Lecture 3

22/04/2021

THEOREM (Completion of a space with inner product)
(pre-Hilbert space)

Let $(\mathcal{V}, \langle , \rangle_{\mathcal{V}})$ be an inner product space on $\mathbb{F}$ $(\mathbb{F} = \mathbb{R}$ or $\mathbb{C})$

Then $\exists$ Hilbert space $(\mathcal{H}, \langle , \rangle_{\mathcal{H}})$ and $\mathcal{U}: \mathcal{V} \to \mathcal{H}$ st

(i) $\mathcal{U}$ is injective

(ii) $\mathcal{U}$ is linear

(iii) $\langle \mathcal{U}(x), \mathcal{U}(y) \rangle_{\mathcal{H}} = \langle x, y \rangle_{\mathcal{V}} \qquad \forall x, y \in \mathcal{V}$

(iv) $\mathcal{U}(\mathcal{V}) = \{\mathcal{U}(x) : x \in \mathcal{V}\}$ is dense in $\mathcal{H}$.

If $\mathcal{V}$ is complete, then $\mathcal{U}(\mathcal{V}) = \mathcal{H}$.

Remark Keep in mind how you complete rational numbers to obtain real numbers!

$\mathbb{Q} \hookrightarrow \mathbb{R}$

IDEA Define $\mathcal{H}$ as all possible limits of $\mathcal{V}$

ATTENTION: Different sequences might converge to the same element.

proof Notation: $\|\cdot\|_{\mathcal{V}} \to$ the norm induced by $\langle \cdot, \cdot \rangle_{\mathcal{V}}$

$$\tilde{\mathcal{V}} = \left\{ \{x_n\}_{n \in \mathbb{N}} \text{ Cauchy sequence in } \mathcal{V} \right\}$$

$\hookrightarrow$ $\forall \varepsilon > 0 \ \exists N = N(\varepsilon)$ st if $n, m \geq N$

$\implies \|x_n - x_m\|_{\mathcal{V}} < \varepsilon$

Ex. (vector space over $\mathbb{F}$)

Introduce this relation

$$\{x_n\}_n \sim \{y_n\}_n \iff \lim_{n \to \infty} \|x_n - y_n\|_{\mathcal{V}} = 0$$

$\hookrightarrow$ this is an equivalence relation

Reflexivity: $\{x_n\}_n \sim \{x_n\}_n$

Transitivity: $\{x_n\} \sim \{y_n\}$ & $\{y_n\} \sim \{z_n\}$

$\implies \{x_n\} \sim \{z_n\}$

TRIANGLE INEQ.

Symmetry: $\{x_n\} \sim \{y_n\} \implies \{y_n\} \sim \{x_n\}$

$\longrightarrow$ TAKE THE QUOTIENT! $\tilde{\mathcal{V}}/\sim$

Define $\mathcal{H} = \tilde{\nu}/_{\sim}$ $\longleftarrow$ quotient space

an element of this quotient is an equivalence

class $[\{x_m\}_m] = \{ \{y_m\}_m \in \tilde{\nu}$ st $\{y_m\}_m \sim \{x_m\}_m \}$

Properties of $\mathcal{H}$:

I. Either $[\{x_m\}_m] = [\{y_m\}]$ or $[\{x_m\}] \cap [\{y_m\}] = \emptyset$

proof: suppose that $[\{x_m\}] \neq [\{y_m\}]$ but

$$\underbrace{[\{x_m\}] \cap [\{y_m\}] \neq \emptyset}$$

$$\Downarrow$$

$$\exists \{z_m\} \in [\{x_m\}] \cap [\{y_m\}]$$

$\Rightarrow \{z_m\} \sim \{x_m\}$ $\quad (\Leftrightarrow \|x_m - z_m\|_\nu \to 0)$

and

$\{z_m\} \sim \{y_m\}$ $\quad (\Leftrightarrow \|y_m - z_m\|_\nu \to 0)$

$\Rightarrow \exists \{v_m\} \in [\{x_m\}]$ $\longleftarrow$ or vice-versa

but $\{v_m\} \notin [\{y_m\}]$

TRIANGLE

$$\|v_m - y_m\|_\nu \le \underbrace{\|v_m - x_m\|_\nu}_{\to 0} + \underbrace{\|x_m - z_m\|_\nu}_{\to 0} + \underbrace{\|z_m - y_m\|_\nu}_{\to 0} \longrightarrow 0$$

$\Rightarrow \|v_m - y_m\|_\nu \longrightarrow 0 \Leftrightarrow \{v_m\} \sim \{y_m\} \Leftrightarrow \{v_m\} \in [\{y_m\}]$

CONTRADICTION.

II • If $\{x_n\}_n$, $\{y_n\} \in \tilde{\nu}$ $\Longrightarrow$ $\exists \lim\limits_{n\to\infty} \overbrace{\langle x_n, y_n\rangle_\nu}^{\in \mathbb{F}}$

proof. We want to show that the sequence $\langle x_n, y_n\rangle$ is Cauchy in $\mathbb{F}$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ $\leftarrow$ it is complete)

Fix $\varepsilon > 0$

$$|\langle x_n, y_n\rangle_\nu - \langle x_m, y_m\rangle_\nu| \leq$$

$$\leq |\langle x_n - x_m, y_n\rangle_\nu| + |\langle x_m, y_n - y_m\rangle_\nu|$$

CAUCHY SCHWARZ

$$\leq \underbrace{\|x_n - x_m\|_\nu}_{\substack{\text{CAUCHY SEQUENCE} \\ \text{(ALL SEQ. IN } \tilde{\nu} \\ \text{ARE CAUCHY)}}} \underbrace{\|y_n\|_\nu}_{\text{BOUNDED}} + \underbrace{\|x_m\|_\nu}_{\substack{\text{BOUNDED} \\ \text{(CAUCHY SEQ} \\ \text{ARE BOUNDED)}}} \underbrace{\|y_n - y_m\|_\nu}_{\text{CAUCHY}}$$

$\leq \varepsilon$ $\Longrightarrow$ $\{\langle x_n, y_n\rangle\}_n$ is Cauchy in $\mathbb{F}$

$\uparrow$

$n, m \geq N$

Hence $\exists \lim\limits_{n\to\infty} \langle x_n, y_n\rangle$. $\square$

III

• $\mathcal{H}$ is a vector space on $\mathbb{F}$

$\hookrightarrow$ i) $[\{x_m\}] + [\{y_m\}] = [\{x_m + y_m\}]$

ii) $\alpha[\{x_m\}] = [\{\alpha x_m\}]$ $\quad \forall \alpha \in \mathbb{F}$

proof (i) if $\{x'_m\} \sim \{x_m\}$, $\{y'_m\} \sim \{y_m\}$ $\Rightarrow \{x'_m + y'_m\} \sim \{x_m + y_m\}$

in fact $\|(x'_m + y'_m) - (x_m + y_m)\|_V \leq$

$$\leq \underbrace{\|x'_m - x_m\|_V}_{\downarrow\, 0} + \underbrace{\|y'_m - y_m\|_V}_{\downarrow\, 0} \xrightarrow{m\to\infty} 0 \qquad \square$$

(ii) similarly $\leftarrow$ exercise.

IV

- We define an inner product on $\mathcal{H}$ in the following way:

$$\langle [\{x_n\}], [\{y_n\}] \rangle_{\mathcal{H}} := \underbrace{\lim_{n\to\infty} \langle x_n, y_n \rangle_{\mathcal{V}}}$$

THIS LIMIT EXISTS BY II

↳ check that it is well-defined namely show that the limit does not depend on the chosen representative of the quotient class

$\{x'_n\} \sim \{x_n\}$ $\quad (\Leftrightarrow [\{x'_n\}] = [\{x_n\}])$

$\{y'_n\} \sim \{y_n\}$

$$|\langle x'_n, y'_n\rangle_{\mathcal{V}} - \langle x_n, y_n\rangle_{\mathcal{V}}| \leq$$

$$\leq |\langle x'_n - x_n, y'_n\rangle_{\mathcal{V}}| + |\langle x_n, y'_n - y_n\rangle_{\mathcal{V}}|$$

$$\leq \underbrace{\|x'_n - x_n\|_{\mathcal{V}}}_{\downarrow 0} \, \|y'_n\|_{\mathcal{V}} + \|x_n\|_{\mathcal{V}} \, \underbrace{\|y'_n - y_n\|_{\mathcal{V}}}_{\downarrow 0}$$

CAUCHY_SCHWARZ

BOUNDED (IT IS CAUCHY)

$$\xrightarrow{n\to\infty} 0 \quad \Leftarrow \quad \lim_{n\to\infty} \langle x_n, y_n\rangle_{\mathcal{V}} = \lim_{n\to\infty} \langle x'_n, y'_n\rangle_{\mathcal{V}}$$

$\square$

Let us prove that $\langle\, ,\rangle_{\mathcal{H}}$ is an inner product over $\mathcal{H}$

- $\langle \underbrace{\alpha[\{x_n\}] + \beta[\{y_n\}]}_{=\,[\{\alpha x_n + \beta y_n\}]}, [z_n] \rangle_{\mathcal{H}}$ $\leftarrow$ FROM III

$$= \lim_{n\to\infty} \langle \alpha x_n + \beta y_n, z_n \rangle_{\mathcal{V}} = \lim_{n\to\infty} \alpha \langle x_n, z_n\rangle + \beta\langle y_n, z_n\rangle$$

Definition of $\langle\, ,\rangle_{\mathcal{H}}$

$\langle\,\rangle_{\mathcal{V}}$ is an inner product

$$= \alpha \langle [\{x_n\}], \{z_n\} \rangle_{\mathcal{H}} + \beta \langle [\{y_n\}], \{z_n\}\rangle_{\mathcal{H}}$$

- $\langle [\{z_n\}], \alpha[\{x_n\}] + \beta[\{y_n\}] \rangle_{\mathcal{H}} = \bar{\alpha} \langle [\{z_n\}], [\{x_n\}]\rangle_{\mathcal{H}} + \bar{\beta} \langle [\{z_n\}], [\{y_n\}]\rangle_{\mathcal{H}}$

EXERCISE

- $\langle [\{x_n\}], [\{y_n\}] \rangle_{\mathcal{H}} = \overline{\langle [\{y_n\}], [\{x_n\}] \rangle_{\mathcal{H}}}$

- $\langle [\{x_n\}], [\{x_n\}] \rangle_{\mathcal{H}} = \lim_{n\to\infty} \underbrace{\langle x_n, x_n \rangle_{\mathcal{V}}}_{\geq 0} \geq 0$

$\longrightarrow$ Hence $\langle\, ,\rangle_{\mathcal{H}}$ is a semi-inner product.

- Non degeneracy: $\langle [\{x_n\}], \{x_n\} \rangle_{\mathcal{H}} = 0$

$\iff \lim_{n\to\infty} \langle x_n, x_n \rangle_{\mathcal{V}} = 0 \iff \|x_n\| \to 0$

$\Updownarrow$

$\|x_n - 0\| \to 0$

$\{0\}_n \sim \{x_n\}_n$

$\iff [\{0\}_n] = [\{x_n\}_n]$ null element in $\mathcal{H}$

So: if $\langle [\{x_n\}], [\{x_n\}] \rangle_{\mathcal{H}} = 0 \implies [\{x_n\}] = [0]_{\mathcal{B}}$

$\longrightarrow$ Hence $\langle , \rangle_{\mathcal{H}}$ is an inner product

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Summarizing:

$(\mathcal{H}, \langle , \rangle_{\mathcal{H}})$ is an inner product space

$\hookrightarrow$ Let us prove that this is complete.

V. $(\mathcal{H}, \langle , \rangle_{\mathcal{H}})$ is complete (Hence a Hilbert space)

proof $\{S^{(m)}\} \subseteq H$ Cauchy sequence in H

$$S^{(m)} = [\{x_j^{(m)}\}_j]$$

GOAL: Show that $S^{(m)}$ has a limit in H

- CONSTRUCTION OF A CANDIDATE TO BE THE LIMIT
- START WITH $S^{(1)} = [\{x_j^{(1)}\}]$

$\{x_j^{(1)}\}$ is a Cauchy sequence in $\mathcal{V}$

$$\Rightarrow \exists\, \ell_1 \in \mathbb{N} \quad \forall j \geq \ell_1 \quad \|x_j^{(1)} - x_{\ell_1}^{(1)}\|_{\mathcal{V}} < 1$$

- $S^{(2)} = [\{x_j^{(2)}\}]$

Define $\bar{x}_1 = x_{\ell_1}^{(1)}$

$\{x_j^{(2)}\}$ is a Cauchy sequence in $\mathcal{V}$

$$\Rightarrow \exists\, \ell_2 \in \mathbb{N} \quad \forall j \geq \ell_2 \quad \|x_j^{(2)} - x_{\ell_2}^{(2)}\|_{\mathcal{V}} < \frac{1}{2}$$

⋮

- $S^{(m)} = [\{x_j^{(m)}\}]$

Define $\bar{x}_2 = x_{\ell_2}^{(2)}$

$\{x_j^{(m)}\}$ is a Cauchy sequence in $\mathcal{V}$

$$\Rightarrow \exists\, \ell_m \in \mathbb{N} \quad \forall j \geq \ell_m \quad \|x_j - x_{\ell_m}^{(m)}\|_{\mathcal{V}} < \frac{1}{m} \qquad (*)$$

⋮

Define $\bar{x}_m = x_{\ell_m}^{(m)}$

ITERATE

$\rightarrow$ We have defined a sequence $\{\bar{x}_n\} \in \tilde{\nu}$

Let us prove that $\{\bar{x}_n\}$ is Cauchy in $\tilde{\nu}$

$\rightarrow$ Prove that $\{\bar{x}_n\}$ is a Cauchy sequence: Fix $\varepsilon > 0$

$$\|\bar{x}_n - \bar{x}_m\|_\nu = \|x^{(n)}_{\ell_n} - x^{(m)}_{\ell_m}\|_\nu$$

$$\leq \|x^{(n)}_{\ell_n} - x^{(n)}_j\|_\nu + \|x^{(n)}_j - x^{(m)}_j\|_\nu + \|x^{(m)}_j - x^{(m)}_{\ell_m}\|_\nu$$

$$= \underbrace{\|x^{(n)}_{\ell_n} - x^{(n)}_j\|_\nu}_{\leq \frac{1}{n} \text{ if } j \geq \ell_n} + \underbrace{\left(\|x^{(n)}_j - x^{(m)}_j\|_\nu - \|S^{(n)} - S^{(m)}\|_H\right)}_{(\#)}$$

$$+ \underbrace{\|S^{(n)} - S^{(m)}\|_H}_{\text{Being Cauchy } \leq \frac{\varepsilon}{4} \text{ if } n,m \geq \tilde{N} = \tilde{N}(\varepsilon)} + \underbrace{\|x^{(m)}_j - x^{(m)}_{\ell_m}\|_\nu}_{\leq \frac{1}{m} \text{ if } j \geq \ell_m}$$

(see (*). on the previous page)

(#): $\|S^{(n)} - S^{(m)}\|_H = \lim_{j \to \infty} \|x^{(n)}_j - x^{(m)}_j\|_\nu$

BY DEFINITION

Hence $\exists N_{n,m}$ st if $j \geq N_{nm}$ $(\#) \leq \frac{\varepsilon}{4}$

Summarizing

$$\|\bar{x}_n - \bar{x}_m\| \le \underbrace{\frac{1}{n}}_{\le \frac{\varepsilon}{4}} + \underbrace{\frac{1}{m}}_{\le \frac{\varepsilon}{4}} + \frac{\varepsilon}{2} \le \varepsilon$$

$$\left(\begin{array}{l} n, m \ge \frac{4}{\varepsilon} \\ \text{and } \; n, m \ge \tilde{N} \end{array} \right)$$

$\Rightarrow \; \{\bar{x}_n\}_n$ is Cauchy in ν $\;\Rightarrow\;$ $[\{\bar{x}_n\}] \in \tilde{\nu}$ $\square$

$\longrightarrow$ Remains to prove that $S^{(n)} \longrightarrow [\{\bar{x}_n\}] = \bar{S}$

In fact:

$$\|S^{(n)} - \bar{S}\|_{\mathcal{H}} = \lim_{j\to\infty} \|x_j^{(n)} - \bar{x}_j\|_\nu, \qquad \bar{x}_j = x_{\ell_j}^{(j)}$$

$$\|x_j^{(n)} - x_{\ell_j}^{(j)}\|_\nu \le \underbrace{\|\overset{=\bar{x}_j}{x_{\ell_j}^{(j)}} - \overset{=\bar{x}_n}{x_{\ell_n}^{(n)}}\|_\nu}_{\substack{\text{Being Cauchy} \\ \exists M \text{ if } j,n \ge M \\ \le \frac{\varepsilon}{2}}} + \underbrace{\|x_{\ell_n}^{(n)} - x_j^{(n)}\|_\nu}_{\substack{\le \frac{1}{n} \text{ if } j \ge \ell_n \\ \text{By definition of } \bar{x}_j}}$$

Hence

$$\lim_{j\to\infty} \|x_j^{(m)} - x_{\ell_0}^{(j)}\|_{\vartheta} \le \frac{\varepsilon}{2} + \frac{1}{m}$$

$$\Rightarrow \|S^{(m)} - \bar{S}\|_{H} \le \frac{\varepsilon}{2} + \frac{1}{m}$$

Hence $\limsup_{m\to\infty} \|S^{(m)} - \bar{S}\|_{H} \le \frac{\varepsilon}{2}$

Since ε is arbitrary $\Rightarrow \limsup_{m\to\infty} \|S^{(m)} - \bar{S}\|_{H} \le 0$ (≥ 0)

$$\Rightarrow \exists \lim_{m\to\infty} \|S^{(m)} - \bar{S}\|_{H} = 0 \quad \square$$

VI What is $\mathcal{U} : \vartheta \longrightarrow \mathcal{H}$?

(How to define the inclusion map)

$x \in \vartheta \qquad \mathcal{U}(x) = [\{x\}]$

constant sequence $x_m = x \quad \forall m$

START TO PROVE PROPERTIES IN THE STATEMENT OF THE THEOREM

(... CONTINUE NEXT TIME)