★ Recall that if E is any finite dimensional normed space, then

$K \subset E$ is compact $\Leftrightarrow$ K is closed and bounded.

→ In particular, the closed unit ball

$$B_1 = \{x \in E : \|x\| \leq 1\}$$

is compact if $\dim(E) < \infty$.

★ Riesz Lemma: Let M be a proper closed subspace of a normed space E and let $0 < \theta < 1$. Then $\exists\, y \in E$ tal que $\|y\| = 1$ e

$$\|y - x\| \geq \theta \quad \forall\, x \in M.$$

Proof: Let $y_0 \in E \setminus M$ and

$$d = \operatorname{dist}(y_0, M) := \inf_{x \in M} \|y_0 - x\|$$

→ In general $d > 0$ because M is closed.

→ Let $x_0 \in M$ such that

$$\|y_0 - x_0\| \leq \frac{d}{\theta}$$

and choose $y = \dfrac{y_0 - x_0}{\|y_0 - x_0\|}$

→ So $\|y\| = 1$ and for any $x \in M$

$$\|y - x\| = \left\| \frac{y_0 - (x_0 + \|y_0 - x_0\| x)}{\|y_0 - x_0\|} \right\|$$

$$\geq \frac{d}{\|y_0 - x_0\|} \geq \theta \quad \text{because } x_0 + \|y_0 - x_0\| x \in M$$

So $y \in E$ with $\|y\| = 1$ and

$$\operatorname{dist}(y, M) \geq \theta \quad \blacksquare$$

☆ Theorem: Let E be a normed space.

$$\dim(E) = \infty \Leftrightarrow B_1 \text{ is not compact}$$

Proof: ($\Leftarrow$) This direction is clear.

($\Rightarrow$) Suppose $\dim(E) = \infty$.

→ Choose $x_1 \in E$ with $\|x_1\| = 1$.

→ Since $\dim(E) = \infty$, the subspace $[x_1] := \mathbb{C} x_1$ is closed and proper

→ By Riesz Lemma $\exists\ x_2 \in E \setminus [x_1]$ with $\|x_2\| = 1$ s.t

$$\|x_2 - x_1\| \geq \theta = \frac{1}{2}$$

→ Again $[x_1, x_2] = \mathbb{C} x_1 + \mathbb{C} x_2 \neq E$ and $\exists\ x_3 \in E \setminus [x_1, x_2]$ with $\|x_3\| = 1$

s.t $$\|x_3 - x_j\| \geq \theta = \frac{1}{2}, \quad j = 1, 2$$

→ Continuing this process, we have a sequence $(x_n)_{n\in\mathbb{N}}$ of unit vectors in E s.t

$$\|x_m - x_n\| \geq \frac{1}{2} \quad \text{if } m \neq n.$$

So $(x_n)_{n\in\mathbb{N}}$ is a sequence in B_1 with no convergent subsequence

$\therefore$ B_1 is not compact ▨

☆ Weak Topologies

Let X be a set and Y a topological space. Let $\mathcal{F}$ be a family of maps $f : X \to Y$.

→ The "weak topology" on X "induced" by $\mathcal{F}$ is the weakest or smallest topology $\mathcal{T}$ on X s.t each $f \in \mathcal{F}$ is continuous.

→ So $\mathcal{T}$ is "generated" by the sets

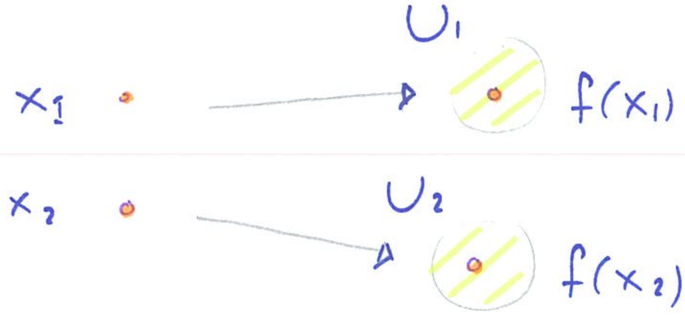
$$\{ f^{-1}(U) : f \in \mathcal{F}, U \text{ open in } Y \}$$

→ Convergence of nets in this weak topology :

$$\lim_{\alpha \in A} x_\alpha = x \iff \lim_{\alpha \in A} f(x_\alpha) = f(x)$$

$\forall f \in \mathcal{F}$. "Topology of pointwise convergence"

→ If Y is Hausdorff and $\mathcal{F}$ "separates points" of X, then the weak topology is Hausdorff.



$X \longrightarrow Y$

$\exists f \in \mathcal{F}$ s.t $f(x_1) \neq f(x_2)$

$\Rightarrow f^{-1}(U_1) \cap f^{-1}(U_2) = \emptyset$

☆ Weak-star ω^* topology:

For each $f \in \mathcal{H}$ in a Banach space, let $\hat{f} : \mathcal{H}^* \longrightarrow \mathbb{C}$ be the function

$$\hat{f}(\omega) = \omega(f) \quad \forall \omega \in \mathcal{H}^*.$$

The w^*-topology on $\mathcal{H}^*$ is the weak topology on $\mathcal{H}^*$ induced by the family $\mathcal{F} = \{\hat{f} : f \in \mathcal{H}\}$.

→ w^*-topology on $\mathcal{H}^*$ is Hausdorff:

If $\varphi_1 \neq \varphi_2$ in $\mathcal{H}^*$ $\Rightarrow \exists f \in \mathcal{H}$ s.t

$$\varphi_1(f_1) \neq \varphi_2(f)$$

$\Rightarrow \hat{f}(\varphi_1) \neq \hat{f}(\varphi_2)$

$\Rightarrow \mathcal{F}$ separates points of $\mathcal{H}^*$. ▨

→ In general w^*-topology is not metrizable or first-countable.

☆ A net $(\varphi_\alpha)_{\alpha \in A} \to \varphi$ in $\mathcal{H}^*$ with respect to the w^*-topology iff

$$\lim_{\alpha \in A} \hat{f}(\varphi_\alpha) = \hat{f}(\varphi) \quad \text{or}$$

$$\lim_{\alpha \in A} \varphi_\alpha(f) = \varphi(f) \quad \forall f \in \mathcal{H}.$$

☆ Proposition 1.21: Suppose M is a dense subset of $\mathcal{H}$ and $(\varphi_\alpha)_{\alpha \in A}$ is a uniformly bounded net in $\mathcal{H}^*$ s.t

$$\lim_{\alpha \in A} \varphi_\alpha(f) = \varphi(f) \quad \forall f \in M.$$

$\Rightarrow$ $\varphi_\alpha \to \varphi$ in the w^*-topology.

Proof: Given $g \in \mathcal{H}$ and $\varepsilon > 0$, choose $f \in M$ s.t $\|f - g\| < \frac{\varepsilon}{3C}$ where

$$C = \sup\{\|\varphi\|, \|\varphi_\alpha\| : \alpha \in A\}.$$

$\rightarrow$ If $\alpha_0 \in A$ is chosen so that

$$\alpha \geq \alpha_0 \Rightarrow |\varphi_\alpha(f) - \varphi(f)| < \varepsilon/3$$

then for $\alpha \geq \alpha_0$ we have

$$|\varphi_\alpha(g) - \varphi(g)| \leq |\varphi_\alpha(g) - \varphi_\alpha(f)|$$

$$+ |\varphi_\alpha(f) - \varphi(f)| + |\varphi(f) - \varphi(g)|$$

$$\leq \|\varphi_\alpha\| \|f - g\| + \frac{\varepsilon}{3} + \|\varphi\| \|f - g\| < \varepsilon \quad \blacksquare$$