Lecture 2

19/4/2021

H vector space over $\mathbb{F}$ $\quad (\mathbb{F} = \mathbb{R}$ or $\mathbb{C})$

$\langle \, , \, \rangle : H \times H \longrightarrow \mathbb{F}$ inner product

$\hookrightarrow$ • $|\langle x, y \rangle| \leq \langle x, x \rangle \langle y, y \rangle \quad \forall x, y \in H$

(CAUCHY - SCHWARZ INEQ)

• Define a norm $\|x\| := \sqrt{\langle x, x \rangle} \geq 0$

PROPERTIES DEFINING A NORM:

- $\|x+y\| \leq \|x\| + \|y\|$ (TRIANGLE) $\forall x, y \in H$
- $\|\alpha x\| = |\alpha| \, \|x\|$ $\quad \forall \alpha \in \mathbb{F}$, $\forall x \in H$
- $\|x\| = 0 \iff x = 0$

$(H, \langle , \rangle) \rightsquigarrow$ A space with a norm

$\longrightarrow$ DISTANCE: $d(x, y) = \|x - y\|$

$\longrightarrow$ TOPOLOGY $\longrightarrow$ TOPOLOGY INDUCED BY THIS DISTANCE

(when we refer to topological properties of a Hilbert space, we mean with respect to this topology)

PROPOSITION $\|\cdot\|: H \longrightarrow [0, +\infty)$

$\langle , \rangle : H \times H \longrightarrow \mathbb{F}$

ARE CONTINUOUS FUNCTIONS

(w.r.t THE TOPOLOGY INDUCED BY THE DISTANCE ASSOCIATED TO $\|\ \|$)

PROOF: Let's prove that $\langle , \rangle$ is continuous

(THE PROOF THAT $\|\cdot\|$ IS CONTINUOUS IS ANALOGOUS)

HYPOTHESES

if $x_m \to x$ in H
$y_m \to y$ in H

$\Updownarrow$

$$\begin{pmatrix} \|x_m - x\| \to 0 & m \to \infty \\ \|y_m - y\| \to 0 & m \to \infty \end{pmatrix}$$

$\Longrightarrow$

THESIS

$\langle x_m, y_m \rangle \longrightarrow \langle x, y \rangle$ in $\mathbb{F}$

$\Updownarrow$

$|\langle x_m, y_m \rangle - \langle x, y \rangle| \xrightarrow{m \to \infty} 0$

$$|\langle x_m, y_m \rangle - \langle x, y \rangle| = |\langle x_m - x, y_m \rangle + \langle x, y - y_m \rangle|$$

linearity

TRIANGLE INEQ OF $|\cdot|$

$$\leq |\langle x_m - x, y_m \rangle| + |\langle x, y - y_m \rangle|$$

$$\leq \underbrace{\|x_m - x\|}_{\to 0} \|y_m\| + \|x\| \underbrace{\|y - y_m\|}_{\to 0} \xrightarrow{m \to \infty} 0 \qquad \square$$

CAUCHY-SCHWARZ

is BOUNDED

(EVERY CONVERGENT SEQ. IS BOUNDED $\leftarrow$ exercise)

PROPERTIES OF $\|\cdot\|$:

$(H, \langle , \rangle)$ inner product space

- **POLAR IDENTITY**

$$\|x+y\|^2 \underset{\text{DEF.}}{=} \langle x+y, x+y \rangle \overset{\text{linearity}}{=} \underbrace{\langle x,x\rangle}_{\|x\|^2} + \langle x,y\rangle + \underbrace{\langle y,x\rangle}_{\overline{\langle x,y\rangle}} + \underbrace{\langle y,y\rangle}_{\|y\|^2}$$

$\Vert \leftarrow$ DEF   PROP. OF $\langle , \rangle$ DEF $\rightarrow \Vert$

$$= \|x\|^2 + \|y\|^2 + 2\,\mathrm{Re}(\langle x,y\rangle) \qquad \forall x,y \in H$$

$\left(z \in \mathbb{C} \quad \frac{z+\bar{z}}{2} = \mathrm{Re}(z) \right)$

- **PARALLELOGRAM IDENTITY**

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right) \qquad \forall x,y \in H$$

Proof:

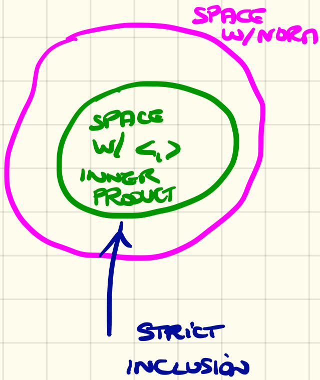
$$\|x+y\|^2 + \|x-y\|^2 \overset{\text{DEF}}{=} \langle x+y, x+y\rangle + \langle x-y, x-y\rangle$$

$$\underset{\text{linearity}}{=} \|x\|^2 + \|y\|^2 + \cancel{\langle x,y\rangle} + \cancel{\langle y,x\rangle} + \|x\|^2 + \|y\|^2 - \cancel{\langle x,y\rangle} - \cancel{\langle y,x\rangle}$$

$$= 2\|x\|^2 + 2\|y\|^2 \qquad \square$$

Remark: NOT ALL NORMS SATISFY THE PARALLELOGRAM IDENTITY!

$\Rightarrow$ NOT ALL NORMS ARISE FROM AN INNER PRODUCT!



Example:

WITH Lebesgue measure

$$H = L^\infty(\mathbb{R})$$

$\|\cdot\|_\infty \qquad f : \mathbb{R} \to \mathbb{R}$ measurable

$$\|f\|_\infty = \operatorname{ess\,sup} |f|$$

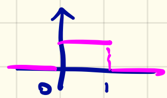
essential sup

$$L^\infty(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{R} \text{ measurable st } \|f\|_\infty < \infty \}$$

$\|\cdot\|_\infty$ is a norm on $L^\infty(\mathbb{R})$

$\hookrightarrow$ DOESN'T SATISFY THE PARALLELOGRAM IDENTITY.

$$f(x) = \chi_{[0,1]} = \begin{cases} 1 & x \in [0,1] \\ 0 & \text{OTHERWISE} \end{cases} \qquad \|f\|_\infty = 1$$



CHARACTERISTIC FCT

$$g(x) := \chi_{[2,3]} = \begin{cases} 1 & x \in [2,3] \\ 0 & \text{OTHERWISE} \end{cases}$$

check:

$\|f+g\|_\infty = 1 \qquad \|f\|_\infty = 1 \qquad$ PARALLELOGRAM IDENTITY

$\|f-g\|_\infty = 1 \qquad \|g\|_\infty = 1 \qquad 1+1 \neq 2(1+1)$

PROPOSITION (Parallelogram Identity is sufficient and necessary condition for a norm to arise from inner product)

Let $(X, \|\cdot\|)$ BE A NORMED VECTOR SPACE OVER $\mathbb{F}$

$\|\cdot\|$ SATISFIES THE PARALLELOGRAM IDENTITY $\iff$

$\|\cdot\|$ ARISES FROM AN INNER PRODUCT (i.e., $\exists \langle , \rangle : X \times X \to \mathbb{F}$ st $\|x\| = \sqrt{\langle x, x\rangle}$ $\forall x \in X$)

proof ($\Leftarrow$) WE ALREADY PROVED IT

($\Rightarrow$) SEE EXERCISE SHEET N° 1

Def (HILBERT SPACE) $(H, \langle , \rangle)$ BE A VECTOR SPACE OVER $\mathbb{F}$ w/ INNER PRODUCT $\langle , \rangle$.

H IS SAID TO BE A HILBERT SPACE IF IT IS COMPLETE w.r.t THE NORM INDUCED BY $\langle , \rangle$.

(EVERY CAUCHY SEQUENCE CONVERGES TO AN ELEMENT OF H)

Recall (Cauchy sequence)

$\{x_n\} \subset H$ is Cauchy iff $\forall \varepsilon > 0$ $\exists N = N(\varepsilon)$ st $\forall n, m \geq N$ $\|x_n - x_m\| < \varepsilon$

Examples:

① $\mathbb{F}^d$ with the standard inner product $\longrightarrow$ it's an Hilbert space

with Lebesgue measure

② $L^2([0,1])$ with inner product $\langle f,g\rangle := \int_0^1 f\bar{g}\,dx$

$\quad \| \quad \{f:[0,1]\to \overset{\mathbb{C}}{\mathbb{R}}$ measurable st $\int_0^1 |f|^2 < \infty\}$

$(L^2([0,1]), \langle\cdot,\cdot\rangle)$ complete $\longrightarrow$ Hilbert space

$$\|f\|_{L^2} = \sqrt{\int_0^1 |f|^2\,dx}$$

③ $C[0,1]$ with inner product $\langle f,g\rangle = \int_0^1 f\bar{g}\,dx$

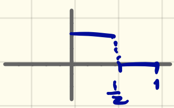
$\quad \| \quad \{f:[0,1]\to \overset{\mathbb{C}}{\mathbb{R}}$ continuous$\}$

$C[0,1] \subseteq L^2[0,1]$ it's a vector subspace (exercise)

$\longrightarrow (C[0,1], \langle,\rangle)$ is a space with an inner product

↳ This space is not complete! $\qquad f\in L^2[0,1] \quad f\notin C[0,1]$

$C[0,1]$ is dense in $L^2[0,1]$! $\qquad$ Take $f(x) = \chi_{[0,\frac12]} = \begin{cases} 1 & x\in[0,\frac12] \\ 0 & \text{otherwise} \end{cases}$



Since $C[a,1]$ is dense in $L^2[a,1]$

$\Rightarrow \exists \{f_m\} \subseteq C[a,1]$ st $\|f_m - f\|_{L^2} \xrightarrow{m\to\infty} 0$

$\Rightarrow \{f_m\}$ is CAUCHY SEQUENCE IN $L^2[a,1]$ and hence in $C[a,1]$

$\Rightarrow f_m \longrightarrow f$ in $L^2[0,1]$ But f_m does not converge in $C[a,1]$

EXERCISE: complete this proof

(if $f_m \to g$ in $C[0,1] \Rightarrow g = f$ a.e. $\ast$)

PROBLEM: $C[a,1]$ IS NOT CLOSED IN $L^2[a,1]$!

PROPOSITION

$(H, \langle , \rangle)$ BE AN HILBERT SPACE

AND $V \subseteq H$ A CLOSED SUBSPACE OF H

$\Rightarrow (V, \langle , \rangle|_{V\times V})$ IS AN HILBERT SPACE

proof WE NEED TO PROVE THAT $(V, \langle , \rangle|_{V\times V})$ IS COMPLETE

$\Leftrightarrow$ every Cauchy sequence converges to an element of V

$\{v_m\}_m \subseteq V$ Cauchy sequence

$\left(\Leftrightarrow \forall \varepsilon > 0 \ \exists N = N(\varepsilon) : \forall n, m \geq N \quad \|v_n - v_m\| < \varepsilon \right)$

$\{v_n\} \subseteq V \subseteq H \implies \{v_n\}$ is a Cauchy sequence in H

$\implies \exists\, h \in H$ st $v_n \to h$ in H

H is complete

$\implies h \in V$ $\square$

V is a closed
(every convergent sequence in V must converge to an element of V)

Remark: (X, Ω, μ) measure space $L^p(X, \Omega, \mu)$ $p \geq 1$ $p = \infty$

$L^2(X, \Omega, \mu)$ is an Hilbert space $\langle f, g \rangle = \int_X f \bar{g} \, d\mu$

Does any $\| \cdot \|_{L^p}$ arise from an inner product?

ONLY FOR $p = 2$!

↳ L^2 IS THE ONLY HILBERT SPACE AMONG L^p- SPACES

see Example sheet 1

Theorem (Completion of an inner product space)

Let $(V, \langle , \rangle_V)$ BE A VECTOR SPACE WITH AN INNER PRODUCT

$\Rightarrow \exists (H, \langle , \rangle_H)$ HILBERT SPACE AND A MAP $\mathcal{U}: V \to H$

such that:

"$(H, \langle , \rangle_H)$ is called Completion of $(V, \langle , \rangle_V)$"

① $\mathcal{U}$ is injective

② $\mathcal{U}$ is linear

$\mathcal{U}$ preserves the inner product

③ $\langle \mathcal{U}(x), \mathcal{U}(y) \rangle_{\mathcal{H}} = \langle x, y \rangle_V \quad \forall x, y \in V$

④ $\mathcal{U}(V) = \{\mathcal{U}(x) : x \in V\}$ is dense in $\mathcal{H}$.

Moreover, if $(V, \langle , \rangle_V)$ WAS ALREADY COMPLETE $\Rightarrow \mathcal{U}(V) = H$.

IDEA TO KEEP IN MIND

(HOW TO COMPLETE THE RATIONAL NUMBERS TO GET THE REAL NUMBERS.)

(.... NEXT LECTURE)