☆ Hahn-Banach Theorem for Banach spaces

Let M be a subspace of the Banach space $\mathcal{H}$. If φ is a bounded linear functional on M, then $\exists\ \Phi \in \mathcal{H}^*$ s.t

$$\Phi(f) = \varphi(f) \text{ for } f \in M \text{ and } \|\Phi\| = \|\varphi\|.$$

Proof: Denote by $\tilde{\mathcal{H}}$ the real linear space $\mathcal{H}$ with scalar field $\mathbb{R}$ instead of $\mathbb{C}$.

→ Then the norm $\|\cdot\| : \tilde{\mathcal{H}} \to \mathbb{R}$ is a sublinear functional on $\tilde{\mathcal{H}}$.

→ Also $\psi = \operatorname{Re}\varphi$ is an $\mathbb{R}$-linear functional on the real subspace $\tilde{M}$ with $\|\psi\| \le \|\varphi\|$ $\quad (\varphi = \psi + i \operatorname{Im}\varphi)$

→ Setting $p(f) := \|\varphi\| \|f\|$

$$\Rightarrow \psi(f) \le \|\psi\| \|f\| \le p(f) \quad \forall f \in \tilde{M}$$

where p is also sublinear on $\tilde{\mathcal{H}}$.

$\therefore$ By the first Hahn-Banach theorem

$\exists$ an $\mathbb{R}$-linear functional Ψ on $\tilde{\mathcal{H}}$ s.t

$$\Psi(f) = \psi(f) \text{ on } \tilde{M} \text{ and}$$

$$\Psi(f) \leq p(f) = \|\varphi\| \, \|f\| \quad \forall f \in \mathcal{H} \quad \text{--} \; (A)$$

$\rightarrow$ Define $\Phi : \mathcal{H} \rightarrow \mathbb{C}$ by

$$\Phi(f) = \Psi(f) - i\Psi(if)$$

$\rightarrow$ We want to show that $\Phi \in \mathcal{H}^*$ (bounded $\mathbb{C}$-linear functional) that extends φ with norm $\|\varphi\|$.

$\rightarrow$ For $f, g \in \mathcal{H}$, we have

$$\Phi(f+g) = \Psi(f+g) - i\Psi(i(f+g))$$

$$= \Psi(f) + \Psi(g) - i\Psi(if) - i\Psi(ig)$$

$$= \Phi(f) + \Phi(g)$$

$\rightarrow$ For $\lambda_1, \lambda_2 \in \mathbb{R}$

$$\Phi((\lambda_1 + i\lambda_2) f) = \Phi(\lambda_1 f) + \Phi(i\lambda_2 f)$$

$= \lambda_1 \Phi(f) + i\lambda_2 \Phi(f) = (\lambda_1 + i\lambda_2)\Phi(f)$

because $\Phi(if) = \Psi(if) + i\Psi(f) = i\Phi(f)$.

$\therefore$ Φ is a complex linear functional on $\mathcal{H}$. For $f \in M$, we have

$\Phi(f) = \Psi(f) - i\Psi(if) = \mathrm{Re}\,\varphi(f) - i\,\mathrm{Re}\,\varphi(if)$

$= \mathrm{Re}\,\varphi(f) - i\,\mathrm{Re}(i\varphi(f))$

$= \mathrm{Re}\,\varphi(f) - i(-\mathrm{Im}\,\varphi(f)) = \varphi(f)$

because $\mathrm{Im}\, z = -\mathrm{Re}(iz)$, $z = x + iy$

So Φ extends φ from M to $\mathcal{H}$.

→ By (A) we have $\|\Psi\| \le \|\varphi\|$ and also $\|\varphi\| \le \|\Phi\|$. Therefore to prove $\|\Phi\| = \|\varphi\|$ it suffices to prove $\|\Phi\| \le \|\Psi\|$.

→ For $f \in \mathcal{H}$ write $\Phi(f) = re^{i\theta}$, then

$|\Phi(f)| = r = e^{-i\theta}\Phi(f) = \Phi(e^{-i\theta} f)$

$= \Psi(e^{-i\theta} f) = |\Psi(e^{-i\theta} f)| \leq \|\varphi\| \, \|f\|$

$\therefore \ \|\Phi\| \leq \|\varphi\| \implies \|\Phi\| = \|\varphi\|$

and $\Phi \in \mathcal{H}^*$ that extends φ ▨

Corollary: If $f \in \mathcal{H}$, then $\exists \ \varphi \in \mathcal{H}^*$ with $\|\varphi\| = 1$ so that $\varphi(f) = \|f\|$.

Proof: Assume $f \neq 0$. Let

$$M = \{\lambda f : \lambda \in \mathbb{C}\} = \mathbb{C} f$$

and define ψ on M by $\psi(f\lambda) = \lambda \|f\|$

$\rightarrow \psi$ is a bound $\mathbb{C}$-linear functional on M with $\|\psi\| = \sup\limits_{\|\lambda f\| \leq 1} |\psi(\lambda f)|$

$= \sup\limits_{|\lambda| \|f\| \leq 1} |\lambda| \|f\| = 1$

$\therefore$ By H-B Theorem $\exists \ \varphi \in \mathcal{H}^*$ with $\|\varphi\| = \|\psi\| = 1$ s.t

$$\varphi(f) = \psi(f) = \|f\| \quad ▨$$

Corollary: If $\omega(f) = 0 \ \forall \omega \in \mathcal{H}^*$ then $f = 0$.

☆ Recall the "evaluation map"

$$J : \mathcal{H} \longrightarrow \mathcal{H}^{**} \quad \text{where } (Jf)(\omega) = \hat{f}(\omega)$$
$$f \longmapsto \hat{f} \qquad\qquad\qquad\qquad = \omega(f)$$

$$\rightarrow \|Jf\| = \sup_{\substack{\|\omega\| \le 1 \\ \omega \in \mathcal{H}^*}} |\hat{f}(\omega)| = \sup_{\substack{\|\omega\| \le 1 \\ \omega \in \mathcal{H}^*}} |\omega(f)|$$

$= \|f\|$ in general $\forall f \in \mathcal{H}$ by previous corollaries

$\therefore$ J is an isometry (hence injective) ▨

☆ $\mathcal{C}(X)$ is a universal Banach space

Every Banach space $\mathcal{H}$ is isometrically isomorphic to a closed subspace of $\mathcal{C}(X)$ for some compact Hausdorff space X.

Proof: Let $X = B_1^*$ the unit ball of $\mathcal{H}^*$ with the w^*-topology.

→ So X is a compact Hausdorff space.

→ Define $\beta : \mathcal{H} \to \mathcal{C}(X)$ by

$$(\beta f)(\varphi) = \varphi(f)$$

→ For $f_1, f_2 \in \mathcal{H}$, $\lambda_1, \lambda_2 \in \mathbb{C}$, we have

$$\beta(\lambda_1 f_1 + \lambda_2 f_2)(\varphi) = \varphi(\lambda_1 f_1 + \lambda_2 f_2)$$

$$= \lambda_1 \varphi(f_1) + \lambda_2 \varphi(f_2)$$

$$= \lambda_1 \beta(f_1)(\varphi) + \lambda_2 \beta(f_2)(\varphi)$$

$\therefore$ β is a linear map.

→ For $f \in \mathcal{H}$ we have

$$\|\beta(f)\|_\infty = \sup_{\varphi \in B_1^*} |\beta(f)(\varphi)| = \sup_{\varphi \in B_1^*} |\varphi(f)|$$

$$\leq \sup_{\varphi \in B_1^*} \|\varphi\| \, \|f\| \leq \|f\| \quad \text{— (B)}$$

But by the Corollaries $\exists\ \varphi \in \mathcal{B}_1^*$ s.t

$\varphi(f) = \|f\|$, so we have in (B)

$$\| \beta(f) \|_\infty = \| f \|$$

$\therefore\ \beta : \mathcal{H} \xrightarrow[\cong]{} \mathcal{C}(X)$ is an

" isometric isomorphism " $\blacksquare$