☆ Quotient Spaces

Let $\mathcal{H}$ be a Banach space and M be a closed subspace of $\mathcal{H}$.

→ There is a norm on the "quotient space" $\mathcal{H}/M$ that makes it into a Banach space.

→ Let $\mathcal{H}/M$ denote the linear space of equivalence classes

$$\{[f] : f \in \mathcal{H}\} \quad \text{where } [f] = \{f+g : g \in M\}$$

$$= \{f + M : f \in \mathcal{H}\} \qquad\qquad = f + M$$

→ Define a norm on $\mathcal{H}/M$ by

$$\|[f]\| = \inf_{g \in M} \|f+g\| = \inf_{h \in [f]} \|h\|$$

(a) $\|[f]\| = 0$

$\Rightarrow \exists$ a sequence $(g_n)_{n=1}^{\infty}$ in M with

$$\lim_{n\to\infty} \|f+g_n\| = 0$$

$\Rightarrow$ $f \in M$ because M is closed

$\therefore$ $[f] = [0]$ "zero element" in $\mathcal{H}/M$

$\rightarrow$ Conversely if $[f] = [0]$ then $f \in M$

and $\quad 0 \leq \|[f]\| \leq \|f-f\| = 0$

$\therefore$ $\|[f]\| = 0 \Leftarrow [f] = [0]$

(b) If $f_1, f_2 \in \mathcal{H}$ and $\lambda \in \mathbb{C}$, then

$$\|\lambda[f_1]\| = \|[\lambda f_1]\| = \inf_{g\in M} \|\lambda f_1 + g\|$$

$$= |\lambda| \inf_{h\in M} \|f_1 + h\| = |\lambda| \, \|[f_1]\| \text{, and}$$

$$\|[f_1] + [f_2]\| = \|[f_1+f_2]\| = \inf_{g\in M} \|f_1+f_2+g\|$$

$$= \inf_{g_1, g_2 \in M} \|f_1 + g_1 + f_2 + g_2\|$$

$$\leq \inf_{g_1\in M} \|f_1 + g_1\| + \inf_{g_2 \in M} \|f_2 + g_2\|$$

$\leq \| [f_1] \| + \| [f_2] \|$

$\therefore \| \cdot \|$ is a norm on $\mathcal{H}/\mathcal{M}$.

© Completeness of $\mathcal{H}/\mathcal{M}$

Let $([f_n])_{n=1}^{\infty}$ be a Cauchy sequence in $\mathcal{H}/\mathcal{M}$.

$\rightarrow$ $\exists$ a subsequence $(f_{n_k})_{k=1}^{\infty}$ s.t

$$\| [f_{n_{k+1}}] - [f_{n_k}] \| < 1/2^k$$

$\rightarrow$ Choose $h_k \in [f_{n_{k+1}} - f_{n_k}]$ s.t

$\| h_k \| < 1/2^k \Rightarrow \sum_{k=1}^{\infty} \| h_k \| < 1$

$\Rightarrow h = \sum_{k=1}^{\infty} h_k$ exists by Proposition 1.9

$\rightarrow [f_{n_k} - f_{n_1}] = \sum_{i=1}^{k-1} [f_{n_{i+1}} - f_{n_i}]$

$= \sum_{i=1}^{k-1} [h_i] \Rightarrow \lim_{k \to \infty} [f_{n_k} - f_{n_1}] = [h]$

$\therefore \lim_{k \to \infty} [f_{n_k}] = [h + f_{n_1}]$

$\therefore ([f_n])_{n=1}^{\infty}$ converges in $\mathcal{H}/M$

and $\mathcal{H}/M$ is a Banach space ▨

☆ The "natural map" $\pi : \mathcal{H} \to \mathcal{H}/M$

is a contraction and an open map.

Proof: Since $\|\pi(f)\| = \|[f]\|$

$= \inf_{h \in [f]} \|h\| \leq \|f\|$ "contraction"

→ Suppose $f \in \mathcal{H}$, $\varepsilon > 0$ and

$N_\varepsilon(f) = \{g \in \mathcal{H} : \|f-g\| < \varepsilon\}$.

→ If $[h] \in N_\varepsilon([f]) = \{[k] \in \mathcal{H}/M : \|[f]-[k]\| < \varepsilon\}$

then $\exists\, h_0 \in [h]$ s.t $\|f - h_0\| < \varepsilon$

→ In other words $[h] \in \pi(N_\varepsilon(f))$

$\Rightarrow N_\varepsilon([f]) \subset \pi(N_\varepsilon(f))$

$\therefore \pi$ is an open map ▨

☆ **Definition**: Let $\mathcal{H}$ and $\mathcal{Y}$ be Banach spaces. A linear map $T : \mathcal{H} \to \mathcal{Y}$ is "bounded" if

$$\|T\| := \sup_{f \neq 0} \frac{\|Tf\|}{\|f\|} < \infty$$

→ The set of bounded linear maps of $\mathcal{H}$ to $\mathcal{Y}$ is denoted $\mathcal{L}(\mathcal{H}, \mathcal{Y})$ with $\mathcal{L}(\mathcal{H}) := \mathcal{L}(\mathcal{H}, \mathcal{H})$.

→ $T \in \mathcal{L}(\mathcal{H}, \mathcal{Y})$ is bounded

$\Leftrightarrow$ it is continuous.

→ $\mathcal{L}(\mathcal{H}, \mathcal{Y})$ is a Banach space.

☆ **The Baire Category Theorem**

Let S be a topological space. A set $E \subset S$ is "nowhere dense" if $\overline{E}$ has empty interior.

The sets of "first category" in S are the countable unions of nowhere dense sets. "Meager sets"

→ A subset of S is of "second category" if it is not of first category. "nonmeager sets"

Theorem: If S is a metric space that is complete, then S is of second category.

Proof: Suppose $E_1, E_2, \dots$ are nowhere dense sets. Then $V_1, V_2, \dots$ are dense open sets of S. ($V_i = \overline{E_i}^c$)

→ Let B_0 be an open set in S.

→ If $n \geq 1$ and an open $B_{n-1} \neq \emptyset$ has been chosen, then $\exists$ an open $B_n \neq \emptyset$ with $\overline{B}_n \subset V_n \cap B_{n-1}$

Put $K = \bigcap_{n=1}^{\infty} \bar{B}_n$ and assume

B_n are just balls of radius $< 1/n$

$\Rightarrow K \neq \emptyset$ and $K \subset B_0$ and

$K \subset V_n \ \forall n \in \mathbb{N}$

$\therefore B_0 \cap \left(\bigcap_{n=1}^{\infty} V_n \right) \neq \emptyset$

$\Rightarrow \bigcap_{n=1}^{\infty} V_n \neq \emptyset \Rightarrow S \neq \bigcup_{n=1}^{\infty} E_n$

$\therefore$ S is not of first category ▨