☆ Isomorphic Banach Spaces

Two Banach spaces X and Y are "isometrically isomorphic" if $\exists$ a a linear bijection $T: X \to Y$ such that

Ⓐ T and T^{-1} are continuous. "isomorphism"

Ⓑ $\|Tf\| = \|f\| \ \forall f \in X$ "isometric"

→ So X and Y are "structurally" the same.

☆ Some Examples of Dual spaces

① If $\mathcal{H}$ is a Hilbert space, then

$\mathcal{H}^* \cong \mathcal{H}$ "isometrically isomorphic" to it's own dual.

→ A Banach space X is "reflexive" if $X \cong X^{**}$ "second dual" via the iso. isom. $J: X \to X^{**}$ defined by

$J : f \longrightarrow \hat{f}$ "evaluation map"

where $\hat{f}(\omega) = \omega(f) \quad \forall \; \omega \in X^*$

→ In general J is "only injective"

and an "isometry", but not "surjective"

$X \hookrightarrow X^{**}$ " X is embedded in X^{**} "

→ So X is essentially a subspace of X^{**}

→ $X \cong X^{**}$ if J is also surjective.

→ So Hilbert spaces are always reflexive.

(2) Let $\ell^\infty(\mathbb{Z}^+) = \{ f : \mathbb{Z}^+ \to \mathbb{C} \mid f \text{ is bounded} \}$

→ $\ell^\infty(\mathbb{Z}^+)$ is the collection of all bounded complex sequences on $\mathbb{Z}^+$

$$f = (a_0, a_1, a_2, \dots) \quad ; \; a_k \in \mathbb{C}$$

Define addition and scalar mult. pointwise and set

$$\|f\|_\infty := \sup\{|f(n)| : n \in \mathbb{Z}^+\}$$

→ $\ell^\infty(\mathbb{Z}_+)$ is a Banach space with norm $\|\cdot\|_\infty$.

→ Consider also the space

$$c_0(\mathbb{Z}^+) = \{f : \mathbb{Z}^+ \to \mathbb{C} \mid \lim_{n\to\infty} f(n) = 0\}$$

"vanish at infinity"

→ $c_0(\mathbb{Z}^+)$ is a closed subspace of $\ell^\infty(\mathbb{Z}^+)$

⇒ $c_0(\mathbb{Z}^+)$ is a Banach space

→ Finally let

$$\ell^1(\mathbb{Z}^+) = \{f : \mathbb{Z}^+ \to \mathbb{C} \mid \sum_{n=0}^{\infty} |f(n)| < \infty\}$$

→ With norm $\|f\|_1 = \sum_{n=0}^{\infty} |f(n)|$

$\ell^1(\mathbb{Z}^+)$ becomes a Banach space

$\longrightarrow$ $\ell^1(\mathbb{Z}^+) \subsetneq c_0(\mathbb{Z}^+) \subsetneq \ell^\infty(\mathbb{Z}^+)$

as subsets (not as Banach space)

☆ $c_0(\mathbb{Z}^+)^* \cong \ell^1(\mathbb{Z}^+)$

Proof: For $\varphi \in \ell^1$ we define the "functional" $\hat{\varphi} : c_0(\mathbb{Z}^+) \longrightarrow \mathbb{C}$ by

$$\hat{\varphi}(f) = \sum_{n=0}^{\infty} \varphi(n) f(n) \quad \forall f \in c_0$$

$\rightarrow$ $\hat{\varphi}$ is well-defined:

$$|\hat{\varphi}(f)| = \left|\sum_{n=0}^{\infty} \varphi(n) f(n)\right| \le \sum_{n=0}^{\infty} |\varphi(n)|\,|f(n)|$$

$$\le \|f\|_\infty \sum_{n=0}^{\infty} |\varphi(n)| = \|f\|_\infty \|\varphi\|_1 \quad \text{— (A)}$$

$\rightarrow$ $\hat{\varphi}$ is obviously linear.

$\rightarrow$ $\hat{\varphi}$ is bounded: (A) gave us

$$|\hat{\varphi}(f)| \leq \|\varphi\|_1 \|f\|_\infty \qquad (M = \|\varphi\|_1)$$

$\therefore$ $\hat{\varphi} \in c_0(\mathbb{Z}^+)^*$ and $\|\hat{\varphi}\| \leq \|\varphi\|_1$

→ So we can define the map

$\alpha : \ell_1(\mathbb{Z}^+) \to c_0(\mathbb{C}^+)^*$ by $\alpha(\varphi) = \hat{\varphi}$

→ $\|\alpha(\varphi)\| \leq \|\varphi\|_1$ implies α is well-defined and "contractive"

→ We want to show α is isometric and onto $c_0(\mathbb{Z}^+)^*$.

→ Let $L \in c_0(\mathbb{Z}^+)^*$ and define $\varphi_L : \mathbb{Z}^+ \to \mathbb{C}$

by $$\varphi_L(n) = L(e_n) \quad \forall\, n \in \mathbb{Z}^+$$

where $e_n \in c_0$ s.t $e_n(m) = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$

→ We want to show

$$\hat{\varphi}_L = L \quad \text{and} \quad \|\varphi_L\|_1 \leq \|L\|.$$

For each $N \in \mathbb{Z}^+$, consider

$$f_N = \sum_{n=0}^{N} \frac{\overline{L(e_n)}}{|L(e_n)|} \cdot e_n \quad \text{in} \quad c_0$$

→ So $\|f_N\|_\infty \leq 1$ and

$$\|L\| \geq |L(f_N)| = \left| \sum_{n=0}^{N} \frac{\overline{L(e_n)}}{|L(e_n)|} \cdot L(e_n) \right|$$

$$= \sum_{n=0}^{N} |L(e_n)| = \sum_{n=0}^{\infty} |\varphi_L(n)|$$

$\Rightarrow \|\varphi_L\|_1 \leq \|L\| < \infty \Rightarrow \varphi_L \in \ell_1(\mathbb{Z}^+)$

→ Thus $\beta : c_0^* \to \ell_1$ defined by

$\beta(L) = \varphi_L$ is well-defined

and contractive

→ Note that for any $g \in c_0$

$$\sum_{n=0}^{N} g(n) e_n \to g \quad \text{in } c_0 \quad \text{as } N \to \infty$$

→ Hence for any $L \in c_0^*$

$$L(g) = \lim_{N\to\infty} \sum_{n=0}^{N} g(n)\, L(e_n)$$

$$= \lim_{N\to\infty} \sum_{n=0}^{N} g(n)\, \varphi_L(n)$$

$$= \sum_{n=0}^{\infty} g(n)\, \varphi_L(n) = \hat{\varphi}_L(g) \quad \forall g \in c_0$$

$\therefore$ $L = \hat{\varphi}_L$ and $\alpha \circ \beta = Id$ on c_0^*

$$c_0^* \xrightarrow{\beta} \ell_1 \xrightarrow{\alpha} c_0^*$$

$$L \longrightarrow \varphi_L \longrightarrow \hat{\varphi}_L = L$$

$\rightarrow$ Note $\hat{\varphi} = 0 \Rightarrow \varphi = 0$

$\Rightarrow$ α is injective (and surjective)

so $\alpha^{-1} = \beta$

$\rightarrow$ Since $\|\beta(L)\|_1 \leq \|L\|$ and

$\|\alpha(\varphi)\| \leq \|\varphi\|_1$ but $\alpha \circ \beta = Id$

$\Rightarrow$ α is an isometric isomorphism $\blacksquare$