International Mathematics Master – Academic Year 2020/21

Functional Analysis - Part II

Lecturer: Prof. Alfonso Sorrentino

Problem sheet 3

Exercise 1. Consider $\mathcal{H}$ a real Hilbert space. Let L be a bounded linear functional on $\mathcal{H}$ and K a non-empty closed convex subset of $\mathcal{H}$. Prove that there exists in K a unique minimizer of $J(x) := \frac{\|x\|^2}{2} + L(x)$; namely, there is a unique $x_0 \in K$ such that

$$J(x_0) = \inf_{x \in K} J(x).$$

Exercise 2. Let $\mathcal{H}$ be a Hilbert space and let M, N be closed subspaces of $\mathcal{H}$.
(**i**) Prove that

$$(M+N)^\perp = M^\perp \cap N^\perp \qquad \text{and} \qquad (M \cap N)^\perp = \overline{M^\perp + N^\perp}.$$

(**ii**) Prove that the following statements are equivalent (P_M and P_N denote, respecitvely, the orthogonal projections onto M and N):

a) $M \perp N$;

b) $P_M \circ P_N = P_N \circ P_M$, *i.e.* they commute;

c) $P := P_M + P_N$ coincides with the projection onto $M+N$.

(**iii**) Is it true that $M+N$ is necessarily a closed vector subspace?

Exercise 1. (**i**) (Gram-Schmidt Orthogonalization process) If $\mathcal{H}$ is a Hilbert space and $\{h_n : n \in \mathbb{N}\}$ is a subset of linearly independent elements of $\mathcal{H}$ (*i.e.*, every finite subset consists of linearly independent elements), then there is an orthonormal set $\{e_n : n \in \mathbb{N}\}$ such that for every n the linear span of $\{e_1, \ldots, e_n\}$ equals the linear span of $\{h_1, \ldots, h_n\}$.

Note: Recall that a set $\{e_n : n \in \mathbb{N}\}$ is said to be orthonormal if

$$\langle e_n, e_m \rangle = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m. \end{cases}$$

(**ii**) Consider the (complex) Hilbert space $L^2([-1,1])$ and apply the Gram-Schmidt orthogonalization process to the three polynomials $\{1, x, x^2\}$. Use this to find the distance of the polynomial x^3 from the vector subspace generated by $\{1, x, x^2\}$, namely:

$$\min_{a,b,c \in \mathbb{C}} \int_{-1}^{1} |x^3 - a - bx - cx^2|^2 dx.$$

(**iii**) The Gram-Schmidt process applied to the sequence of monomials $\{1, x, x^2, x^3, \ldots\}$ yields to the so-called *Legendre polynomials* $(P_n)_{n \geq 0}$. Show that with the normalization $P_n(1) = 1$ for all $n \geq 0$, then we obtain $P_0(x) = 1$, $P_1(x) = x$ and the recursion formula

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x) \qquad n \geq 1.$$