☆ **Theorem:** If $\mathcal{H}$ and $\mathcal{Y}$ are Banach spaces and $T \in \mathcal{L}(\mathcal{H}, \mathcal{Y})$ is one-to-one and onto, then T^{-1} exists and is bounded.

Proof: The map $T^{-1} : \mathcal{Y} \to \mathcal{H}$ is well defined and we must show it is bounded.

→ For $r > 0$ let $B_r^{\mathcal{H}} = \{ f \in \mathcal{H} : \| f \| \leq r \}$

and $B_r^{\mathcal{Y}} = \{ g \in \mathcal{Y} : \| g \| \leq r \}$

→ For boundedness of T^{-1}, it is enough to prove $T^{-1}(B_1^{\mathcal{Y}}) \subset B_r^{\mathcal{H}}$ for some $r > 0$

$\Leftrightarrow$ $B_1^{\mathcal{Y}} \subset T(B_N^{\mathcal{H}})$ for some $N \in \mathbb{N}$

→ Sine T is onto, we have

$$\bigcup_{n=1}^{\infty} T[B_n^{\mathcal{H}}] = \mathcal{Y}$$

→ The Baire Category Theorem states that $\mathcal{Y}$ is not the countable union of nowhere dense sets.

→ $\exists\, N \in \mathbb{N}$ s.t $\text{clos}\{T(B_N^{\mathcal{H}})\}$ contains a non-empty open set.

→ So $\exists\, h \in B_N^{\mathcal{H}}$ and $\varepsilon > 0$ s.t

$$Th + B_\varepsilon^{\mathcal{Y}} = \{f \in \mathcal{Y} : \|f - Th\| < \varepsilon\} \subset \text{clos}\{T(B_N^{\mathcal{H}})\}$$

$$\therefore\ B_\varepsilon^{\mathcal{Y}} \subset -Th + \text{clos}\{T(B_N^{\mathcal{H}})\}$$

$$\subset \text{clos}\{T(B_{2N}^{\mathcal{H}})\}$$

$$\Rightarrow B_1^{\mathcal{Y}} \subset \text{clos}\{T(B_r^{\mathcal{H}})\}, \quad r = \frac{2N}{\varepsilon}$$

→ We want to remove the closure.

→ Let $f \in B_1^{\mathcal{Y}}$. Then $\exists\, g_1 \in B_r^{\mathcal{H}}$ with

$$\|f - Tg_1\| < \frac{1}{2}$$

→ Since $f - Tg_1 \in B_{1/2}^{\mathcal{Y}}$, $\exists\, g_2 \in B_{r/2}^{\mathcal{H}}$ with

$$\|f - Tg_1 - Tg_2\| < 1/4$$

→ Continuing by induction, we obtain a sequence $(g_n)_{n=1}^{\infty}$ s.t

$$\|g_n\| \leq \frac{r}{2^{n-1}} \quad \text{and} \quad \|f - \sum_{i=1}^{n} Tg_i\| < \frac{1}{2^n}$$

→ Since $\sum_{n=1}^{\infty} \|g_n\| \leq \sum_{n=1}^{\infty} \frac{r}{2^{n-1}} = 2r$

⇒ $\sum_{n=1}^{\infty} g_n$ converges to some $g \in B_{2r}^{\mathcal{X}}$

→ $Tg = T\left(\lim_{n\to\infty} \sum_{k=1}^{n} g_k\right) = \lim_{n\to\infty} \sum_{k=1}^{n} Tg_k$

$= f$

$\therefore \quad B_1^{\mathcal{Y}} \subset T(B_{2r}^{\mathcal{X}})$ ▨

☆ Open Mapping Theorem:

If $\mathcal{X}$ and $\mathcal{Y}$ are Banach space, and T is an onto operator in $\mathcal{L}(\mathcal{X}, \mathcal{Y})$, then T is an open map.

Proof: Since T is continuous

$$M = \{ f \in \mathcal{H} : Tf = 0 \}$$

is a closed subspace of $\mathcal{H}$.

→ We want to define a map S from $\mathcal{H}/M \to \mathcal{Y}$ as follows:

For $[f] \in \mathcal{H}/M$ set $S[f] = Tg$ for $g \in [f]$

→ Since $g_1, g_2 \in [f] \Rightarrow g_1 - g_2 \in M$

so $T(g_1 - g_2) = 0 \Rightarrow Tg_1 = Tg_2$

$\Rightarrow S$ is well-defined.

→ S is obviously linear and

$$\|S[f]\| = \inf_{g \in [f]} \|Tg\| \leq \|T\| \inf_{g \in [f]} \|g\|$$

$$= \|T\| \, \|[f]\|$$

$\Rightarrow S : \mathcal{H}/M \to \mathcal{Y}$ is bounded

with $\|S\| \leq \|T\|$.

$$\begin{tikzcd} \mathcal{H} \arrow[r, "T"] \arrow[d, "\pi"'] & \mathcal{Y} \\ \mathcal{H}/M \arrow[ur, dashed, "S"'] & \end{tikzcd}$$

→ If $S[f] = 0$ then $Tf = 0$

$\Rightarrow f \in M$ and $[f] = 0$

$\Rightarrow S$ is injective

→ S is surjective because T is.

→ So S^{-1} exists and is continuous

$\Rightarrow S$ is an open map

→ Finally $T = S\pi$ is open

because $\pi : \mathcal{H} \to \mathcal{H}/M$ is open ▨

☆ Uniform Boundedness Principle

If $\mathcal{H}$ is a Banach space and $(e_n)_{n=1}^{\infty}$ is a sequence in $\mathcal{H}^*$ such that

$$\sup\{|\varphi_n(f)| : n \in \mathbb{N}\} < \infty$$

for all $f \in \mathcal{H}$, then

$$\sup\{\|\varphi_n\| : n \in \mathbb{N}\} < \infty.$$

Proof: Let $u : \mathcal{H} \to \mathbb{R}$ be defined by

$$u(f) := \sup\{|\varphi_n(f)| : n \in \mathbb{N}\}.$$

→ Let $L_k = \{f \in \mathcal{H} : u(f) \leq k\}$, $k \in \mathbb{N}$

→ L_k is closed.

→ If $k \geq u(f)$ then $f \in L_k$ and

$$\bigcup_{k \in \mathbb{N}} L_k = \mathcal{H}.$$

→ By Baire's Theorem $\exists$ some L_{k_0} that contains an open ball

$$\{f \in \mathcal{H} : \|f - f_0\| < \delta\}$$

for $f_0 \in \mathcal{H}$ and $\delta > 0$.

→ Now we obtain

$$\|\varphi_n\| = \sup_{g \in B_\delta} \frac{1}{\delta} |\varphi_n(g)|$$

$$\leq \frac{1}{\delta} \sup_{g \in B_\delta} |\varphi_n(g+f_0)| + \frac{1}{\delta} |\varphi_n(f_0)|$$

$$\leq \frac{1}{\delta} k_0 + \frac{1}{\delta} u(f_0)$$

$$\therefore \quad \sup_{n \in \mathbb{N}} \{ \|\varphi_n\| : n \in \mathbb{N} \} \leq \frac{k_0 + u(f_0)}{\delta} \quad \blacksquare$$