☆ Abstract Banach Space

A Banach space is a complex linear space $\mathcal{H}$ with a norm $\|\cdot\|$ satisfying

① $\|f\| = 0$ iff $f = 0$

② $\|\lambda f\| = |\lambda| \, \|f\| \quad \forall \lambda \in \mathbb{C}, f \in \mathcal{H}$

③ $\|f+g\| \leq \|f\| + \|g\| \quad \forall f, g \in \mathcal{H}$

such that $\mathcal{H}$ is complete in the metric

$$d(f,g) := \|f-g\|$$

induced by the norm.

☆ The functions

$a : \mathcal{H} \times \mathcal{H} \to \mathcal{H} \quad ; \quad a(f,g) := f+g$

$s : \mathbb{C} \times \mathcal{H} \to \mathcal{H} \quad ; \quad s(\lambda, f) := \lambda f$

$n : \mathcal{H} \to \mathbb{R}^+ \quad ; \quad n(f) := \|f\|$

are continuous.

→ Possible norms on $\mathcal{H} \times \mathcal{H}$ and $\mathbb{C} \times \mathcal{H}$:

$$\|(f,g)\| := \|f\| + \|g\| \quad \text{and}$$

$$\|(\lambda, f)\| := |\lambda| + \|f\|$$

☆ Directed Sets and Nets

→ A directed set A is a partially ordered set such that for each pair $\alpha, \beta \in A$ $\exists\, \gamma \in A$ such that $\gamma \geq \alpha$ and $\gamma \geq \beta$

→ A net is a function

$\alpha \longmapsto \lambda_\alpha$ on a directed set A

$(\lambda_\alpha)_{\alpha \in A}$ "another way to write"

→ If $A = \mathbb{N}$, then $(\lambda_n)_{n \in \mathbb{N}}$ is a sequence.

→ In general, one may use $A = \mathbb{R}$ or any set of much larger cardinality.

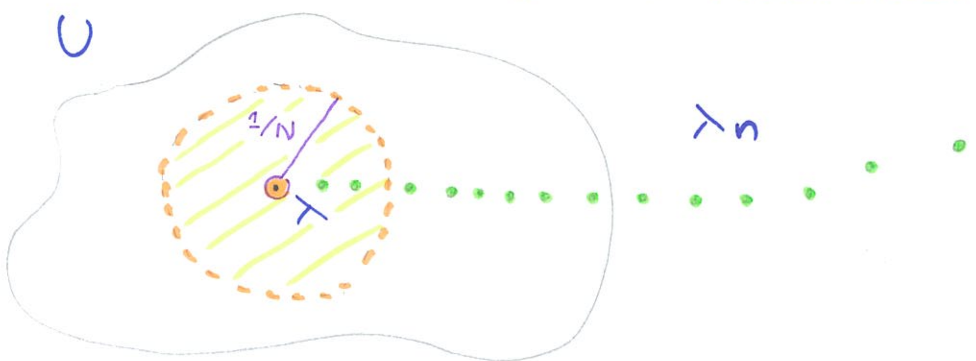
→ If $(\lambda_\alpha)_{\alpha \in A} \subset X$ "topological space"

then we say that λ_α "converges" to some $\lambda \in X$, if for each neighborhood U of λ $\exists\ \alpha_U \in A$ such that $\lambda_\alpha \in U$ whenever $\alpha \geq \alpha_U$.

→ Recall: If (X, d) is a metric space then a sequence $(\lambda_n)_{n \in \mathbb{N}}$ converges $\lambda_n \longrightarrow \lambda$ as $n \to \infty$ if for each nbd. U of λ $\exists\ N \in \mathbb{N}$ such that

$$\lambda_n \in B_{1/N}(\lambda) = \{x \in X : d(x, \lambda) < \tfrac{1}{N}\}$$

$$\subset U \quad \text{whenever } n \geq N.$$



→ In metric spaces we have a "countable" base of open sets at each point.

→ In general topological spaces this is "not true". So we need "convergent nets" to describe the topology.

Definition: A net $(f_\alpha)_{\alpha \in A}$ in a Banach space X is a "Cauch net" if

$$\forall \varepsilon > 0 \quad \exists \alpha_0 \in A \quad \text{s.t} \quad \alpha_1, \alpha_2 \geq \alpha_0$$

$$\Rightarrow \quad \| f_{\alpha_1} - f_{\alpha_2} \| < \varepsilon .$$

Proposition 1.7: In a Banach space $\mathcal{H}$ each Cauchy net is convergent.

Proof: Let $(f_\alpha)_{\alpha \in A} \subset \mathcal{H}$ be a Cauchy net.

→ Choose $\alpha_1 \in A$ such that $\alpha \geq \alpha_1$

$\Rightarrow \|f_\alpha - f_{\alpha_1}\| < 1$

→ Inductively, having chosen $(\alpha_n)_{n=1}^{n}$ in A, we choose $\alpha_{n+1} \geq \alpha_n$ s.t

$$\alpha \geq \alpha_{n+1} \Rightarrow \|f_\alpha - f_{\alpha_{n+1}}\| < \frac{1}{n+1}$$

→ This way we forma sequence $(f_{\alpha_n})_{n=1}^{\infty}$ which is "Cauchy".

→ Since $\mathcal{H}$ is complete $\exists f \in \mathcal{H}$ s.t $\lim_{n\to\infty} f_{\alpha_n} = f$.

→ We want to prove that $(f_\alpha)_{\alpha \in A}$ converges to f as a net:

$$\lim_{\alpha \in A} f_\alpha = f.$$

→ Given $\varepsilon > 0$, choose $n \in \mathbb{N}$ s.t $\frac{1}{n} < \frac{\varepsilon}{2}$ and $\|f_{\alpha_n} - f\| < \varepsilon/2$.

→ Then for $\alpha \geq \alpha_n$ we have

$$\| f_\alpha - f \| \leq \| f_\alpha - f_{\alpha_n} \| + \| f_{\alpha_n} - f \|$$

$$< \frac{1}{n} + \frac{\varepsilon}{2} < \varepsilon \quad \blacksquare$$