Lecture 4

26/4/2021

(...it continues from lecture 3)

$$\mathcal{U}: \mathcal{V} \longrightarrow \mathcal{H}$$

$$x \longmapsto [\{x\}]$$

$\hookrightarrow \{x\} \leftarrow$ constant sequence $x_n = x \ \forall n \in \mathbb{N}$

VII $\mathcal{U}$ is linear

$$\mathcal{U}(\alpha x) = [\{\alpha x\}] = \alpha[\{x\}] \overset{\text{(see III)}}{=} \alpha\,\mathcal{U}(x) \qquad \forall \alpha \in \mathbb{F}$$

$$\mathcal{U}(x+y) = [\{x+y\}] = [\{x\}] + [\{y\}] = \mathcal{U}(x) + \mathcal{U}(y)$$

$\mathcal{U}$ preserves inner product:

$$\forall x, y \in \mathcal{V} \qquad \langle \mathcal{U}(x), \mathcal{U}(y) \rangle_{\mathcal{H}} = \langle x, y \rangle_{\mathcal{V}}$$

proof. $\langle \mathcal{U}(x), \mathcal{U}(y) \rangle_{\mathcal{H}} = \langle [\{x\}], [\{y\}] \rangle_{\mathcal{H}}$

$$= \lim_{n\to\infty} \langle x, y \rangle_{\mathcal{V}} = \langle x, y \rangle_{\mathcal{V}} \qquad \circ$$

$x_n = x \quad \forall n$
$y_n = y$

VIII $\mathcal{U}$ is injective

proof:

$$\mathcal{U}(x) = \mathcal{U}(y) \iff [\{x\}] = [\{y\}]$$

$$x,y \in \mathcal{V} \qquad \iff \lim_{m\to\infty} \|x-y\|_{\mathcal{V}} = 0$$

definition of $\sim$

$x_m \equiv x$

$y_m \equiv y$

def of norm

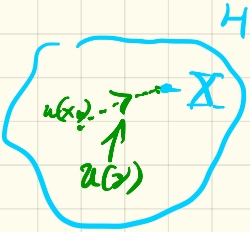
$$\iff \|x-y\|_{\mathcal{V}} = 0 \iff x = y$$

IX $\mathcal{U}(\mathcal{V}) = \{\mathcal{U}(x) : x \in \mathcal{V}\}$ is dense in $\mathcal{H}$

equivalently

$$\iff \forall X = [\{x_m\}] \in \mathcal{H} \quad \exists \{y_k\} \subset \mathcal{V} \quad \text{st}$$

$$\mathcal{U}(y_k) \to X \text{ as } k \to +\infty$$



proof $X = [\{x_m\}] \in \mathcal{H}$

$y_k = x_k$ $\qquad \mathcal{U}(y_k) \to X$ in $\mathcal{H}$

$\iff$ we want to show: $\|\mathcal{U}(y_k) - X\|_{\mathcal{H}} \xrightarrow{k\to\infty} 0$

$$\|\underset{\substack{\| \\ [\{y_k\}] \\ \| \\ [\{x_k\}]}}{\mathcal{U}(y_k)} - \underset{\substack{\| \\ [\{x_m\}]}}{X}\|_{\mathcal{H}} = \lim_{m\to\infty} \|x_k - x_m\|_2$$

↑ constant sequence

Since $\{x_m\}_m$ is a Cauchy sequence

$\forall \varepsilon > 0 \quad \exists N = N(\varepsilon) : \quad$ if $k, m \geq N(\varepsilon)$

$\Rightarrow \|x_k - x_m\|_2 \leq \varepsilon$

$\leq \varepsilon$

↳ $\forall \varepsilon > 0$ if $k \geq N(\varepsilon)$

$\Rightarrow \|\mathcal{U}(y_k) - X\|_{\mathcal{H}} \leq \varepsilon$

$\underset{\text{def of limit}}{\Longleftrightarrow} \quad \mathcal{U}(y_k) \longrightarrow X$ in $\mathcal{H}$ as $k \to \infty$

□

X If $(\mathcal{V}, \langle , \rangle_{\mathcal{V}})$ was already complete

$\Rightarrow \mathcal{U}(\mathcal{V}) = \mathcal{H}$

proof $X \in \mathcal{H}$

Goal: $\exists \bar{x} \in \mathcal{V}$ st $\mathcal{U}(\bar{x}) = X$

We know from **IX** $\mathcal{U}(\mathcal{V})$ is dense

$\Rightarrow \exists \{x_k \in \mathcal{V}\}_k$ st $X = \lim \mathcal{U}(x_k)$

$\Rightarrow \{\mathcal{U}(x_k)\}$ is a Cauchy sequence in $\mathcal{H}$

$\Rightarrow \{x_k\}$ is Cauchy in $\mathcal{V}$

$\mathcal{U}$ preserves inner product (VII)

$\Rightarrow$ Since $\mathcal{V}$ is complete $\exists \bar{x} = \lim_{k\to\infty} x_k$ in $\mathcal{V}$

$\Rightarrow U(\bar{x}) = \lim_{k\to\infty} \mathcal{U}(x_k)$

$\Rightarrow \mathcal{U}(\bar{x}) \xleftarrow[k\to\infty]{} \mathcal{U}(x_k) \xrightarrow[k\to\infty]{} X$

$\Rightarrow$ by uniqueness of limit $\mathcal{U}(\bar{x}) = X$ $\square$

ORTHOGONALITY

Def. $(\mathcal{H}, \langle , \rangle)$ Hilbert space

- $x, y \in \mathcal{H}$ are orthogonal $(x \perp y) \iff \langle x, y \rangle = 0$

Remarks: $\forall y \in \mathcal{H} \quad 0 \perp y \iff \langle y, 0 \rangle = 0)$

$x \perp y \iff y \perp x$ (Symmetric)

$x \perp y \iff \alpha x \perp \beta y \quad \forall \alpha, \beta \in \mathbb{F}$

- $A, B \subseteq \mathcal{H}$ $\quad$ A is orthogonal to B $\iff x \perp y$
$\forall x \in A$
$\forall y \in B$

- if $A \subseteq \mathcal{H}$ we define $A^\perp = \{x \in \mathcal{H} : \langle x, a \rangle = 0 \ \forall a \in A\}$

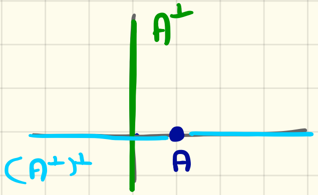
Proposition / Exercise

1) $A^\perp$ is a closed vector subspace of $\mathcal{H}$ $\longrightarrow$ see proof

2) $(A^\perp)^\perp \supseteq A$ $\quad$ ← Exercise

↓

3) $(A^\perp)^\perp = A \iff$ A is a closed vector subspace of $\mathcal{H}$



$A = \{(1,0)\}$ $\qquad$ $A^\perp = \{(0,y) : y \in \mathbb{R}\}$

$A \underset{\neq}{\subseteq} (A^\perp)^\perp = \{(x,0) : x \in \mathbb{R}\}$

proof . $A^\perp$ is a vector space

$$\text{if } x, y \in A^\perp \overset{\text{WE NEED TO PROVE}}{\Longrightarrow} \alpha x + \beta y \in A^\perp \qquad \forall\, \alpha \in \mathbb{F},\ \beta \in \mathbb{F}$$

$$\Updownarrow$$

$$\langle \alpha x + \beta y , a \rangle = 0 \qquad \forall\, a \in A$$

$$\Updownarrow$$

$$\alpha \underbrace{\langle x, a \rangle}_{\overset{\parallel}{0} \ \uparrow\ x \in A^\perp} + \beta \underbrace{\langle y, a \rangle}_{\overset{\parallel}{0} \ \uparrow\ y \in A^\perp} = 0 \qquad \forall\, a \in A \qquad \square$$

- $A^\perp$ is closed

observe that: $$A^\perp = \bigcap_{a \in A} \underbrace{\{ x \in \mathcal{H} : \langle x, a \rangle = 0 \}}$$

ALTERNATIVELY:

Prove that if $\{x_n\} \to A^\perp$

and $x_n \to \bar{x}$

$\Rightarrow \bar{x} \in A$

$$(\langle \cdot , a \rangle)^{-1}(\{0\})$$

$\uparrow$

continuous function

$\Rightarrow$ THIS SET IS CLOSED

$\Rightarrow$ the intersection of closed sets is closed

Hence $A^\perp$ is closed $\square$

Proposition (Pythagorean theorem)

If $x_1 \dots x_m \in \mathcal{H}$ pairwise orthogonal $\left(\langle x_i , x_j \rangle = 0 \text{ if } i \neq j\right)$

$$\Longrightarrow \quad \|x_1 + \dots + x_m\|^2 = \|x_1\|^2 + \dots + \|x_m\|^2$$

proof $m=2$ $\quad x_1, x_2 \in \mathcal{H} \quad \langle x_1, x_2 \rangle = 0$

$$\|x_1 + x_2\|^2 = \langle x_1 + x_2 , x_1 + x_2 \rangle =$$

$$= \|x_1\|^2 + \|x_2\|^2 + 2 \underbrace{\operatorname{Re} \langle x_1, x_2 \rangle}_{\substack{\shortparallel \\ 0}}$$

$\uparrow$
polar identity

by induction you can extend to $m \geq 3$ (← exercise)

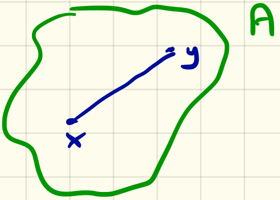
$\square$

Def Let $\mathcal{H}$ be a vector space and $A \subseteq \mathcal{H}$

We say that A is convex :

$$\forall x,y \in A \qquad \underbrace{tx+(1-t)y \in A \qquad \forall\ 0 \le t \le 1}_{\downarrow}$$

geometrically this is the segment joining x to y



Examples

- any subspace of $\mathcal{H}$ is a convex set
- if $\mathcal{H}$ is a normed space $\Rightarrow$ any ball is convex

$$B(x,r) = \{y \in \mathcal{H} : \|y-x\| < r\}$$

Exercise

Ball centred at x w/ radius $r>0$

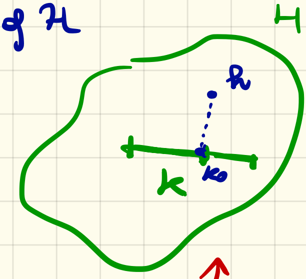
Theorem: Let $(\mathcal{H}, \langle , \rangle)$ be an Hilbert space

Let $K \subseteq \mathcal{H}$ closed and convex subset of $\mathcal{H}$
$K \neq \emptyset$

and $h \in \mathcal{H}$

$\Rightarrow$ $\exists!$ $k_0 \in K$ st

$$\|h-k_0\| = \underbrace{\inf\{\|h-k\| : k \in K\}}_{\overset{\shortparallel}{dist(h, K)}}$$



If K is convex and closed $\exists!$ a unique element minimizing the distance from h

Exercise: Show that the theorem does not hold if K is not closed or not convex

proof Observe that we can assume $h=0$

(Otherwise substitute K with the set $K-h = \{k-h : k \in K$

EXERCISE check that this is closed and convex

We need to prove that

$h=0$

$$\exists! \ k_0 \in K : \quad \|k_0\| = \inf\{\|k\| : k \in K\} =: d$$

Find the element of minimal norm in K

By def of inf $\quad \exists \{x_n\} \subseteq K \quad \|x_n\| \longrightarrow d$

Apply Parallelogram law:

$$\left\|\frac{x_n-x_m}{2}\right\|^2 = \frac{1}{2}\left(\|x_n\|^2+\|x_m\|^2\right) - \overbrace{\left\|\underset{K}{\boxed{\frac{x_n+x_m}{2}}}\right\|^2}^{\geq d^2}$$

by convexity of K $(t=\frac{1}{2})$

Since $\|x_n\| \to d \Rightarrow \forall \varepsilon>0\ \exists N=N(\varepsilon)$

st $\forall n\geq N\quad \|x_n\|^2 \leq d^2+\frac{\varepsilon^2}{4}$

$n,m\geq N$

$$\hookrightarrow \left\|\frac{x_n-x_m}{2}\right\|^2 \leq \frac{1}{2}\left(2d^2+\frac{\varepsilon^2}{2}\right) - d^2 = \frac{\varepsilon^2}{4}$$

$$\Rightarrow \|x_n-x_m\| \leq \varepsilon \quad \forall n,m\geq N$$

Hence $\{x_n\}$ is a Cauchy sequence

$$\underset{\text{completeness of } \mathcal{H}}{\Rightarrow} x_n \to \underset{K \text{ is closed}}{x_0 \in K}$$

$$\Rightarrow \|x_n - x_0\| \xrightarrow{n\to\infty} 0$$

$$d \underset{\substack{\uparrow \\ k_0 \in K}}{\le} \|k_0\| = \| k_0 - k_m + k_m \|$$

$$\underset{\substack{\uparrow \\ \text{TRIANGLE} \\ \text{INEQ}}}{\le} \underset{\substack{\downarrow \, m\to\infty \\ 0}}{\|k_0 - k_m\|} + \underset{\substack{\downarrow \, m\to\infty \\ d}}{\|k_m\|}$$

$$\Rightarrow \quad \|k_0\| = d \qquad \square$$

Let us prove uniqueness:

Let $h_0 \in K$ be another element of norm d

$\|h_0\| = d$

K convex $\Rightarrow \dfrac{h_0 + k_0}{2} \in K$

$$d \le \left\| \frac{h_0 + k_0}{2} \right\| \overset{\text{TRIANGLE}}{\le} \frac{1}{2} \|\overset{d}{h_0}\| + \frac{1}{2} \|\overset{d}{k_0}\| = d$$

$$\Rightarrow \left\| \frac{h_0 + k_0}{2} \right\| = d$$

Apply parallelogram law

$$d^2 = \left\| \frac{h_0 + k_0}{2} \right\|^2 = d^2 - \left\| \frac{h_0 - k_0}{2} \right\|^2 \Rightarrow \left\| \frac{h_0 - k_0}{2} \right\| = 0$$

$$\Rightarrow \|h_0 - k_0\| = 0 \quad \Rightarrow h_0 = k_0 \qquad \square$$