International Mathematics Master – Academic Year 2020/21

Functional Analysis - Part II

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Problem sheet 1

Exercise 1. Let us consider the set of square-summable complex sequences

$$\ell^2(\mathbb{C}) := \Big\{\{z_n\}_{n\in\mathbb{N}} \subset \mathbb{C} \quad \text{such that} \quad \sum_{n=0}^{+\infty} |z_n|^2 < \infty\Big\}.$$

a) Prove that $\ell^2(\mathbb{C})$ is a vector space over $\mathbb{C}$ and that

$$\begin{aligned} \langle\cdot,\cdot\rangle_{\ell^2} : \ell^2(\mathbb{C}) \times \ell^2(\mathbb{C}) &\longrightarrow \mathbb{C} \\ \big(\{z_n\}_{n\in\mathbb{N}}, \{w_n\}_{n\in\mathbb{N}}\big) &\longmapsto \sum_{n=0}^{+\infty} z_n \overline{w_n} \end{aligned}$$

defines an inner product on $\ell^2(\mathbb{C})$.

b) Discuss whether $(\ell^2(\mathbb{C}), \langle\cdot,\cdot\rangle_{\ell^2})$ is a Hilbert space.

c) Consider

$$\mathcal{S} := \{\{z_n\}_{n\in\mathbb{N}} \subset \mathbb{C} \quad \text{such that only finitely many } z_n\text{'s are non-zero}\}.$$

Prove that $\mathcal{S}$ is a vector subspace of $\ell^2(\mathbb{C})$. Is it a Hilbert space with respect to $\langle\cdot,\cdot\rangle_{\ell^2(\mathbb{C})|\mathcal{S}}$ (restricted to $\mathcal{S}$)? If it is not, determine a completion.

Exercise 2. Let $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ be a real Hilbert space. Show that there exists a complex Hilbert space $(\mathcal{K}, \langle\cdot,\cdot\rangle_{\mathcal{K}})$ and a map $U : \mathcal{H} \longrightarrow \mathcal{K}$ such that

a) U is linear;

b) $\langle U(h_1), U(h_2)\rangle_{\mathcal{K}} = \langle h_1, h_2\rangle_{\mathcal{H}}$ for every $h_1, h_2 \in \mathcal{H}$;

c) for every $k \in \mathcal{K}$, there exist unique $h_1, h_2 \in \mathcal{H}$ such that $k = U(h_1) + iU(h_2)$.

Remark: $(\mathcal{K}, \langle\cdot,\cdot\rangle_{\mathcal{K}})$ is called a *complexification* of $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$.

Exercise 3.

(i) Let $\mathcal{H}$ be a real vector space and $\|\cdot\|$ be a norm on it. Prove that if $\|\cdot\|$ satisfies the parallelogram identity

$$\|x+y\|^2 + \|x-y\|^2 = 2\big(\|x\|^2 + \|y\|^2\big) \qquad \forall\, x, y \in \mathcal{H},$$

then $\|\cdot\|$ arises from an inner product, namely there exists an inner product $\langle\cdot,\cdot\rangle : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{R}$ such that $\|x\| = \sqrt{\langle x, x\rangle}$ for every $x \in \mathcal{H}$.
(*See some hints on next page*)

(ii) Let us consider $\mathbb{R}$ with the Lebesgue measure. Prove that the norm $\|\cdot\|_{L^p}$, with $p \geq 1$ or $p = \infty$, satisfies the parallelogram identity if and only if $p = 2$ (in other words, $L^2(\mathbb{R})$ is the only Hilbert space among the spaces $L^p(\mathbb{R})$).

(iii) (*Facultative*) Prove the statement in item **(i)** in the case of a complex vector space. (*See some hints below*)

Some Hints:

- **Exercise 3 (i).** Define $\langle x, y\rangle := \frac{1}{4}\big(\|x+y\|^2 - \|x-y\|^2\big)$ and prove that it is an inner product and that it generates $\|\cdot\|$. One could proceed as follows:
  - **Step 1**: Prove that $2\langle \frac{x+y}{2}, z\rangle = \langle x, z\rangle + \langle y, z\rangle$ for every $x, y, z \in \mathcal{H}$.
  - **Step 2**: Deduce that $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$ for every $x, y, z \in \mathcal{H}$.
  - **Step 3**: Prove that $\langle \frac{m}{n}\, x, y\rangle = \frac{m}{n}\langle x, y\rangle$ for every $x, y \in \mathcal{H}$ and $\frac{m}{n} \in \mathbb{Q}$.
  - **Step 4**: Prove that $\langle \alpha\, x, y\rangle = \alpha\langle x, y\rangle$ for every $x, y \in \mathcal{H}$ and $\alpha \in \mathbb{R}$.
- **Exercise 3 (iii).** Define $\langle x, y\rangle := \frac{1}{4}\Big[\big(\|x+y\|^2 - \|x-y\|^2\big) + i\big(\|x+iy\|^2 - \|x-iy\|^2\big)\Big]$.