☆ Let X and Y be any topological spaces and $f: X \to Y$ be a map. Consider the following conditions:

Ⓐ f is continuous

Ⓑ Given any "sequence"

$$x_n \longrightarrow x \text{ in } X \implies f(x_n) \longrightarrow f(x) \text{ in } Y$$

→ In general

Ⓐ ⟹ Ⓑ but Ⓑ ⇏ Ⓐ

→ Ⓐ is equivalent to Ⓑ if X and Y are "first-countable": each point of X and Y have a "countable neighborhood base" or "local base"

→ X has a "countable local base" if $\forall x \in X$, $\exists$ a sequence of nds.

$N_1, N_2, \cdots$ of x such that

for any nbd. N of x $\exists\, i \in \mathbb{N}$

such that $N_i \subset N$.

→ "Almost all" spaces in mathematics are first-countable, in particular all metric spaces.

☆ If we introduce

(B') For any "net" $(x_\alpha)_{\alpha \in A}$ in X

$$\lim_{\alpha \in A} x_\alpha = x \quad \Rightarrow \quad \lim_{\alpha \in A} f(x_\alpha) = f(x)$$

in X in Y

→ Then (A) and (B') are "always" equivalent.

☆ General Notion of Summability

Let $(f_\alpha)_{\alpha \in A}$ be a set of vectors in a Banach space $\mathcal{H}$.

Let $\mathcal{F} := \{F \subset A : F \text{ finite}\}$

→ With the "partial order"

$$F_1 \leq F_2 \Leftrightarrow F_1 \subset F_2$$

$\mathcal{F}$ is a directed set.

→ For each $F \in \mathcal{F}$, let

$$g_F = \sum_{\alpha \in F}' f_\alpha \quad \text{"partial sum"}$$

⇒ $(g_F)_{F \in \mathcal{F}}$ is a "net" of partial sums.

→ If $(g_F)_{F \in \mathcal{F}}$ converges to some $g \in \mathcal{H}$, then we say the "sum" $\sum_{\alpha \in A}' f_\alpha$ "converges" and write

$$g = \sum_{\alpha \in A}' f_\alpha .$$

Proposition 1.9: If $(f_\alpha)_{\alpha \in A}$ is a set of vectors in a Banach space $\mathcal{H}$, then

$$\sum_{\alpha \in A}' \|f_\alpha\| \text{ converges in } \mathbb{R}$$

$$\Rightarrow \sum_{\alpha \in A}' f_\alpha \text{ converges in } \mathcal{H}.$$

Proof: It suffices to show that the net of partial sums $(g_F)_{F \in \mathcal{F}}$, where

$$g_F = \sum_{\alpha \in F}' f_\alpha \,,$$ is a "Cauchy net".

→ Since $\sum_{\alpha \in A}' \|f_\alpha\|$ converges in $\mathbb{R}$

$\forall\, \varepsilon > 0$, $\exists\, F_0 \in \mathcal{F}$ such that

$$F \geq F_0 \Rightarrow \sum_{\alpha \in F}' \|f_\alpha\| - \sum_{\alpha \in F_0}' \|f_\alpha\| < \varepsilon \quad \text{— A}$$

→ Hence for $F_1, F_2 \geq F_0$ we have

$$\|g_{F_1} - g_{F_2}\| = \Big\| \sum_{\alpha\in F_1}' f_\alpha - \sum_{\alpha\in F_2}' f_\alpha \Big\|$$

$$= \Big\| \sum_{\alpha\in F_1\setminus F_2}' f_\alpha - \sum_{\alpha\in F_2\setminus F_1}' f_\alpha \Big\|$$

$$\leq \sum_{\alpha\in F_1\setminus F_2}' \|f_\alpha\| + \sum_{\alpha\in F_2\setminus F_1}' \|f_\alpha\|$$

$$\leq \sum_{\alpha\in F_1\cup F_2}' \|f_\alpha\| - \sum_{\alpha\in F_0}' \|f_\alpha\| < \varepsilon$$

by (A) because $F_1 \cup F_2 \geq F_0$.

$\therefore$ $(g_F)_{F\in\mathcal{F}}$ is Cauchy and $\sum_{\alpha\in A}' f_\alpha$ converges ▨

Corollary: A normed linear space $\mathcal{H}$ is a Banach space if and only if

$$\sum_{n=1}^{\infty}' \|f_n\| < \infty \Rightarrow \sum_{n=1}^{\infty}' f_n \text{ converges in } \mathcal{H}$$

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for any sequence $(f_n)_{n=1}^{\infty}$ in $\mathcal{H}$.

Proof: ($\Rightarrow$) This direction of the proof is Proposition 1.9.

($\Leftarrow$) Assume $(g_n)_{n=1}^{\infty}$ is a Cauchy sequence in a normed space $\mathcal{H}$ where condition (B) is valid.

→ Choose subsequence $(g_{n_k})_{k=1}^{\infty}$ as follows: Choose $n_1 \in \mathbb{N}$ s.t

$$i, j \geq n_1 \Rightarrow \|g_i - g_j\| < 1$$

→ Having chosen $(n_k)_{k=1}^{N}$, choose $n_{N+1} > n_N$ s.t $i, j > n_{N+1}$ implies

$$\|g_i - g_j\| < 2^{-N}$$

→ Setting $f_k := g_{n_k} - g_{n_{k-1}}$ for $k > 1$ and $f_1 = g_{n_1}$, then we get

$$\sum_{k=1}^{\infty} \|f_k\| = \sum_{k=1}^{\infty} \|g_{n_k} - g_{n_{k-1}}\| \lesssim \sum_{k=1}^{\infty} 2^{-k} < \infty$$

$\therefore \sum_{k=1}^{\infty} f_k$ converges in $\mathcal{H}$ by our hypothesis

$\Rightarrow g_{n_k} = \sum_{j=1}^{k} f_j$ converges in $\mathcal{H}$

$\Rightarrow (g_{n_k})_{k=1}^{\infty}$ converges in $\mathcal{H}$

$\Rightarrow (g_n)_{n=1}^{\infty}$ converges in $\mathcal{H}$

$\therefore \mathcal{H}$ is complete $\blacksquare$