☆ Hahn-Banach Theorem

Definition: Let $\mathcal{E}$ be a real linear space and p an $\mathbb{R}$-valued function on $\mathcal{E}$. Then p is "sublinear" functional on $\mathcal{E}$ if

$$p(f+g) \leq p(f) + p(g)$$

$\forall f, g \in \mathcal{E}$ and $p(\lambda f) = \lambda p(f)$ $\forall \lambda > 0$.

Theorem 1.25: Let $\mathcal{E}$ be $\mathbb{R}$-linear space and p be a sublinear functional on $\mathcal{E}$. Let F be a subspace of $\mathcal{E}$ and φ be an $\mathbb{R}$-linear functional on F s.t

$$\varphi(f) \leq p(f) \quad \forall f \in F.$$

Then $\exists$ an $\mathbb{R}$-linear functional Φ on $\mathcal{E}$ s.t $\Phi(f) = \varphi(f)$ for $f \in F$ and

$$\Phi(g) \leq p(g) \quad \forall g \in \mathcal{E}.$$

Proof: Assume $F \neq \{0\}$. Take $f \notin F$ and let
$$G = \{g + \lambda f : \lambda \in \mathbb{R}, g \in F\}$$

→ We first extend φ to G.

→ We want $\Phi(g+\lambda f) \leq p(g+\lambda f)$ — (A)
$\forall \lambda \in \mathbb{R}, g \in F$

→ Dividing (A) by $|\lambda|$ we can write

$$\Phi(f-h) \leq p(f-h) \quad \text{and}$$

$$\Phi(-f+h) \leq p(-f+h) \quad \forall h \in F$$

Equivalently we get

$$-p(-f+h) + \varphi(h) \leq \Phi(f-h) + \varphi(h) = \Phi(f)$$

$$p(f-h) + \varphi(h) \geq \Phi(f-h) + \varphi(h) = \Phi(f)$$

or

$$-p(-f+h) + \varphi(h) \leq \Phi(f) \leq p(f-h) + \varphi(h)$$

for all $h \in F$.

→ This is equivalent to

$$\sup_{h\in F}\{-p(-f+h)+\varphi(h)\} \le \Phi(f) \le \inf_{k\in F}\{p(f-k)+\varphi(k)\} \quad \text{— (B)}$$

→ So a value for $\Phi(f)$ for $f \notin F$ can be chooen so that (A) is satisfied iff (B) holds.

→ However for $h, k \in F$, we have

$$\varphi(h)-\varphi(k) = \varphi(h-k) \le p(h-k)$$

$$\le p(f-k)+p(h-f) \quad \text{so that}$$

$$-p(h-f)+\varphi(h) \le p(f-k)+\varphi(k)$$

and (B) is satisfied. Therefore φ can be extended to Φ on G s.t

$$\Phi(h) \le p(h) \quad \forall h \in G.$$

→ Our goal is to obtain a "maximal" extension of φ.

→ To do this we use Zorn's lemma.

Let $\mathcal{P}$ denote the class of all extensions of φ to larger subspace satisfying the required inequality.

→ So elements of $\mathcal{P}$ are pairs (G, Φ_G) where $F \subseteq G \subseteq \mathcal{E}$ "subspace" and Φ_G a linear functional on G that extends φ and satisfies

$$\Phi_G(g) \leq p(g) \quad \forall g \in G.$$

→ Partial order on $\mathcal{P}$

$$(G_1, \Phi_{G_1}) \leq (G_2, \Phi_{G_2}) \Leftrightarrow$$

$$G_1 \subseteq G_2 \quad \text{and} \quad \Phi_{G_2}(f) = \Phi_{G_1} \quad \forall f \in G_1$$

→ To apply Zorn's lemma to $\mathcal{P}$, we must show that for every "chain" $\{(G_\alpha, \Phi_{G_\alpha})\}_{\alpha \in A}$ in $\mathcal{P}$ $\exists$ a "maximal" element in $\mathcal{P}$.

If $\{(G_\alpha, \Phi_{G_\alpha})\}_{\alpha \in A}$ is a chain, let

$G = \bigcup_{\alpha \in A} G_\alpha$ and define

$$\Phi(f) := \Phi_{G_\alpha}(f) \text{ if } f \in G_\alpha$$

→ Clearly G is a subspace of $\mathcal{E}$ containing F and Φ is well-defined, linear and $\Phi(f) \leq p(f) \ \forall f \in G$.

→ So $(G_\alpha, \Phi_{G_\alpha}) \leq (G, \Phi)$ in $\mathcal{P}$ $\forall \alpha \in A$.

∴ Every chain has a maximal element.

→ Zorn's Lemma implies that $\mathcal{P}$ itself has a maximal element (G_0, Φ_0).

→ G_0 must be $\mathcal{E}$ since otherwise we can find an element in $\mathcal{E}$ greater than (G_0, Φ_{G_0}) "contradiction".

$\therefore$ $G_0 = \mathcal{E}$ and $\Phi_{G_0} = \Phi$ is the desired extension of φ to $\mathcal{E}$ ▨