☆ Axiom of Choice (AC)

A "choice function" is a function f defined on a collection X of nonempty sets such that

$$A \in X \Rightarrow f(A) \in A .$$

Axiom : For any collection X of nonempty sets, $\exists$ a choice function f on X.

Equivalent form : If $(A_i)_{i \in I}$ is a family of non-empty sets, then their "Cartesian product" $\prod_{i \in I} A_i \neq \emptyset$.

(Cohen 1963) AC is "independant" of set theory (Zermelo-Fraenkel theory)

(Solovay 1970) A "non-measurable" set $A \subset \mathbb{R}$ for Lebesgue measure cannot be "proven" to exist without the AC.

☆ Tychonoff's Theorem: Let $(X_\alpha)_{\alpha \in I}$ be a family of compact topological spaces. Then $\prod_{\alpha \in I} X_\alpha$ is compact in the "product topology"

→ Depends on AC.

☆ Zorn's Lemma: Suppose a "partially ordered" set P has the property that every "chain" in P has an "upper bound" in P. Then P contains at least one "maximal element".

→ "Chain" means a totally ordered set

→ $p \in P$ is an "upper bound" for a set $Q \subset P$ if $q \leq p \;\; \forall q \in Q$

→ An element $m \in P$ is "maximal" if

$m \leq x$ for some $x \in P \implies x = m$

→ ZL can be used to prove that:

every ring $R \neq \{0\}$ with unity contains a maximal ideal.

☆ Banach-Alaoglu Theorem

The unit ball $B_1^* = \{\phi \in \mathcal{H}^* : \|\phi\| \leq 1\}$ of the dual of a Banach space $\mathcal{H}$ is compact in the ω^*-topology.

Proof: For each $f \in B_1 = \{x \in \mathcal{H} : \|x\| \leq 1\}$ let $\mathbb{D}_f$ be a copy of the closed unit disk $\{z \in \mathbb{C} : |z| \leq 1\}$ in $\mathbb{C}$.

→ Let $P = \prod_{f \in B_1} \mathbb{D}_f$ "product space"

→ P is compact by Tychonoff's theorem.

→ Define $\Lambda : B_1^* \longrightarrow P$ by

$$\Lambda(\phi) = \phi|_{B_1} \quad \text{"restricted to } B_1\text{"}$$

→ Λ is injective: $\Lambda(\phi_1) = \Lambda(\phi_2)$

$\Rightarrow \; \phi_1|_{B_1} = \phi_2|_{B_1} \;\Rightarrow\; \phi_1 = \phi_2$

→ A net $(\phi_\alpha)_{\alpha\in A}$ in $\mathcal{H}^*$ converges in the w^*-topology to a ϕ in $\mathcal{H}^*$

$$\Leftrightarrow \lim_{\alpha\in A} \phi_\alpha(f) = \phi(f) \quad \forall f \in \mathcal{H}$$

$$\Leftrightarrow \lim_{\alpha\in A} \Lambda(\phi_\alpha)(f) = \Lambda(\phi)(f) \quad \forall f \in B_1$$

$$\Leftrightarrow \lim_{\alpha\in A} \Lambda(\phi_\alpha) = \Lambda(\phi) \quad \text{in topology of } P.$$

→ Λ is a homeomorphism between B_1^* and the subset $\Lambda(B_1^*)$ of P.

→ It remains to prove that $\Lambda(B_1^*)$ is closed in P.

→ Suppose $(\Lambda(\phi_\alpha))_{\alpha\in A}$ is a net in $\Lambda(B_1^*)$ that converges in the product topology to ψ in P.

→ If $f, g, f+g \in B_1$, then

$$\Psi(f+g) = \lim_{\alpha \in A} \Delta(\phi_\alpha)(f+g)$$

$$= \lim_{\alpha \in A} \Delta(\phi_\alpha)(f) + \lim_{\alpha \in A} \Delta(\phi_\alpha)(g)$$

$$= \Psi(f) + \Psi(g)$$

→ If $f, \lambda f \in B_1$, then

$$\Psi(\lambda f) = \lim_{\alpha \in A} \Delta(\phi_\alpha)(\lambda f) = \lim_{\alpha \in A} \phi_\alpha(\lambda f)$$

$$= \lambda \lim_{\alpha \in A} \phi_\alpha(f) = \lambda \Psi(f)$$

→ So Ψ determines an element $\tilde{\Psi} \in \mathcal{H}^*$ by $\underline{\tilde{\Psi}(f) = \|f\| \, \Psi\left(\frac{f}{\|f\|}\right)}$

→ $\tilde{\Psi}$ is clearly linear on $\mathcal{H}$ and

$$|\tilde{\Psi}(f)| = \|f\| \left| \Psi\left(\frac{f}{\|f\|}\right) \right| \leq \|f\|$$

$\Rightarrow \|\tilde{\Psi}\| \leq 1 \quad \therefore \tilde{\Psi} \in B_1^*$

→ Since $\tilde{\Psi}(f) = \Psi(f) \quad \forall f \in B_1$

$\Rightarrow \tilde{\Psi}|_{B_1} = \Psi \quad \Rightarrow \Delta(\tilde{\Psi}) = \Psi$

$\therefore \quad \psi \in \Delta(B_1^*)$

$\rightarrow$ So $\Delta(B_1^*)$ is closed and hence compact in P.

$\therefore$ B_1^* is compact in the w^*-topology of H^* ▨