp$

by properties of projection

- if $P(h)=0 \implies h - \underbrace{P(h)}_{=0} \in M^\perp$

$\implies h \in M^\perp$

$\ker P \subseteq M^\perp$

- if $h \in M^\perp$ then 0 is the unique vector in M st $h-0 \perp M$

$\implies P(h)=0$

The fact that range$(P)=M$ is clear from the definition of $P: H \to M$

$\square$

Rmk It was crucial in the proof that M was a closed vector subspace

↳ in infinite dim not all vector subspaces are closed

Ex. $H = L^2([0,1], \text{leb})$ ← Hilbert space

$$\langle f, g \rangle = \int_0^1 f \overline{g} \, dx$$

Take $M = C[0,1]$

↑ vector subspace

→ IT IS NOT CLOSED! IT IS DENSE IN $L^2[0,1]$

$$\overline{M} = L^2[0,1]$$

Proposition

I) M is a closed vector subspace of H

$$\Leftrightarrow (M^\perp)^\perp = M$$

II) Let $S \subseteq H$ subset of H

$$\Rightarrow (S^\perp)^\perp = \overline{\operatorname{span}(S)}$$

(exercise using I).

Im particular S is dense in H $\Leftrightarrow$ $S^\perp = \{0\}$

Remark: $A \subseteq H$ is a subset of H

$\Rightarrow A^\perp$ is a closed vector subspace of H

it was proved last lecture

$A = M^\perp$

proof I) $(\Leftarrow)$ If $M = (M^\perp)^\perp \Rightarrow M$ is a closed vector subspace

$(\Rightarrow)$ $x \in (M^\perp)^\perp$

since M is a closed vector subspace, consider P_M orthog. proj on M

$$\boxed{x = \underbrace{P(x)}_{\in M}^{=x_0} + \underbrace{(x - P(x))}_{\in M^\perp}^{=y_0}}$$

$$\Rightarrow \underbrace{\langle x, y_0\rangle}_{=0} = \underbrace{\langle x_0, y_0\rangle}_{=0} + \langle y_0, y_0\rangle = \|y_0\|^2 \Rightarrow \|y_0\| = 0$$

$x \perp M^\perp$ $\qquad$ $x_0 \perp y_0$

$$\Rightarrow y_0 = 0$$

$$\Rightarrow x = P(x) \in M$$

$\Rightarrow \quad x = P(x) \quad \Rightarrow \quad x \in M$

$(M^\perp)^\perp \subseteq M$

Observe that: $M \subseteq (M^\perp)^\perp$

$\Rightarrow \quad M = (M^\perp)^\perp$

exercise (it is essentially the definition)

II) $\overline{\mathrm{Span}(S)}$ ← the smallest closed subspace containing S

observe

$S \subseteq \overline{\mathrm{Span}(S)}$

exercise
if $A \subseteq B$
$\Rightarrow B^\perp \subseteq A^\perp$

$\Rightarrow \quad \left(\overline{\mathrm{Span}(S)}\right)^\perp \subseteq S^\perp$

Item (I)

$\Rightarrow \quad (S^\perp)^\perp \subseteq \left(\overline{\mathrm{Span}(S)}^\perp\right)^\perp = \overline{\mathrm{Span}(S)}$

$\cup\!\!\mid$

S

easy to check as above

$M = \overline{\mathrm{Span}(S)}$

closed
a vector subspace containing S
being $\overline{\mathrm{Span}(S)}$ the smallest
$\Rightarrow (S^\perp)^\perp = \overline{\mathrm{Span}(S)}$ $\square$