☆ $\ell^1(\mathbb{Z}^+) \subsetneq c_0(\mathbb{Z}^+) \subsetneq \ell^\infty(\mathbb{Z}^+)$

→ $c_0^* \cong \ell^1$ "already proved"

→ $\alpha : \ell^1 \xrightarrow[\cong]{} c_0^*$ defined by

$\alpha(\varphi) = \hat{\varphi}$ where $\hat{\varphi}(f) = \sum_{n=0}^{\infty} \varphi(n) f(n)$

→ Similarly $\ell^{1*} \cong \ell^\infty$ via the isometric isomorphism $\gamma : \ell^\infty \xrightarrow[\cong]{} \ell^{1*}$

$\gamma(f) = \hat{f}$ where $\hat{f}(\varphi) = \sum_{n=0}^{\infty} f(n) \varphi(n)$

☆ $c_0^{**} \cong (\ell^1)^* \cong \ell^\infty$

Question: Is c_0 "reflexive"?

→ Is $J : c_0 \to c_0^{**} = \ell^\infty$ defined by

$J(f) = \hat{f}$ where $\hat{f} \in (\ell^1)^*$

and $\hat{f}(\varphi) = \varphi(f) = \sum_{n=0}^{\infty} f(n) \varphi(n)$

an isometric isomorphism?

Answer: No. Because the map

$J: c_0 \to (\ell^1)^*$ is the "restriction" of

$\gamma: \ell^\infty \underset{\cong}{\longrightarrow} (\ell^1)^*$ to c_0.

$\Rightarrow$ c_0 is "not" reflexive ▨

$\rightarrow$ In principle it is still possible that

$$c_0 \cong \ell^\infty \ ?$$

Recall: A metric space X is "separable" if $\exists$ a countable set $D \subset X$ that is "dense" in X $(\bar{D} = X)$.

☆ c_0 is a separable Banach space.

Proof: Define

$$c_{00} = \{ f: \mathbb{Z}^+ \to \mathbb{C} \mid f(n) \neq 0 \text{ for only finitely many } n \in \mathbb{Z}^+ \}$$

$\rightarrow$ $c_{00} = \bigcup_{n=1}^{\infty} \mathbb{C}^n$ "topologically same"

→ Let $D = \{ f \in c_{00} \mid \text{each } f(n) \in \mathbb{Q}^2 \}$

"sequences with rational terms"

→ $D = \bigcup_{n=1}^{\infty} (\mathbb{C} \cap \mathbb{Q}^2)^n$ is a countable union of countable sets

$\Rightarrow$ D is countable

→ Lets prove D is dense in c_0

→ Let $f \in c_0$ and given an $\varepsilon > 0$.

→ Since $f(n) \to 0$ as $n \to \infty$, $\exists N \in \mathbb{N}$ s.t

$$|f(n)| < \frac{\varepsilon}{2} \quad \forall\, n > N$$

→ Let $g : \mathbb{Z}^+ \to \mathbb{C}$ be defined as

$$g = (f(0), f(1), \dots, f(N), 0, 0, \dots)$$

and $h = (\xi_0, \xi_1, \xi_2, \dots, \xi_N, 0, 0, \dots)$

with $\xi_0, \dots, \xi_N \in \mathbb{Q}^2$ so that

$$\|g - h\|_\infty = \sup_j |f(j) - \xi_j| < \frac{\varepsilon}{2}$$

Clearly $g \in c_{00}$ and $h \in D$. So

$$\|f-h\|_\infty \le \|f-g\|_\infty + \|g-h\|_\infty$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\therefore$ c_0 is separable ▨

☆ ℓ^∞ is "not" separable.

Proof: Let $D = (f_n)_{n=1}^{\infty}$ be any countable set in ℓ^∞.

→ We must show that ∃ some $h \in \ell^\infty$ s.t. $h \notin \bar{D}$. Define $h \in \ell^\infty$ as follows

$$h(j) = f_j(j) + 1 \quad \text{if } |f_j(j)| \le 1$$

and $$h(j) = 0 \quad \text{if } |f_j(j)| > 1$$

→ Since $\sup_{j \in \mathbb{N}} |h(j)| \le 2 \Rightarrow \|h\|_\infty \le 2$

$\Rightarrow h \in \ell^\infty$.

→ But $\|h - f_j\|_\infty \ge |h(j) - f_j(j)| \ge 1$

for all $j \in \mathbb{N}$. $\therefore h \notin \bar{D}$

$\Rightarrow$ No countable set D is dense in ℓ^∞

$\Rightarrow$ ℓ^∞ is not separable ▨

$\rightarrow$ $\therefore c_0 \not\cong \ell^\infty$

☆ In general, for $1 \le p < \infty$

$$\ell^p = \ell^p(\mathbb{Z}^+) = \{ f : \mathbb{Z}^+ \to \mathbb{C} \mid \sum_{n=0}^{\infty} |f(n)|^p < \infty \}$$

is a Banach space with norm

$$\|f\|_p = \left(\sum_{n=0}^{\infty} |f(n)|^p \right)^{1/p}.$$

$\rightarrow$ $\ell^1 \subsetneq \ell^2 \subsetneq \cdots \subsetneq \ell^p \subsetneq \cdots \subsetneq c_0 \subsetneq \ell^\infty$

$\rightarrow$ So $(\ell^p)^* \cong \ell^q$ where $1 < q \le \infty$

with $\frac{1}{p} + \frac{1}{q} = 1$ "p, q are conjugate exponents"

$\rightarrow$ $(\ell^2)^* = \ell^2$

☆ Banach space of measures

Let X be a compact Hausdorff space, and $\mathcal{S}$ a σ-algebra of subsets of X.

→ A countable collection $(E_i)_{i=1}^{\infty} \subset \mathcal{S}$ is a "partition" of $E \in \mathcal{S}$ if

$$E_i \cap E_j \neq \emptyset \text{ for } i \neq j \text{ and } E = \bigcup_{i=1}^{\infty} E_i$$

→ A "complex measure" μ on $\mathcal{S}$ is a function $\mu : \mathcal{S} \to \mathbb{C}$ such that

$$\mu(E) = \sum_{i=1}^{\infty} \mu(E_i) \quad ; \; E \in \mathcal{S}$$

for every partition $(E_i)_i$ of E.

→ Let $\mathcal{M}_{\mathcal{S}}(X) = \mathcal{M}(X)$ denote the vector space of complex measures on the measurable space $(X, \mathcal{S})$:

$$(\nu + \mu)(E) = \nu(E) + \mu(E) \quad , \quad (\alpha\mu)(E) = \alpha\mu(E)$$

☆ $\mathcal{M}(X)$ equipped with the "total variation" norm

$$\|\mu\| := |\mu|(X) = \sup \sum_{i=1}^{\infty} |\mu(X_i)|$$

the sup taken over all partitions $(X_i)_{i=1}^{\infty}$ of X.

→ $\mathcal{M}(X)$ is a Banach space

→ Let λ be finite positive measure on $(X, \mathcal{S})$ and define

$$L^p(\lambda) = \{ f : X \to \mathbb{C} \mid \int_X |f|^p d\lambda < \infty \}$$

for $1 \leq p < \infty$. Then $L^p(\lambda)$ is a Banach space with norm

$$\|f\|_p = \left(\int_X |f|^p d\lambda \right)^{1/p}$$

→ $L^\infty(\lambda)$ "essentially bounded" is also a Banach space.

☆ $\mathcal{C}(X) \subsetneq L^\infty \subsetneq \cdots \subsetneq L^p \subsetneq \cdots \subsetneq L^1 \subsetneq \mathcal{M}(X)$

→ For any $f \in L^1(\lambda)$, we can define the complex measure $f\,d\lambda$ by

$$(f\,d\lambda)(E) = \int_E f\,d\lambda$$

→ $f \longmapsto f\,d\lambda$ is the embedding

$$L^1 \hookrightarrow \mathcal{M}(X)$$

☆ **Riesz Representation Theorem :**

for $L^p(\lambda)$: $(L^p)^* \cong L^q$

where $1 < q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$

$$g \in L^q \overset{\cong}{\longmapsto} \Phi_g(f) = \int_X f g\,d\lambda$$

☆ **Riesz - Markov Theorem :**

$\mathcal{C}^*(X) \cong \mathcal{M}(X)$ where

$$\mu \in \mathcal{M}(X) \overset{\cong}{\longmapsto} \Phi_\mu(f) = \int_X f\,d\mu$$