☆ Linear Functionals

Let $\mathcal{H}$ be a Banach space. A function $\varphi : \mathcal{H} \to \mathbb{C}$ is a "bounded linear functional" if :

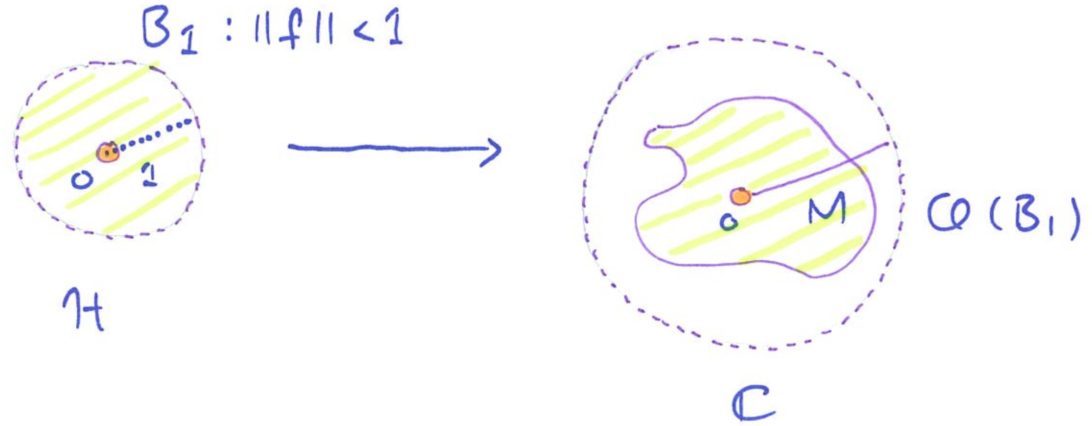
① $\varphi(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 \varphi(f_1) + \lambda_2 \varphi(f_2)$

$\forall\ f_1, f_2 \in \mathcal{H}\ ,\ \lambda_1, \lambda_2 \in \mathbb{C}$ "linearity"

② $\exists\ M$ s.t $|\varphi(f)| \leq M \|f\|\ \ \forall\ f \in \mathcal{H}$

"boundedness"

→ ② implies $|\varphi(f)| \leq M$ if $\|f\| \leq 1$



The "image" of the open unit ball B_1 in $\mathcal{H}$ under φ is "bounded" in $\mathbb{C}$.

Proposition 1.12: Let φ be a linear functional on $\mathcal{H}$. The following are equivalent:

① φ is bounded

② φ is continuous

③ φ is continuous at 0.

Proof: ① implies ②. If $(f_\alpha)_{\alpha \in A}$ is a net in $\mathcal{H}$ converging to f, then

$$\lim_{\alpha \in A} |\varphi(f_\alpha) - \varphi(f)| = \lim_{\alpha \in A} |\varphi(f_\alpha - f)|$$

$$\leq \lim_{\alpha \in A} M \| f_\alpha - f \| = 0$$

$\therefore$ The net $(\varphi(f_\alpha))_{\alpha \in A}$ converges to $\varphi(f)$.

$\Rightarrow$ φ is continuous.

② implies ③. Trivial.

③ implies ①. If φ is continuous at 0, then $\exists\ \delta > 0$ such that $\|f\| < \delta \Rightarrow |\varphi(f)| < 1$

Hence for any $g \neq 0$ in $\mathcal{H}$, we have

$$|\omega(g)| = \frac{2\|g\|}{\delta} \left| \omega\left(\frac{\delta}{2\|g\|} \cdot g \right) \right| < \frac{2}{\delta} \|g\|$$

$\Rightarrow$ is bounded ▨

✩ Dual Space of $\mathcal{H}$

Let $\mathcal{H}^*$ be the space of all bounded linear functionals on the Banach space $\mathcal{H}$. For $\omega \in \mathcal{H}^*$, let

$$\|\omega\| := \sup \left\{ \frac{|\omega(f)|}{\|f\|} : f \neq 0 \right\} \quad \text{—— (A)}$$

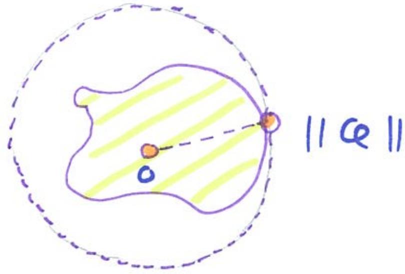
$$= \sup \{ |\omega(f)| : \|f\| = 1 \}$$

$$= \sup \{ |\omega(f)| : \|f\| \leq 1 \}$$

Observation: For all $f \in \mathcal{H}$, we have

$$|\omega(f)| = \|f\| \frac{|\omega(f)|}{\|f\|} \leq \|\omega\| \|f\|$$

$\longrightarrow$ $\|\omega\|$ is the "smallest possible" upper bound M such that $|\varphi(f)| \le M\|f\|$ $\forall f \in \mathcal{H}$



"Ball of radius $\|\varphi\|$"

→ So $\|\varphi\|$ is the radius of the "smallest" open ball in which $\varphi(B_1)$ "fits".

→ $\mathcal{H}^*$ is called the "conjugate" or "dual" space of $\mathcal{H}$.

☆ **Proposition 1.14**: $\mathcal{H}^*$ is a Banach space.

Proof: $\mathcal{H}^*$ is clearly a "linear space".

If $\varphi_1, \varphi_2 \in \mathcal{H}^*$, then

① $\varphi_1 + \varphi_2$ is linear

② $|(\varphi_1 + \varphi_2)(f)| \le |\varphi_1(f)| + |\varphi_2(f)|$

$\le M_1\|f\| + M_2\|f\| = (M_1 + M_2)\|f\| \quad \forall f \in \mathcal{H}$

where M_1, M_2 are the bounds for φ_1, φ_2 respectively.

$\therefore \quad \varphi_1 + \varphi_2 \in \mathcal{H}^*$

$\rightarrow$ $\|\cdot\|$ defined by (A) is a norm:

(a) $\|\varphi\| = 0 \iff \varphi = 0$ in $\mathcal{H}^*$

(b) $\|\lambda \varphi\| = |\lambda| \|\varphi\| \quad \forall \lambda \in \mathbb{C}, \varphi \in \mathcal{H}^*$

(c) $$\|\varphi_1 + \varphi_2\| = \sup_{f \neq 0} \frac{|(\varphi_1 + \varphi_2)(f)|}{\|f\|}$$

$$\leq \sup_{f \neq 0} \frac{|\varphi_1(f)|}{\|f\|} + \sup_{f \neq 0} \frac{|\varphi_2(f)|}{\|f\|}$$

$$= \|\varphi_1\| + \|\varphi_2\| \quad \forall \varphi_1, \varphi_2 \in \mathcal{H}^*$$

Completeness: Suppose $(\varphi_n)_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{H}^*$.

$\rightarrow$ $|\varphi_n(f) - \varphi_m(f)| \leq \|\varphi_n - \varphi_m\| \|f\|$

$\forall f \in \mathcal{H} \implies (\varphi_n(f))_{n=1}^{\infty}$ is Cauchy in $\mathbb{C}$

$\forall f \in \mathcal{H}$

$\therefore$ Define $\varphi(f) := \lim_{n\to\infty} \varphi_n(f) \quad \forall f \in \mathcal{H}$

→ φ is clearly linear

→ Choose $N \in \mathbb{N}$ s.t

$$n, m \geq N \implies \|\varphi_n - \varphi_m\| < 1$$

--> So for $f \in \mathcal{H}$, we have

$$|\varphi(f)| \leq |\varphi(f) - \varphi_N(f)| + |\varphi_N(f)|$$

$$= \lim_{n\to\infty} |\varphi_n(f) - \varphi_N(f)| + |\varphi_N(f)|$$

$$\leq \lim_{n\to\infty} \|\varphi_n - \varphi_N\| \|f\| + \|\varphi_N\| \|f\|$$

$$\leq (1 + \|\varphi_N\|) \|f\| \implies \varphi \in \mathcal{H}^*$$

→ It remains to show $\varphi_n \to \varphi$ in $\mathcal{H}^*$.

→ Given $\varepsilon > 0$, choose N s.t

$$n, m \geq N \implies \|\varphi_n - \varphi_m\| < \varepsilon$$

→ Then for $f \in \mathcal{H}$ and $m, n \geq N$, we get

$$|(\varphi - \varphi_n)(f)| \leq |(\varphi - \varphi_m)(f)| + |(\varphi_m - \varphi_n)(f)|$$

$$\leq |(\varphi - \varphi_m)(f)| + \varepsilon \|f\| \leq 2\varepsilon \|f\|$$

$\therefore$ $\|\varphi - \varphi_n\| \leq 2\varepsilon$ by definition

$\Rightarrow$ $\varphi_n \longrightarrow \varphi$ in $\mathcal{H}^*$ and $\mathcal{H}^*$ is complete

So $\mathcal{H}^*$ is a Banach space ▨