Exam

You are allowed to use any theorem from the course, provided that you give its full statement and check all the assumptions of the theorem.

1. **[20 points]** Consider a dynamical system $\Phi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ generated by a system of ODE's (also called Lorenz system):

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = xy - bz$$

for $\sigma, b, r > 0$. Here $p = (0, 0, 0)$ is a fixed point.

(a) Show that if $r < 1$ then $p = (0, 0, 0)$ is the unique fixed point.

(b) Find linearization at fixed point. Prove that if $r < 1$ then $p = (0, 0, 0)$ is asymptotically stable.

(c) Find the derivative along the system of the function $V(x, y, z) = \frac{1}{\sigma}x^2 + y^2 + z^2$.

(d) Using result of (c) prove that for any point (x, y, z) we have that $\Phi(t, (x, y, z)) \to 0$ as $t \to \infty$.
Note that in item (b) you proved this statement only for $(x, y, z) \in B_\delta(0)$ for some $\delta > 0$.

2. **[20 points]** Let maps $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) = x^2 + x, \qquad g(x) = x^2 + ax + b.$$

(a) For $a = 2$, $b = 1$ find a homeomorphism $h : \mathbb{R} \to \mathbb{R}$ such that $f \circ h = h \circ g$.

(b) Prove that for $a = 3$, $b = 4$ there is no homeomorphism $h : \mathbb{R} \to \mathbb{R}$ satisfying $f \circ h = h \circ g$.

3. **[20 points]** For $a \in \mathbb{R}$ consider map $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined as:

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a(x + x^5) + y \\ 0.5(y + y^3) \end{pmatrix}$$

(a) Prove that for $|a| < 1$ the point $x = y = 0$ is asymptotically stable.

(b) Find all a for which the Grobman-Hartman theorem is applicable for f in the neighbourhood of the fixed point $x = y = 0$?

(c) Prove if the point $x = y = 0$ is Lyapunov stable or not for $a = 1$.

Qualitative Theory of Differential Equations

Final Exam

11 January 2020

Answer all questions from Part A and Part B.

Part A

1. [**40 points**] The following system of differential equations models a predator-prey interaction:

$$\frac{dx}{dt} = r\left(1 - \frac{x}{k}\right)x - axy \tag{0.1}$$

$$\frac{dy}{dt} = bxy - dy \tag{0.2}$$

where r, k, a, b and d are all positive constants.

(a) Show that the variables x, y and t can be scaled to give a system of equations of the following form:

$$\frac{du}{ds} = (1-u)\,u - uv \tag{0.3}$$

$$\frac{dv}{ds} = Buv - Dv \tag{0.4}$$

where B and D are both positive. Express B and D as functions of r, k, a, b and d.

(b) Sketch the nullclines of the system (0.3, 0.4) for the two cases (a) $B < D$ and (b) $B > D$.

(c) Determine the types of each of the equilibrium points of the system (0.3, 0.4) in each of the cases (a) $B < D$ and (b) $B > D$.

(d) Sketch a phase portrait of $u \geq 0$, $v \geq 0$ for the system (0.3, 0.4) in each of the cases (a) $B < D$ and (b) $B > D$.

(e) What do the above results say about the behaviour of the prey and predator populations in this model? What is the difference between the two cases (a) $B < D$ and (b) $B > D$?

(f) Suppose that B and D are both much smaller than 1 and $B > D$. Sketch graphs of the prey and predator populations as functions of time if they are both very small when $t = 0$.

Part B

You are allowed to use any theorem from the course, provided that you give its full statement and check all the assumptions of the theorem.

2. [**20 points**] Consider a dynamical system $\Phi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ generated by a system of ODE's (also called Lorenz system):

$$\begin{aligned}\dot{x} &= \sigma(y - x)\\ \dot{y} &= rx - y - xz\\ \dot{z} &= xy - bz\end{aligned}$$

for $\sigma, b, r > 0$. Here $p = (0, 0, 0)$ is a fixed point.

(a) Show that if $r < 1$ then $p = (0, 0, 0)$ is the unique fixed point.

(b) Find linearization at fixed point. Prove that if $r < 1$ then $p = (0, 0, 0)$ is asymptotically stable.

(c) Find the derivative along the system of the function $V(x, y, z) = \frac{1}{\sigma}x^2 + y^2 + z^2$.

(d) Using result of (c) prove that for any point (x, y, z) we have that $\Phi(t, (x, y, z)) \to 0$ as $t \to \infty$.
Note that in item (b) you proved this statement only for $(x, y, z) \in B_\delta(0)$ for some $\delta > 0$.

3. [**20 points**] Let maps $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) = x^2 + x, \qquad g(x) = x^2 + ax + b.$$

(a) For $a = 2$, $b = 1$ find a homeomorphism $h : \mathbb{R} \to \mathbb{R}$ such that $f \circ h = h \circ g$.

(b) Prove that for $a = 3$, $b = 4$ there is no homeomorphism $h : \mathbb{R} \to \mathbb{R}$ satisfying $f \circ h = h \circ g$.

4. [**20 points**] For $a \in \mathbb{R}$ consider map $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined as:

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a(x + x^5) + y \\ 0.5(y + y^3) \end{pmatrix}$$

(a) Prove that for $|a| < 1$ the point $x = y = 0$ is asymptotically stable.

(b) Find all a for which the Grobman-Hartman theorem is applicable for f in the neighbourhood of the fixed point $x = y = 0$?

(c) Prove if the point $x = y = 0$ is Lyapunov stable or not for $a = 1$.