International Mathematics Master – Academic Year 2020/21

Functional Analysis - Part II

Lecturer: Prof. Alfonso Sorrentino

Problem sheet 2

Exercise 1. Let $(\mathcal{H}, \langle \cdot, \cdot, \rangle_{\mathcal{H}})$ be a Hilbert space, $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ a Banach space and $T : \mathcal{H} \longrightarrow \mathcal{B}$ an isometric linear isomorphism. Prove that $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is also a Hilbert space.

(*Recall that T being an isometry means that $\|T(h)\|_{\mathcal{B}} = \|h\|_{\mathcal{H}}$ for every $h \in \mathcal{H}$, where $\|\cdot\|_{\mathcal{H}}$ denotes the norm induced by $\langle \cdot, \cdot, \rangle_{\mathcal{H}}$.*)

Exercise 2. Let $\lambda = \{\lambda_n\}_{n\in\mathbb{N}}$ be a real sequence with $0 < \lambda_n < 1$ for all $n \in \mathbb{N}$. On the space of square-summable complex sequences (see also Problem sheet 1, exercise 1)

$$\ell^2(\mathbb{C}) := \Big\{ \{z_n\}_{n\in\mathbb{N}} \subset \mathbb{C} \quad \text{such that} \quad \sum_{n=0}^{+\infty} |z_n|^2 < \infty \Big\}$$

define the inner product

$$\big\langle \{z_n\}_{n\in\mathbb{N}}, \{w_n\}_{n\in\mathbb{N}} \big\rangle_\lambda := \sum_{n=0}^{\infty} \lambda_n z_n \overline{w_n}.$$

Is it true that $(\ell^2(\mathbb{C}), \langle \cdot, \cdot \rangle_\lambda)$ is a Hilbert space ?

Exercise 3. Let $(\mathcal{X}, \langle \cdot, \cdot, \rangle)$ be an inner product space over $\mathbb{F}$ (where $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$). Given $x, y \in \mathcal{X}$, prove that the following statements are equivalent:

(a) $x \perp y$ (*i.e.*, they are orthogonal, namely $\langle x, y \rangle = 0$).

(b) For all $\lambda \in \mathbb{F}$, $\|x + \lambda y\| = \|x - \lambda y\|$.

(c) For all $\lambda \in \mathbb{F}$, $\|x + \lambda y\| \geq \|x\|$.

Exercise 4. Let $(\mathcal{H}, \langle \cdot, \cdot, \rangle)$ be a Hilbert space and M a closed vector subspace of H. Prove that the quotient space H/M is also a Hilbert space and that it is isometrically isomorphic to $M^\perp = \{h \in \mathcal{H} : \ \langle h, m \rangle = 0 \ \forall \, m \in M\}$.
What happens if M is not closed?

(*Recall that H/M is the set of equivalence classes with respect to the equivalence relation: $x \sim y$ if and ony if $x - y \in M$.*)